Baryon fluctuations and the QCD phase transition

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The dynamic separation into phases of high and low baryon density in a heavy ion collision can enhance fluctuations of the net rapidity density of baryons compared to model expectations. We demonstrate that event-by-event proton and antiproton measurements can be used to observe this phenomenon. We then perform real-time lattice simulations to show how these fluctuations arise and how they can survive through freeze-out.

If the QCD phase transition is first order, matter at the appropriate temperatures and densities can form a mixed phase consisting of plasma droplets in equilibrium with a surrounding hadronic fluid. If formed in ion collisions, this mixed phase can produce large event-by-event fluctuations as the system hadronizes [1]. In particular, extraordinary baryon number fluctuations [2] can accompany the first order transition expected at high baryon density [3].

In this Rapid Communication we explore the dynamics of phase separation in nuclear collisions. The aims of this paper are twofold. First, we study the role of baryon number fluctuations as a probe of the order of the QCD transition. We focus on the high baryon density regime, where theory [3] and lattice simulations [4] suggest that the QCD phase transition is first order in a strict thermodynamic sense with baryon density as an order parameter. Our work may also apply to RHIC collisions, if the low baryon density systems produced at the highest energies *approximate* a first order transition [5,6]. Second, we generalize techniques from condensed matter physics [7] to confront phase separation in the highly-nonequilibrium context of nuclear collisions. Our framework can be used to systematically address other probes as experimental information and theoretical understanding evolve.

To begin, we describe the character of mixed-phase baryon fluctuations and show how they can be measured. Measurement is not completely straightforward as, e.g., neutrons are not easily observed on an event-by-event basis. We then formulate a dissipative-hydrodynamic model of phase separation and perform numerical simulations for that model.

QCD with two massless flavors can exhibit a first order transition whose coexistence curve culminates in a tricritical point at temperature T_c and baryon chemical potential μ_c [3]. For $T > T_c$ and $\mu < \mu_c$, a second order phase transition breaks/restores chiral symmetry. If the quark masses are sufficiently large, the second order transition is replaced by a smooth transformation (since chiral symmetry is explicitly broken). The first order line remains, however, with the tricritical point replaced by a critical point in the same universality class as a liquid-gas transition.

At RHIC, baryon density may also serve as an approximate order parameter for the nearly first order transition at small net baryon density. Lattice simulations [8,5] and general arguments [6,8,9] show that the baryon susceptibility χ at $\mu = 0$ can increase suddenly as temperature is increased near $T_h \sim 160$ MeV, where the chiral order parameter and,

e.g., the energy density change sharply. Jumps in the susceptibility commonly accompany first order transitions. For a liquid-gas transition, $\chi = \partial \rho / \partial \mu$ is proportional to the compressibility: steam is much more compressible than water.

Large fluctuations in baryon number occur during phase separation in a first order transition. Figure 1(b) shows the phase diagram in the $T - \rho$ plane [3], where ρ is the baryon density. A uniform system quenched into the outer parabolic region will separate into droplets at the high baryon density ρ_q surrounded by matter at density ρ_h. The net baryon number N_B in a subvolume of the system varies depending on the number of droplets in the subvolume. The variance of the baryon number $V_B = \langle N_B^2 \rangle - \langle N_B \rangle^2$ can exceed the equilibrium expectation by an amount

$$\Delta V_B \approx f(1 - f)(\Delta N_B)^2, \tag{1}$$

where f is the fraction of the high density phase in the subvolume V and $\Delta N_B = (\rho_q - \rho_h)V$. In contrast, an equilibrium system follows Poisson statistics, so that $V_B = V + \bar{V} = \langle N + \bar{N} \rangle$, where N, V and $\bar{N}$, $\bar{V}$ are the numbers and variances of baryons and antibaryons and $N_B = N - \bar{N}$.

Experimenters can search for a "super-Poissonian" variance such as Eq. (1) by measuring

$$\Omega_p = \frac{V_{p-\bar{p}} - \langle N_p + N_{\bar{p}} \rangle}{\langle N_p + N_{\bar{p}} \rangle^2}, \tag{2}$$

where N_p and $N_{\bar{p}}$ are the numbers of protons and antiprotons in a rapidity interval and $V_{p-\bar{p}}$ is the variance of the net proton number $N_p - N_{\bar{p}}$. This quantity vanishes in equilibrium and is related to the more familiar scaled variance ω_p
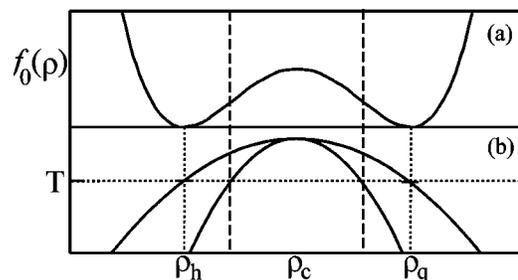

FIG. 1. Free energy (a) and phase diagram (b) vs baryon density for Eq. (4).

$=\langle N_p+N_{\bar{p}}\rangle(1+\Omega_p)$. Most importantly, Ω_p is ideal for our application because of the property

$$\Omega_p=\Omega_B\equiv\frac{V_B-\langle N+\bar{N}\rangle}{\langle N+\bar{N}\rangle^2}, \qquad (3)$$

where N and $\bar{N}$ are the numbers of baryons and antibaryons—including unseen neutrons and antineutrons (the proof follows). The conditions for which Eq. (3) holds are met by a range of thermal and Glauber models that respect isospin symmetry. Isospin fluctuations can alter Eq. (3) near the tricritical point or in the presence of a disoriented chiral condensate, but those effects will be evident from pion measurements.

We demonstrate Eq. (3) by writing the joint probability for N_p and $N_{\bar{p}}$ as $\Sigma_{N,\bar{N}}p(N_p|N)p(N_{\bar{p}}|\bar{N})P(N,\bar{N})$. The distribution $P(N,\bar{N})$, which determines Ω_B, is modified by phase separation; we make no assumptions about its form. We assume that the conditional probability $p(N_p|N)$ for measuring N_p given N baryons is binomial, with q the chance that any individual baryon is a proton (see [10] for notation). We further take $p(N_{\bar{p}}|\bar{N})$ for antiprotons to be binomial with the same q. These assumptions hold for most thermal and multiple-scattering models. The average of the joint distribution is $\langle N_p+N_{\bar{p}}\rangle=\Sigma_{N,\bar{N}}p(N,\bar{N})(\nu+\bar{\nu})$, where the binomial averages $\nu=\Sigma_{N_p}p(N_p|N)N_p$ and $\bar{\nu}=\Sigma_{N_{\bar{p}}}p(N_{\bar{p}}|\bar{N})N_{\bar{p}}$ yield $\langle N_p+N_{\bar{p}}\rangle=q\langle N+\bar{N}\rangle$. The quantity $\langle N+\bar{N}\rangle$ depends only on $P(N,\bar{N})$. Similarly, we find $\langle(N_p-N_{\bar{p}})^2\rangle=q^2\langle(N-\bar{N})^2\rangle+q(1-q)\langle N+\bar{N}\rangle$. We combine these moments to obtain Eq. (3).

The antiproton contribution to Eqs. (2) and (3) is large only at RHIC, where $N_{\bar{p}}/N_p\sim0.6$ at $\sqrt{s}=130A$ GeV [11]. At the top SPS energy, we estimate $\bar{p}$ contributions to Eq. (2) to be at the few percent level in Au+Au at $\sqrt{s}=17.5A$ GeV, since $N_{\bar{p}}/N_p\sim6\%$ [12]. The highest baryon density—and the greatest potential for observing a first order transition—is perhaps at lower energies.

We remark that Jeon and Koch and Asakawa *et al.* have proposed that hadronization may change the character of charge and baryon number fluctuations even in the absence of a phase transition [13]. This effect is essentially Poissonian, however, so it is not clear that it would cause Ω_p to differ from zero, the equilibrium value, or that it could be tested without measuring neutrons. The effect on charge fluctuations is much more dramatic [13].

We now turn to describe the process of phase separation. To describe the state of the mixed phase, we follow the standard condensed matter practice [7] and write a Ginzburg-Landau free energy $f=\kappa(\nabla\rho)^2/2+f_0$, where

$$f_0=-m^2(\rho-\rho_c)^2/2+\lambda(\rho-\rho_c)^4/4 \qquad (4)$$

describes the excursions of the baryon density ρ from its equilibrium value in the uniform matter. For $m^2\propto T_c-T$ we find the correct liquid-gas critical exponents. The values ρ_h and ρ_q in Fig. 1 correspond to the equilibrium densities at $T<T_c$: $\rho_h=\rho_c-\Delta\rho$ and $\rho_q=\rho_c+\Delta\rho$, where $\Delta\rho=\sqrt{m^2/\lambda}$.

The κ term describes the droplet surface tension. For our f_0, we compute $\sigma=(8\kappa m^3/9\lambda^2)^{1/2}\propto\kappa^{1/2}$ [14].

To describe the dynamics of the system, we must account for the fact that baryon number is conserved. Furthermore, it is crucial to include dissipation to describe this strongly fluctuating system. The simplest equations that meet these criteria are

$$\partial\rho/\partial t=M\nabla^2\mu, \quad \mu=f_0'-\kappa\nabla^2\rho; \qquad (5)$$

model B in Ref. [7]. We illustrate that Eq. (5) describes diffusion in a stable liquid by considering fluctuations about the equilibrium density $\rho=\rho_h+\delta\rho_k\exp(-i\mathbf{k}\cdot\mathbf{x})$, where $\delta\rho_k\ll\rho_h$. A system at this density is near the minimum of f_0, so that $f_0'\approx f_0''(\rho_h)\delta\rho_k=2m^2\delta\rho_k$. Therefore, Eq. (5) is standard diffusion equation at linear order in $\delta\rho_k$. We identify the baryon diffusion coefficient at ρ_h as $D=2m^2M$. In general, diffusion drives the system towards homogeneity at all density for which $f_0''(\rho)>0$.

Phase separation is most dramatic if the rapid expansion of the heavy ion system drives the system into the unstable region, i.e., the inner parabolic region in Fig. 1(b), corresponding to $f_0''(\rho)<0$ in Fig. 1(a). Droplets form from runaway density fluctuations in a process known as spinodal decomposition. We estimate the time scale τ_R for this process by considering the time evolution of small fluctuations near ρ_c. We take $\rho=\rho_c+\delta\rho_k\exp(-i\mathbf{k}\cdot\mathbf{x})$ to find

$$\frac{d}{d\tau}\delta\rho_k=m^2M\left(k^2-\frac{\kappa}{m^2}k^4\right)\delta\rho_k\equiv\frac{\delta\rho_k}{\tau_k} \qquad (6)$$

to linear order in $\delta\rho$. Long wavelength disturbances corresponding to $0<k<\sqrt{(m^2/\kappa}$ grow with time, while the surface-tension term stabilizes the shorter wavelength modes. The time scale for growth τ_k is shortest at $k_R=\sqrt{(m^2/2\kappa)}$.

The fastest-growing mode at wave number k_R dominates the early evolution of the system in the unstable regime. The time scale for the growth of this mode is

$$\tau_R=8\xi^2/D, \qquad (7)$$

where $\xi=\kappa^{1/2}/m$ is the correlation length. The values of ξ and τ_R determine the time and spatial scales for the onset of spinodal decomposition. We plausibly estimate the correlation length to be $\xi\sim1$ fm, roughly the value of the inverse sigma mass. For a value of $D\sim8$ fm consistent with calculations in Ref. [17], we find $\tau_R\sim1$ fm as in Ref. [2]. We remark that the large magnitude of D suggested by Ref. [17] is consistent with our assumption in Eq. (5) that baryon diffusion is the dominant transport mode for baryons at high density. Our model superficially suggests a slower onset of the instability for a substantially smaller value of D. However, if D were truly small then it would be necessary to include transport mechanisms involving convection and viscosity. In fact, viscosity must dominate near $\mu=0$, where diffusion precesses are strictly irrelevant [18].

To describe nuclear collisions, we extend Eq. (5) to include drift due to Bjorken longitudinal flow:

$$\partial \rho / \partial \tau + \rho / \tau = M \nabla^2 \mu, \tag{8}$$

where τ is the proper time and μ is given by Eqs. (4) and (5). The new drift term forces the average density to decrease as $\langle \rho \rangle \propto \tau^{-1}$, driving the system through the phase coexistence region. Fluctuations grow when densities approach ρ_c (see Fig. 1). To derive the drift term, observe that Eq. (5) follows from baryon current conservation, which more generally implies $\partial_\mu j^\mu = 0$. The current is $j^\mu = \rho u^\mu + j_d^\mu$ [19], where u^μ is a fluid velocity that includes a contribution from the meson flow, and j_d is the diffusion current, $\propto \nabla \mu$ when $u = (1,0,0,0)$. The left and right sides of Eq. (8), respectively, follow from $\partial_\mu (\rho u^\mu)$ and $\partial_\mu j_d^\mu$ for Bjorken flow.

For times $t \gg \tau_R$, the system undergoes a nonlinear evolution in which droplets merge, reducing their surface energy. To study this regime, we write the evolution equation (8) in the dimensionless form

$$\frac{\partial \psi}{\partial \tau} + \frac{\epsilon + \psi}{\tau} = -\frac{1}{2} \hat{\nabla}^2 (\psi - \psi^3 + \hat{\nabla}^2 \psi), \tag{9}$$

where we use the dimensionless coordinates $\hat{\tau} = 8\tau / \tau_R$ and $\hat{\mathbf{x}} = \mathbf{x}/\xi$. The dimensionless order parameter $\psi \equiv (\rho - \rho_c)/\Delta \rho$ equals ± 1 when $\rho = \rho_{h,q} = \rho_c \pm \Delta \rho$. The only remaining parameter is $\epsilon = \rho_c / \Delta \rho$, which controls the strength of the first order transition. Here, we take $\epsilon = 1$ corresponding to a strongly first order transition. Observe that Eq. (9) depends on the temperature and density scale only through ϵ. This is an artifact of our very simplistic quadratic f_0; we will introduce a more realistic free energy density in a later work to study the role of temperature in the evolution.

We solve Eq. (9) numerically on a $2+1$ dimensional lattice following Grant *et al.* [15]. We use a forward Euler method to evolve the system in time for a time step $\Delta \hat{\tau} = 0.05$. We study the evolution in the transverse plane and in the rapidity η-x_T plane, where x_T is a Cartesian transverse coordinate. The Laplacian in the η-x_T case is

$$\hat{\nabla}^2 = \frac{1}{\hat{\tau}^2} \frac{\partial^2}{\partial \eta^2} + \hat{\nabla}_\perp^2. \tag{10}$$

To treat the higher spatial derivatives we extend the next-nearest-neighbor algorithm developed by Oono and Puri [16] and used in [15] to account for the asymmetric η-x_T lattice. We write

$$\hat{\nabla}^2 \psi = \frac{1}{2(\Delta x)^2} \left(\sum_{NN} \psi + \frac{1}{4} \sum_{NNNN} \psi - \frac{5}{2} \psi \right), \tag{11}$$

where the first sum runs over the four nearest neighbors (NN) and the second over the four adjacent next-next-nearest neighbors (NNNN). Oono and Puri use the diagonal next-nearest-neighbors instead in Eq. (11)—a formulation that requires a symmetric lattice. We take $\Delta x = 1$. We find that our results are practically indistinguishable from NNN results [15] for this spacing on a symmetric lattice. To study longitudinal expansion, it suffices to replace one coordinate Δx for $\hat{\tau} \Delta \eta$.
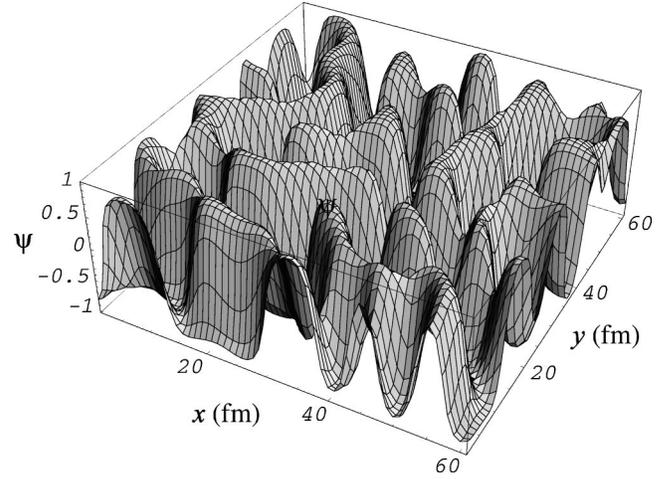

FIG. 2. Order parameter in the transverse plane in the absence of expansion. Droplets tend to merge.

Figures 2 and 3 show $2+1$ dimensional numerical simulations of Eq. (9) in the transverse plane. Only longitudinal expansion is considered so the coordinates are Cartesian with periodic boundary conditions. For comparison, Fig. 2 shows results in which expansion is neglected by omitting the term $(\epsilon + \psi)/\hat{\tau}$ in Eq. (9). Expansion shown in Fig. 3 prevents droplets from merging as in Fig. 2. The expanding system reaches ρ_c at $\tau_0 = 5$ fm. Because this is a dissipative system, we must apply thermal noise at each lattice site at τ_0 to seed phase separation (noise at earlier times is dampened). The memory of the initial conditions is essentially lost for $\tau - \tau_0 > \tau_R$.

We now study the rapidity dependence of baryon number fluctuations. Figure 3 shows the computed variance for two different initial times and for two rapidity intervals. The variance is computed from a sample of 5000 simulated events, each unique due to the thermal noise. We see that the super-Poissonian fluctuations grow appreciably by $\tau \sim 2\tau_0$. This variance drops as the rapidity interval is increased. We find that variance is governed by the ratio τ_0 / τ_R, which compares the expansion and droplet-growth time scales.
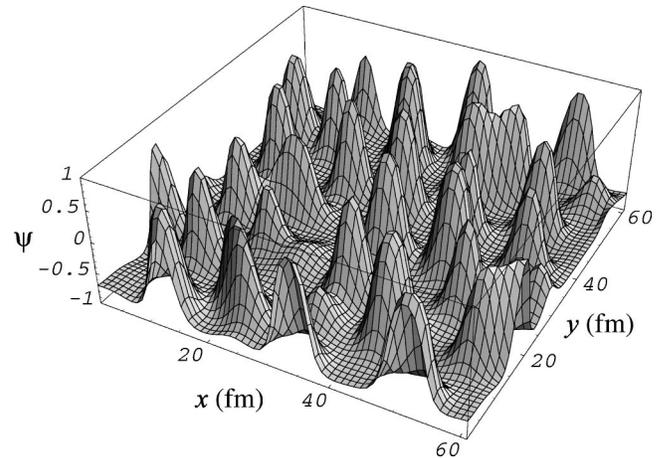

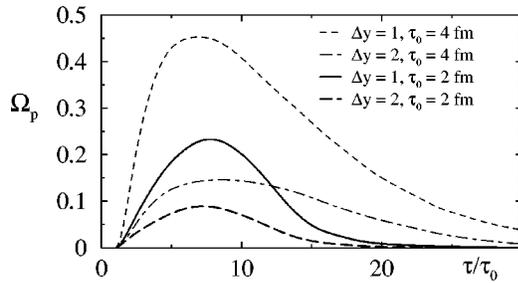FIG. 3. Order parameter in the transverse plane including expansion. Expansion prevents droplets from merging.

FIG. 4. Enhanced variance vs time for two rapidity windows, Eq. (1).

These schematic calculations serve to illustrate the impact of droplet formation on baryon fluctuations. To obtain more quantitative predictions, one must use a more realistic form of the free energy f_0. Our quadratic $f_0(\rho)$ strictly applies only near T_c and yields compressibilities that are equal at ρ_h and ρ_c. This result is unchanged if linear and cubic terms are added. In contrast, lattice QCD calculations suggest that the compressibility may jump across the transition [5]. The bag model equation of state describes a first order transition and predicts a jump $\Delta(\partial\rho/\partial\mu)\sim 2T_c^2$ for two light flavors, but is not analytic in the two-phase region. We will discuss a more sophisticated parametrization of $f_0(\rho)$ in future work.

Turning now to the interpretation of experiments, we emphasize that the identification of phase-transition induced fluctuations requires a systematic comparison of Ω_p in pp, p-nucleus (pA), and nucleus-nucleus (AA) collisions. Benchmark RQMD and HIJING simulations for central Au $+$ Au collisions yield $\Omega_p\approx 0$ in the absence of a phase transition [20], in marked contrast with Fig. 4. However, we have been unable to find experimental results on net-proton fluctuations in pp or pA collisions in the literature, so it is not clear whether these benchmark estimates are reliable. It is therefore important that AA experimenters study fluctuations of identified baryons in pp and pA collisions. Measurements of fluctuations of unidentified charged particles in hadronic collisions [21] and strange baryon production in e^+e^- collisions [22] hint at substantial proton-antiproton correlations in pp collisions. If it turns out that light and heavy ion fluctuations are similar, it may be necessary to correlate baryon measurements with other signals to extract phase transition information, as in Ref. [23].

Nevertheless, we stress that it is unlikely that super-Poissonian fluctuations in nucleon-nucleon (NN) collisions—if present—result in significant fluctuations in AA interactions unless there is a major source of coherence or collectivity. If we treat the AA collision as a superposition of NN subcollisions, then $\Omega_p(AA)=\Omega_p(NN)/N(b)$, where $N(b)$ is the number of participant nucleons. To obtain a rough upperbound on $\Omega_p(NN)$, we take the total charge fluctuations measured to be ~ 0.6 in 200 GeV pp collisions from Whitmore's review [21]. For Au$+$Au collisions at b <10 fm, we estimate $\Omega_p(\text{Au}+\text{Au})=\Omega_p(NN)/N(0)<0.01$,

where we use the wounded nucleon model to compute $N(b)\approx 59$ for $b=10$ fm and 372 for $b=0$. RQMD Au$+$Au simulations for impact parameters fall below this bound [20].

We expect Ω_p to dramatically increase in heavy ion systems compared to light ones. In central S$+$S we expect the NN contribution to Ω_p to be below 1%, as implied by our wounded nucleon model estimate. Since there is no evidence of a phase transition in such light systems at AGS or SPS, the appearance of fluctuations at the level of Fig. 4 in Au$+$Au would be impressive. But is there any source of coherence or collectivity other than a phase transition? Gluon junction effects [24] can lead to correlated baryon production in pp, pA, and AA collisions. This effect is only partially included in RQMD [25]. We are currently studying how gluon junctions can effect Ω_p [20].

In summary, we have studied the phenomenological impact of baryon density, a proposed order parameter of the putative first order QCD phase transition at high baryon density [3]. We have shown that phase separation in the nonequilibrium heavy ion system can lead to large baryon fluctuations. These fluctuations are super-Poissonian and, consequently, can be extracted by measuring protons alone. For Eq. (4) with $\langle\rho\rangle\propto\tau^{-1}$, the system is unstable only for $\tau<2.3\tau_0$. We extend the calculations to much longer times to demonstrate that the fluctuations in rapidity survive well past the freeze-out time, of order 10–30 fm, in accord with [2].

For sufficiently large τ_0, final state fluctuations can be substantial. However, we have seen that a more rapid expansion corresponding to smaller τ_0 leads to an ''inflation'' that prevents the fluctuations from having a large impact on the final state. If experiments find that the non-Poissonian component of fluctuations is small, we must use information from flow signals to ascertain the degree of this inflation.

We emphasize that these calculations include diffusion, which dampens the fluctuations once the system becomes stable. While diffusion is the primary mechanism for dampening fluctuations at high density, viscosity becomes more important at small net baryon density. Several key questions remain: At what energy do heavy ion collisions reach a baryon density where the phase transition is strongly first order? Is there a residual modification of fluctuations due to the near transition at zero baryon density? To what extent does cooling, convection, viscosity, and collision-geometry alter Ω_p compared to our estimates? Finally, we note that our mixed-phase effect may be compensated to some extent by the effect due to the difference between fluctuations in a plasma compared to a hadron gas [13]. Nevertheless, the strength of the signal in our exploratory calculations invites further work.

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