

## Parameter symmetries of quantum many-body systems

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We analyze the occurrence of dynamically equivalent Hamiltonians in the parameter space of general many-body interactions for quantum systems, particularly those that conserve the total number of particles. As an illustration of the general framework, the appearance of parameter symmetries in the interacting boson model 1 and their absence in the Ginocchio  $SO_8$  fermionic model are discussed.

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### I. INTRODUCTION

It is generally accepted that symmetry belongs to the most fundamental concepts in physics. In particular, the generalization of the standard invariance groups in terms of dynamical (spectrum generating) groups and dynamical symmetries [1] seems to provide a rather general framework for describing both classical and quantum physical systems [2]. The role of dynamical symmetries in problems of quantum integrability is reviewed by Zhang and Feng in Ref. [3]. One of the well-known examples of the algebraic approach in non-relativistic quantum physics is the family of so-called interacting boson models (IBM), introduced by Arima and Iachello [4] and extensively employed in phenomenological nuclear physics. The dynamical groups of these models are easily tractable and they neatly decompose into separate dynamical-symmetry chains, each having a clear geometric interpretation and an associated set of nuclei conforming to the various symmetry dictated predictions.

There is, however, a certain ambiguity in the definition of some of the IBM dynamical symmetries resulting from possible gauge transformations of boson operators in the symmetry limits [4,5]. This ambiguity applies even to the simplest version of the model, the IBM 1, where the choice of the boson gauge was for long considered as a mere convention. It was, however, recently recognized as a deeper and universal property of general algebraic systems [6]. Because the twin symmetries resulting from the gauge transformation can be located “between” standard symmetries in the parameter space, i.e., seemingly in transitional regions, they were referred to as “hidden” [6]. The consequences of these hidden symmetries for the problem of quantum chaos were emphasized in Ref. [7].

In the work by Shirokov *et al.* [8] gauge transformations of boson operators, and associated hidden symmetries, were studied from the more general perspective of what these authors call “parameter symmetries.” It was shown that each IBM Hamiltonian has an isospectral partner located at a different point in parameter space. Hidden symmetries emerge as special cases of these parameter symmetries—they arise when the parameter symmetry partner is constructed for a

Hamiltonian possessing a dynamical symmetry. Subsequently the idea was also utilized within the two-component proton-neutron interacting boson model, IBM 2 [9]. It is clear, however, that parameter symmetries can be explored in a much wider class of parameter-dependent systems. It is therefore the aim of the present work to discuss the occurrence of parameter symmetries in more general situations.

We first analyze some generic features of parameter symmetries (Sec. II) and their realization in many-body systems which conserve the total number of particles (Sec. III). Two concrete examples are then considered in detail, namely the interacting boson model 1 (Sec. IV) and the Ginocchio  $SO_8$  model (Sec. V). From the point of view of a link between these models, an interesting comparison between the two analyses can be made.

### II. PARAMETER SYMMETRIES OF GENERAL HAMILTONIANS

Following Refs. [8,9], we define the parameter symmetry  $\mathcal{P}$  of a given Hamiltonian  $H(\lambda)$  depending on a set of  $m$  real parameters  $\lambda \equiv \{\lambda_1, \lambda_2, \dots, \lambda_m\}$  as a mapping of the parameter space onto itself,

$$\mathcal{P}: \lambda \mapsto \lambda' = f(\lambda), \quad (1)$$

such that  $H(\lambda')$  is related to  $H(\lambda)$  by a similarity transformation,

$$H(\lambda') = U_{\lambda',\lambda} H(\lambda) U_{\lambda',\lambda}^{-1}, \quad (2)$$

where  $U_{\lambda',\lambda}$  is a unitary operator. This is, of course, equivalent with the requirement that Hamiltonians  $H(\lambda)$  and  $H(\lambda')$  are isospectral: (i) from Eq. (2) it follows that  $U_{\lambda',\lambda}$  transforms the  $H(\lambda)$  eigenvectors  $|\psi_k(\lambda)\rangle$  into the  $H(\lambda')$  eigenvectors with the same energy  $E_k(\lambda) = E_k(\lambda')$ , and (ii) an equality of energies in  $H(\lambda) = \sum_k |\psi_k(\lambda)\rangle E_k(\lambda) \langle \psi_k(\lambda)|$  and  $H(\lambda') = \sum_k |\psi_k(\lambda')\rangle E_k(\lambda') \langle \psi_k(\lambda')|$  ensures that Eq. (2) is fulfilled with  $U_{\lambda',\lambda}$  defined through  $|\psi_k(\lambda')\rangle = U_{\lambda',\lambda} |\psi_k(\lambda)\rangle$ .

It is apparent that if  $\mathcal{P}_1 \equiv f_1(\lambda)$  and  $\mathcal{P}_2 \equiv f_2(\lambda)$  are two parameter symmetries, then  $\mathcal{P}_3 \equiv \mathcal{P}_1 \circ \mathcal{P}_2: \lambda \mapsto \lambda' = f_2(f_1(\lambda)) \equiv f_3(\lambda)$  is also a parameter symmetry. The parameter space of a system that exhibits the parameter symmetry thus decomposes into subsets where points of the same

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subset correspond to Hamiltonians with essentially the same dynamics. Each of these subsets, geometrically represented by isolated points or a continuous manifold (for a smooth dependence of the Hamiltonian on parameters), forms an equivalence class for which any single member fully represents the whole class. For example, if the Hamiltonian  $H(\lambda^0)$  is integrable, i.e., has  $f$  constants of motion  $\{A_1, A_2, \dots, A_f\}$  in involution with  $f$  being the number of quantum degrees of freedom [3], all the Hamiltonians  $H(\lambda)$  within the same equivalence class as  $H(\lambda^0)$  are integrable as well. The integrals of motion are simply given by  $U_{\lambda\lambda^0} A_i U_{\lambda\lambda^0}^{-1}$ . This feature of parameter symmetries is crucial for the study of quantum chaos because it implies that perfectly regular dynamics can be “imported” into parameter regions that might at first be expected to be chaotic [6,7]. There immediately arises a plethora of questions related to the size and topological structure of the equivalence classes in the parameter space. The answers, of course, depend on the particular Hamiltonian under study.

For a fixed pair  $\lambda$  and  $\lambda'$  relation (2) implies a set of  $n^2$  independent real equations— $n$  being the dimension of the Hilbert space—to determine  $n^2$  independent real parameters of the unitary matrix  $U_{\lambda,\lambda'}$ . (Note that we assume the Hamiltonian to be self-adjoint, but complex.) In general, the structure of the set of equations may very well produce no solution. To determine for a given  $\lambda$  the range of  $\lambda'$  for which a solution exists, it is convenient to consider the diagonalized form of Eq. (2), i.e., the isospectral condition. This yields  $n$  equations for  $m$  variables  $\lambda'$ . However, some of these equations might be identical. This is certainly the case if there exists some inherent degeneracy shared by all Hamiltonians regardless of their parameter values. Yet, not all the mutually different eigenvalues of the Hamiltonian can be considered independent (as elaborated in the next section). Therefore the number of relevant equations is given by the number  $\bar{n}$  of *independent* energies. If this is larger than the number of parameters,  $\bar{n} > m$ , no parameter symmetries are generally expected. For  $m = \bar{n}$ , typically single (discrete) solutions  $\mathcal{P}$  should be found. Finally, a continuous variety of similarity transformations may exist in an “overparametrized” case,  $m > \bar{n}$ .

If  $m \geq \bar{n}$ , the expected dimensionality of manifolds representing the equivalence classes of a given Hamiltonian is  $d = m - \bar{n}$ . For instance, for a two-dimensional (nondegenerate) Hamiltonian dependent on three real parameters  $\{\lambda_{ij}\}_{i=1}^3$  the manifold containing the point  $\lambda^0$  is formed by the intersection of two surfaces  $E_i(\lambda) - E_i(\lambda^0) = 0, i=1,2$  in the three-dimensional parameter space, which is a  $d=1$  object, a curve  $C_{\lambda^0}$  crossing  $\lambda^0$ . Note that the assumption was made again that both the surfaces mentioned are continuous; we do not trace here consequences of possible singularities in the parametric dependence of  $H(\lambda)$ . In the case of a four-parameter Hamiltonian in two dimensions, the  $\lambda^0$ -containing manifold will be a surface  $S_{\lambda^0}$ . For two-parameter two-dimensional Hamiltonians, on the other hand, the  $d=0$  equivalence classes in the parameter space will typically con-

sist of countable sets of isolated points  $P_{\lambda^0} = \{\lambda^0, \lambda^{0'}, \lambda^{0''}, \dots\}$  (note that if the set is finite, it must be cyclic under  $\mathcal{P}$ ).

Assume now that there exists one or more mutually commuting and *parameter-independent* constants of motion  $\{A_1, \dots, A_q\}$  so that  $[H(\lambda), A_i] = 0, i=1 \dots q$  for all  $\lambda$ . It is then, of course, natural to require that an arbitrary  $\lambda \mapsto \lambda'$  transform of any of these integrals (which is a new integral valid at  $\lambda'$ ) must coincide with the original. This implies that all similarity operators  $U_{\lambda,\lambda'}$  in Eq. (2) are required to commute with all the  $A_i$ 's. Both the Hamiltonian and unitary matrices thus have the same block-diagonal form, each block being associated with the subspace of the total Hilbert space characterized by a particular set of the  $A_i$  quantum numbers. The above analysis can then be applied either to the Hamiltonian as a whole or to each block separately. Attention must be paid to the fact that some of the submatrices may depend on a reduced number of parameters. As a result, one can consider parameter symmetries in a particular subset of blocks only. Moreover, the whole set of Hamiltonians can possess an underlying symmetry represented by a group  $G$ . In such a case, the unitary transformation should commute not only with the Casimir operators, but also with all generators of  $G$ . For the rotational symmetry, e.g.,  $U$  must commute with  $J^2, J_z$  and also with  $J_+$  and  $J_-$ .

The implications of these general considerations depend very much on the details of a particular situation, as is demonstrated in Secs. IV and V where we explore the existence and nature of possible parameter symmetries for two well-known nuclear models, the interacting boson model (IBM) [4] and the  $SO_8$  Ginocchio model [10]—also in its fermion dynamical symmetry model (FDSM) incarnation [11].

### III. PARAMETER SYMMETRIES OF MANY-BODY HAMILTONIANS

As was pointed out above, the general analysis of parameter symmetries, based solely on the dimensionality of the problem, can often be inconclusive or even misleading, because without knowledge of the specific physics involved in the model it is hardly possible to determine the number  $\bar{n}$  of independent energies. Turning to more specific examples of parameter symmetries in bosonic or fermionic many-body systems, we now consider a general Hamiltonian with one-, two-, three-,  $\dots$ ,  $K$ -body terms,

$$H(\Lambda) = \Lambda^{(0)} + \sum_{ij} \Lambda_{ij}^{(1)} a_i^\dagger a_j + \sum_{ijkl} \Lambda_{ijkl}^{(2)} a_i^\dagger a_j^\dagger a_k a_l + \sum_{ijklmn} \Lambda_{ijklmn}^{(3)} a_i^\dagger a_j^\dagger a_k^\dagger a_l a_m a_n + \dots + \text{H.c.}, \quad (3)$$

where  $a_i^\dagger$  ( $i=1, \dots, s$ ) is the creation operator of a particle in the  $i$ th state (this state can also specify the type of particle, like neutron or proton). Interaction strengths  $\Lambda \equiv \{\Lambda^{(0)}, \{\Lambda_{ij}^{(1)}\}, \{\Lambda_{ijkl}^{(2)}\}, \{\Lambda_{ijklmn}^{(3)}\}, \dots\}$  form the general set of parameters of the  $K$ -body Hamiltonian (they can be complex but the Hermiticity reduces the number of independent

real parameters—for instance, there are not  $2s^2$  but only  $s^2$  independent real one-body strengths  $\Lambda^{(1)}$ .

The Hamiltonian (3) conserves the total number of particles,  $N = \sum_i a_i^\dagger a_i$ , as should any acceptable similarity operator  $U$ . We thus have

$$U a_i^\dagger U^{-1} = \sum_j \alpha_j^i a_j^\dagger + \sum_{jkl} \beta_{jkl}^i a_j a_k^\dagger a_l^\dagger + \sum_{jklmn} \gamma_{jklmn}^i a_j a_k a_l^\dagger a_m^\dagger a_n^\dagger + \dots, \quad (4)$$

where  $\{\alpha_j^i\}, \{\beta_{jkl}^i\}, \{\gamma_{jklmn}^i\} \dots$  are some complex coefficients satisfying constraints imposed by the unitarity. Equation (4) determines the most general particle-number conserving similarity transformation in the many-body Fock space. However, it does not constitute a parameter symmetry of a  $K$ -body Hamiltonian if higher-order terms with coefficients  $\beta_{jkl}^i, \gamma_{jklmn}^i \dots$  are included. This is so because the higher-order terms increase the maximum order  $K$  of interactions in the Hamiltonian. Parameter symmetries of the Hamiltonian (3) with  $K$  finite can thus be specified by the simplified version

$$U a_i^\dagger U^{-1} = \sum_j \alpha_j^i a_j^\dagger \quad (5)$$

of Eq. (4), with the unitarity constraint  $\sum_k \alpha_k^i \alpha_k^{j*} = \delta_{ij}$ . Namely, Eq. (5) clearly yields the following  $\Lambda \mapsto \Lambda'$  mapping:

$$\begin{aligned} \Lambda^{(0)'} &= \Lambda^{(0)}, \\ \Lambda^{(1)'}_{ij} &= \sum_{kl} \alpha_k^i \alpha_j^{l*} \Lambda_{kl}^{(1)}, \\ \Lambda^{(2)'}_{ijkl} &= \sum_{mnpq} \alpha_i^m \alpha_j^n \alpha_k^{p*} \alpha_l^{q*} \Lambda_{mnpq}^{(2)}, \\ \Lambda^{(3)'}_{ijklmn} &= \dots \end{aligned} \quad (6)$$

If there are some additional global integrals of motion besides  $N$ , the right-hand side of Eq. (5) can only mix the creation operators  $a_j^\dagger$  that carry the same values of these integrals as  $a_i^\dagger$  on the left-hand side, i.e.,  $\alpha_j^i = 0$  if  $i$  and  $j$  label states that differ in one or more conserving quantum numbers. This most obviously applies to the angular momentum  $J^2$  and its projection  $J_z$  in the case of underlying rotational symmetry. Yet other conserved quantities (like charge, etc.) may be relevant. Moreover, as all generators of the symmetry group  $G$  must commute with  $U$ , some additional constraints emerge. In the case of rotational symmetry one gets relations of the type  $\alpha_j^i = \alpha_{j_\pm}^{i_\pm}$ , where  $i_\pm$  and  $j_\pm$  represent states obtained by applying  $J_+$  or  $J_-$  to  $i$  and  $j$ , respectively.

As follows from the previous discussion, the particle-number conserving many-body Hamiltonian in its most general parametrization (3) always exhibits parameter symmetries (6). Even if the form of Eq. (3) is further restricted by

imposing additional integrals of motion and symmetries, the similarity transformation can be constructed whenever Eq. (5) allows, while respecting all the above-discussed constraints, the construction of a nontrivial transformation of the single-particle operators. This is so in spite of the fact that the number of independent real parameters composed from the  $\Lambda$ 's may be (and usually is) smaller than the number of mutually different many-body energies. Obviously, not all of these energies are necessarily independent, as illustrated by the simplest example of a Hamiltonian with just single-particle interactions for which all many-particle energies appear as simple combinations of the set of single-particle energies. In fact, the number of independent energies of a given many-body Hamiltonian must—by definition—be smaller than the number of parameters in the *most general* parametrization. If the  $\Lambda$ 's in Eq. (3) are made dependent on a smaller set of parameters  $\{\lambda\}$ , the quest for parameter symmetries in the reduced parameter space translates into the search of those transformations  $\lambda \mapsto \lambda'$  that accommodate the mapping in Eq. (6).

It should be pointed out that besides the standard single-particle transformations in Eq. (5), a formal exchange of creation and annihilation operators (particles and holes) in the Hamiltonian was also considered in Refs. [8,9] as a possible transformation leading to parameter symmetries. Note that the inclusion of such inverted terms on the right-hand side of Eq. (4) would preserve the particle-number conservation of the transformed Hamiltonian. However, the transformation itself with these additional terms is clearly nonunitary (the basis of the  $N=1$  subspace is mapped onto states that all have a nonzero overlap with the vacuum and thus are not orthonormal). That is why we do not include such transformations into our analysis, although they may be relevant if only  $N \geq 2$  subspaces are considered.

#### IV. THE INTERACTING BOSON MODEL 1

As a simple and well-studied example [8], let us consider first the IBM 1 [4]. It is formulated in terms of two kinds of bosons,  $s$  and  $d$ , with angular momenta 0 and 2, respectively, that interact via a Hamiltonian of the type (3) with only one- and two-body terms. In addition, the Hamiltonian is assumed to be invariant under rotations and the time reversal. Its general form is given by the following expression:

$$\begin{aligned} H(\lambda) &= k_0 + k_1 C_1(U_5) + k_2 C_2(U_5) + k_3 C_2(SO_5) \\ &\quad + k_4 C_2(SO_3) + k_5 C_2(SO_6) + k_6 C_2(SU_3), \end{aligned} \quad (7)$$

where  $\lambda \equiv \{k_0, \dots, k_6\}$  are real parameters, weights of Casimir operators corresponding to groups involved in chains connecting the dynamical group  $U_6$  with the symmetry group  $SO_3$  (their explicit form can be found, e.g., in Ref. [9]).

The dimensional analysis would indicate that parameter symmetries can hardly be found in subspaces with large total boson number. Indeed, the number of different energies (there is always the degeneracy associated with  $SO_3$  symmetry) exceeds the number of parameters for  $N \geq 3$ . However, the above analysis related specifically to many-body Hamiltonians shows that the parameter symmetry exists in all  $N$ -

subspaces. Let us consider the boson transformations according to Eq. (5). Clearly, because  $J^2$  and  $J_z$  are parameter-independent constants of motion, we can only consider transformations  $Us^\dagger U^{-1} = e^{i\phi_s}s^\dagger$  and  $Ud_\mu^\dagger U^{-1} = e^{i\phi_\mu}d_\mu^\dagger$ , where  $s^\dagger$  and  $d_\mu^\dagger$  create, respectively, an  $s$  boson and a  $d$  boson with  $J_z$  projection  $\mu = -2 \cdots +2$ , while  $\phi_s$  and  $\{\phi_\mu\}$  represent arbitrary real phases. However, the fact that  $U$  must commute also with the remaining  $SO_3$  generators,  $J_+$  and  $J_-$ , results in the requirement that the  $d$ -boson phases are independent of  $\mu$ , i.e.,  $\phi_\mu = \phi_d$ . Furthermore, since the Hamiltonian (7) is invariant under a global gauge transformation of all creation and annihilation operators, the only remaining parameter is the relative phase between  $s$  and  $d$  bosons,  $\Delta\phi = \phi_s - \phi_d$ . Without any loss of generality we can set  $\phi_d = 0$ , thus

$$Us^\dagger U^{-1} = e^{i\phi_s}s^\dagger, \quad Ud_\mu^\dagger U^{-1} = d_\mu^\dagger \quad (8)$$

(or, equivalently,  $\phi_s = 0$  and  $\phi_d \neq 0$ ). The reality of coefficients in the Hamiltonian (the time-reversal invariance) allows only some discrete values of  $\phi_s$  [4,5,8,9], namely  $\phi_s = 0, \pi$  for  $k_6 \neq 0$  and  $\phi_s = 0, \pm\pi/2, \pi$  for  $k_6 = 0$ .

From our general analysis, we therefore arrive at a discrete set of similarity transformations of the Hamiltonian (7) that exactly coincide with the gauge transformations described in earlier work [4,5]. After some algebra with the Casimir operators in Eq. (7) [6–9], one derives the following mapping corresponding to Eq. (8) with the above-given discrete values of  $\phi_s$ :

$$(k'_0, k'_1, k'_2, k'_3, k'_4, k'_5, k'_6) = \begin{cases} (k_0, k_1 + 2k_6, k_2 + 2k_6, k_3 - 6k_6, k_4 + 2k_6, k_5 + 4k_6, -k_6) & \text{if } k_6 \neq 0, \\ (k_0 + 10Nk_5, k_1 + 4(N+2)k_5, k_2 - 4k_5, k_3 + 2k_5, k_4, -k_5, 0) & \text{if } k_6 = 0. \end{cases} \quad (9)$$

This is the parameter symmetry given by Shirokov *et al.* [8]. It implies, in particular, that any Hamiltonian  $H(\lambda)$  possessing the  $SU_3$  dynamical symmetry,  $\lambda = (k_0, 0, 0, 0, k_4, 0, k_6)$ , has an isospectral partner  $H(\lambda')$  with  $\lambda' = (k_0, 2k_6, 2k_6, -6k_6, k_4 + 2k_6, 4k_6, -k_6)$ , which is therefore integrable in spite of having nonzero admixtures of all the  $U_5$ ,  $SO_6$ , and  $SU_3$  dynamical symmetries [the Hamiltonian  $H(\lambda')$  is said to have the so-called  $SU_3$  or  $SU_3^*$  hidden dynamical symmetry]. Similarly, the  $SO_6$  Hamiltonians with  $\lambda = (k_0, 0, 0, k_3, k_4, k_5, 0)$  have the  $\overline{SO}_6$  (or  $SO_6^*$ ) isospectral partners at  $\lambda' = [k_0 + 10Nk_5, 4(N+2)k_5, -4k_5, k_3 + 2k_5, k_4, -k_5, 0]$ , i.e., in the  $U_5$ – $SO_6$  transitional region.

In fact, Eq. (9) represents a single mapping of the parameter space onto itself, a mapping discontinuous at  $k_6 = 0$ . However, since the origin of this discontinuity is the extension of the allowed  $\phi_s$  values at  $k_6 = 0$  (see above), Eq. (9) is more appropriately viewed as two separate continuous mappings, the first valid in the whole parameter space and yielding just the identity for  $k_6 = 0$ , the second applicable only in the  $k_6 = 0$  subspace. The fact that the subset of Hamiltonians with no admixture of the  $SU_3$  Casimir operator is invariant under the transformation (9) is important as all these Hamiltonians are known to be integrable [12] and this property is thus not propagated into other regions of the parameter space.

Under the transformation (9), the full seven-dimensional IBM-1 parameter space decomposes into pairs of points that constitute the dynamical equivalence classes of the model. Two consecutive transformations (9) form the identity. Of course, less complex parametrizations (such as the one in Refs. [7,12]) typically contain at most one of the two isospectral Hamiltonians present in the complete parameter space. Let us stress that the spectral equivalence of Hamiltonians connected by Eq. (9) does not imply the same transition rates if a fixed, parameter-independent set of transition

operators is used. However, the transition rates for the associated points  $\lambda$  and  $\lambda'$  will clearly be equal if the transition operators at  $\lambda'$  are chosen to be the  $U_{\lambda',\lambda}$  transforms of the transition operators at  $\lambda$ . A detailed discussion of this point (making a link with the so-called consistent- $Q$  formalism) can be found in Ref. [8]. See also Ref. [13] where it is discussed and stressed that the choice of (phenomenological) transition operators is not necessarily fixed by a given dynamical symmetry of the Hamiltonian.

Our analysis leads us to disagree with the statement made in Ref. [8] concerning an additional IBM-1 parameter symmetry that does not result from a transformation of the type (5). This parameter symmetry is allegedly constructed in three steps: (i) the expansion of the Hamiltonian (7) in terms of the set of Casimir operators where  $C_2(SO_6)$  is replaced by  $C_2(\overline{SO}_6)$  (the group  $\overline{SO}_6$  differs from the “standard”  $SO_6$  by the above gauge transformation with  $\phi_s = \pm\pi/2$ ), (ii) the application of the transformation (9) to the expansion obtained in the first step, and (iii) the reverse decomposition of  $C_2(\overline{SO}_6)$  in the resulting expression into standard Casimir operators. Indeed, when literally following these steps, one finds a Hamiltonian that differs from the one obtained by merely applying Eq. (9). However, this Hamiltonian is not isospectral with the original one because the transformation in Eq. (9) does *not* represent a parameter symmetry for the Hamiltonian decomposition in terms of  $C_2(\overline{SO}_6)$ . If it were so, one could repeatedly apply the new transformation and the one from Eq. (9) yielding a chain of new equivalent Hamiltonians in the parameter space. The dynamical equivalence classes would then be infinite (although countable) sets. However, this is not the case and the mapping (9) represents *the only* parameter symmetry of the IBM 1.

## V. THE GINOCCHIO $SO_8$ MODEL

To explore the microscopic origin of the interacting boson model and its success as a phenomenological model, one has

to link  $s$  and  $d$  bosons to nucleon pairs. A promising perspective was offered by the Ginocchio  $SO_8$  model [10], later generalized to the fermion dynamical symmetry model [11], and formulated in terms of  $s$  and  $d$  bosons by Geyer and Hahne [14]. In the Ginocchio model, an even number ( $2N$ ) of fermions is considered in a shell of single-particle states with total angular momenta  $j$  decomposed as  $\vec{j} = \vec{k} + \frac{\vec{3}}{2}$ , where  $\vec{k}$  is the so-called pseudo-orbital angular momentum ( $k$  is a positive integer) and  $\frac{\vec{3}}{2}$  is termed the pseudospin. Under this restriction, the total angular momentum  $\vec{J} = \vec{j}_1 + \vec{j}_2$  of a nucleon pair can only be  $J=0$  or 2 if  $\vec{k}_1 + \vec{k}_2$  couples to zero in each pair. These  $S$ - and  $D$ -fermion pairs are counterparts of the IBM  $s$  and  $d$  bosons (see Ref. [14]).

The Ginocchio Fermionic Hamiltonian involves the usual one- plus two-body interaction terms. The general form that conserves the total angular momentum is

$$H(\Lambda) = \Lambda^{(0)} + \sum_j \Lambda_j^{(1)} \left( \sum_m (-)^{m-j} a_{jm}^\dagger \tilde{a}_{j-m} \right) + \sum_{j_1 j_2 j_3 j_4} \Lambda_{j_1 j_2 j_3 j_4}^{(2)} \times \left( \sum_M (-)^{M-j_3-j_4} [a_{j_1}^\dagger a_{j_2}^\dagger]_M^J [\tilde{a}_{j_3} \tilde{a}_{j_4}]_{-M}^J \right), \quad (10)$$

with  $[A_{j_1} B_{j_2}]_M^J = \sum_{m_1, m_2} (j_1 m_1 j_2 m_2 | JM) A_{j_1 m_1} B_{j_2 m_2}$ . Here the single-particle operators  $a_{jm}^\dagger$  and  $\tilde{a}_{jm} = (-)^{j+m} a_{j-m}$  are restricted to  $j = |k - \frac{3}{2}|, \dots, (k + \frac{3}{2})$  and  $m = -j, \dots, +j$ . Hermiticity of the Hamiltonian requires  $\Lambda^{(0)}$  and  $\{\Lambda^{(1)}\}$  to be real, while the interaction strengths  $\{\Lambda^{(2)}\}$  satisfy the condition  $\Lambda_{j_1 j_2 j_3 j_4}^{(2)} = \Lambda_{j_4 j_3 j_2 j_1}^{(2)*}$ . We also set  $\Lambda_{j_1 j_2 j_3 j_4}^{(2)} = \Lambda_{j_2 j_1 j_4 j_3}^{(2)}$ , as naturally follows from symmetry properties of the two-body operators in Eq. (10).

Before discussing parameter symmetries of the more specific Ginocchio Hamiltonian, let us consider the ones of the most general Hamiltonian (10). From the previous sections we know that the relevant transformations must be of the following form:

$$U a_{jm}^\dagger U^{-1} = e^{i\phi_j} a_{jm}^\dagger, \quad (11)$$

where  $\phi_j$  are arbitrary real phases. This clearly leads to

$$\Lambda^{(0)'} = \Lambda^{(0)}, \quad (12)$$

$$\Lambda_j^{(1)'} = \Lambda_j^{(1)}, \quad (13)$$

$$\Lambda_{j_1 j_2 j_3 j_4}^{(2)'} = e^{i(\phi_{j_1} + \phi_{j_2} - \phi_{j_3} - \phi_{j_4})} \Lambda_{j_1 j_2 j_3 j_4}^{(2)}. \quad (14)$$

Suppose now that the Hamiltonian (10) is invariant under the time reversal. The time-reversal operator  $T$  is antiunitary and we choose the convention with  $T a_{jm}^\dagger T^{-1} = (-)^{j+m} a_{j-m}^\dagger$ ,  $T \tilde{a}_{jm} T^{-1} = (-)^{j+m} \tilde{a}_{j-m}$ . In addition to the Hermiticity constraints, we then arrive at the further constraint that the co-

efficients  $\Lambda_{j_1 j_2 j_3 j_4}^{(2)}$  are either real (if  $j_1 + j_2 + j_3 + j_4$  is even) or imaginary (if  $j_1 + j_2 + j_3 + j_4$  is odd).

These results lead to a severe restriction of possible values of phases in Eq. (11). Namely, from Eq. (14) we see that the conservation of purely real or imaginary character of the two-body strengths requires  $\phi_{j_1} + \phi_{j_2} - \phi_{j_3} - \phi_{j_4} = n_{j_1 j_2 j_3 j_4} \pi$  with  $n_{j_1 j_2 j_3 j_4} = 0, \pm 1, \pm 2, \dots$  for each  $j_1, j_2, j_3, j_4$ . This will certainly be so if individual phases  $\phi_j$  differ by multiples of  $\pi$ . As the global gauge is irrelevant and as only phase values modulo  $2\pi$  suffice, we end up with transformations generated by various permutations of phases 0 and  $\pi$ . For instance, if  $k \geq 2$ , the four phases  $\{\phi_{k-3/2}, \dots, \phi_{k+3/2}\} \equiv \{\phi_1, \dots, \phi_4\}$  can take any combination of values from the following set:

$$(\phi_1, \phi_2, \phi_3, \phi_4) = (0, 0, 0, \pi), (0, 0, \pi, 0), (0, \pi, 0, 0), (\pi, 0, 0, 0), (0, 0, \pi, \pi), (0, \pi, 0, \pi), (\pi, 0, 0, \pi). \quad (15)$$

Note that the remaining combinations are just  $0 \rightleftharpoons \pi$  conjugates of the ones given above and produce therefore equivalent transformations. Each of the seven possibilities in Eq. (15) generates a specific parameter symmetry that operates in the entire parameter space of the most general Hamiltonian (10). It should be noted, however, that for Hamiltonians with  $\Lambda_{j_1 j_2 j_3 j_4}^{(2)} = 0$  for some particular combinations of angular momenta (i.e., in some parameter subspaces), additional parameter symmetries can be possible. Let us recall that a similar situation was met in the IBM for  $k_6 = 0$ , which in the present language corresponds to  $\Lambda_{2220}^{(2)} = 0$ .

The Ginocchio Hamiltonian is not as general as the one in Eq. (10). It turns out [10] that the  $S$  and  $D$  fermionic pair operators belong to the  $SO_8$  algebra. The model Hamiltonian is thus built exclusively from generators of this algebra, i.e., possesses the  $SO_8$  dynamical symmetry. The pair creation operators are defined in the following way:

$$S^\dagger = \frac{1}{\sqrt{2\Omega}} \sum_j \sqrt{2j+1} [a_j^\dagger a_j]_0^0, \quad (16)$$

$$D_M^\dagger = \frac{1}{\sqrt{\Omega}} \sum_{j_1, j_2} (-)^{j_1+k+3/2} \sqrt{(2j_1+1)(2j_2+1)} \times \begin{Bmatrix} j_1 & j_2 & 2 \\ \frac{3}{2} & \frac{3}{2} & k \end{Bmatrix} [a_{j_1}^\dagger a_{j_2}^\dagger]_M^2, \quad (17)$$

where  $\Omega$  is the maximum number of nucleon pairs in a fully occupied shell,  $2\Omega = \sum_j (2j+1) = 4(2k+1)$ . The corresponding pair annihilation operators are Hermitian conjugates of Eqs. (16) and (17). The remaining  $SO_8$  generators are four multipole operators,

$$P_M^r = 2 \sum_{j_1, j_2} (-)^{r+j_1+k+3/2} \sqrt{(2j_1+1)(2j_2+1)} \\ \times \begin{Bmatrix} j_1 & j_2 & r \\ \frac{3}{2} & \frac{3}{2} & k \end{Bmatrix} [a_{j_1}^\dagger \tilde{a}_{j_2}]_M^r \quad (r=0,1,2,3). \quad (18)$$

The Ginocchio  $SO_8$  Hamiltonian is expressed in terms of the definitions (16)–(18),

$$H(\lambda) = E_0 + G_0 S^\dagger S + G_2 \sum_M D_M^\dagger D_M \\ + \frac{1}{4} \sum_{r=1}^3 b_r \sum_M (-)^M P_M^r P_{-M}^r, \quad (19)$$

where  $\lambda \equiv \{E_0, G_0, G_2, b_1, b_2, b_3\}$  are real control parameters. Expressed in the form of Eq. (10), the Hamiltonian (19) yields the following strength coefficients:

$$\Lambda^{(0)} = E_0, \quad (20)$$

$$\Lambda_j^{(1)} = \sum_{r=1}^3 (2r+1) \sum_{j'} (2j'+1) \begin{Bmatrix} j & j' & r \\ \frac{3}{2} & \frac{3}{2} & k \end{Bmatrix}^2 b_r, \quad (21)$$

$$\Lambda_{j_1 j_2 j_3 j_4}^{(2)} \\ = \delta_{j_0} \delta_{j_1 j_2} \delta_{j_3 j_4} \frac{1}{2\Omega} \sqrt{(2j_1+1)(2j_3+1)} G_0 \\ + \delta_{j_2} (-)^{j_1+j_4+1} \frac{1}{\Omega} \\ \times \sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)} \\ \times \begin{Bmatrix} j_1 & j_2 & 2 \\ \frac{3}{2} & \frac{3}{2} & k \end{Bmatrix} \begin{Bmatrix} j_3 & j_4 & 2 \\ \frac{3}{2} & \frac{3}{2} & k \end{Bmatrix} G_2 \\ + (-)^{j_1+j_4} \sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)} \\ \times \sum_{r=1}^3 (2r+1) \begin{Bmatrix} j_1 & j_3 & r \\ \frac{3}{2} & \frac{3}{2} & k \end{Bmatrix} \begin{Bmatrix} j_2 & j_4 & r \\ \frac{3}{2} & \frac{3}{2} & k \end{Bmatrix} \\ \times \begin{Bmatrix} j_1 & j_2 & J \\ j_4 & j_3 & r \end{Bmatrix} b_r. \quad (22)$$

Note that the one-body terms (21) result from the normal ordering of the last term in Eq. (19). It is also clear that the assumption concerning  $S$  and  $D$  pairs does not restrict the two-body matrix elements (22) to  $J=0,2$  only. Apparently, all the two-body terms (22) fulfill the Hermiticity condition and, in addition, are real. Indeed, because the  $j_1 \rightleftharpoons j_2, j_3 \rightleftharpoons j_4$  symmetry implies that  $(-)^{j_1+j_4} = (-)^{j_2+j_3}$  [Eqs. (17) and (18) are invariant under  $j_1 \rightleftharpoons j_2$ ], the right-hand side of Eq. (22) is nonzero only for even values of the sum  $j_1+j_2+j_3+j_4$ .

It is now simple to see that no parameter mapping  $\lambda \mapsto \lambda'$  can realize the gauge transformation (12)–(14). First, Eqs. (12) and (20) yield  $E_0' = E_0$ , while  $b_r' = b_r$  ( $r=1,2,3$ ) follows

from Eqs. (13) and (21) as coefficients at  $b_r$  in Eq. (21) are positive. Since  $\Lambda_{j_1 j_2 j_2 0}^{(2)'} = \Lambda_{j_1 j_1 j_2 j_2 0}^{(2)}$  also follows from Eqs. (14) and (15), we furthermore find  $G_0' = G_0$ . The only remaining parameter,  $G_2$ , can clearly not fulfill the consistent transformation (14) of *all* two-body strengths (for instance, it can only affect the terms with  $J=2$ ).

These results are interesting from the viewpoint of the known correspondence between the  $SO_8$  model and the IBM 1 [14,15]. Since these models utilize basically the same collective degrees of freedom, which can in fact be formally mapped onto each other, one may wonder why the  $SO_8$  parameter space fails to accommodate isospectral Hamiltonians, in contrast to the IBM-1 space. To answer this question one has to look carefully at how the correspondence between the models is derived, and at the precise status of the correspondence. Among various fermion-boson mapping techniques [16], the Dyson mapping is favored by the fact that it transforms the two-body fermionic Hamiltonian (19) into a two-body bosonic Hamiltonian, while retaining the basic correspondence between a collective fermion pair and a boson. However, a detailed comparison of the boson mapped version of the Ginocchio model (or FDSM) with the phenomenological IBM 1, requires a subsequent Hermitization of the bosonic Hamiltonian—see Refs. [15,17] and especially Ref. [18] in this regard. Without the introduction of three- and more-body boson interactions, or of any approximations, the Hermitization seems to be possible only for a certain subset of the  $SO_8$  parameter space [15]. This implies that the dynamical equivalence of the  $SO_8$  model and the IBM 1 in terms of the link  $(E_0, G_0, G_2, b_1, b_2, b_3) \mapsto (k_0, \dots, k_6)$  between parameters in Eqs. (19) and (7), respectively, can probably be established for this limited  $SO_8$  parameter subset only. We note that some uncertainty in the last statement remains, since there is, strictly speaking, no proof that a Hermitization procedure other than the one in Ref. [15] (and which also preserves the two-body character of interactions) cannot be constructed for the problematic parameter region.

On the other hand, by inspection of the mapping formulas in Refs. [15,17] it becomes apparent that not every Hamiltonian (7) can be mapped from a Hamiltonian of the form (19). In this sense, the  $SO_8$  parameter space is smaller than the IBM-1 space, i.e., the  $SO_8 \rightarrow$  IBM-1 parameter mapping is not onto the complete IBM-1 space, but only into a smaller IBM-1 subspace. The absence of parameter symmetries in the Ginocchio model then indicates that the image of the  $SO_8$  parameter space in the IBM-1 space contains no equivalence classes, or in other words, that out of each pair of the equivalent IBM-1 Hamiltonians *at most one* has a counterpart within the  $SO_8$  space. Note, however, that the bosonic gauge transformations (8) can easily be realized by choosing an appropriate phase convention in the Dyson mapping.

## VI. CONCLUSIONS

We have discussed parameter symmetries of general quantum many-body systems. Identifying such symmetries on the basis of constraints which result from a comparison of

the number of parameters with the number of independent eigenvalues of the Hamiltonian, is not practical because of the difficulty to determine the latter in general. It was shown, however, that the restrictions imposed upon the similarity transformations  $U$ , namely the commutation of  $U$  with (a) all parameter-independent integrals of motion and (b) all generators of the symmetry group, are sometimes sufficient for this determination.

From the above considerations we identified and proposed for many-body Hamiltonians conserving the total number of particles the following procedure: (i) consider single-particle transformations of the type (5) conserving all the model integrals of motion; (ii) apply the commutation rules under (b) to further restrict these transformations; (iii) exclude global gauge transformations that only lead to the trivial mapping  $\lambda \mapsto \lambda$  (such transformations belong to the Abelian symmetry group).

This procedure applied to the interacting boson model 1 showed that the parameter symmetry (9), derived in Ref. [8], is the only parameter symmetry of this model. In fact, since the Hamiltonian (7) is the most general rotationally invariant one- plus two-body Hamiltonian with  $s$ - and  $d$ -boson degrees of freedom, our general analysis already indicates that pa-

rameter symmetries should be a natural ingredient of the model. In contrast, a similar analysis of the Ginocchio  $SO_8$  model disclosed that the parametrization (19) is too restrictive to allow for any parameter symmetries, although such symmetries exist in the more general parametrization (10). These results of course do not contradict any aspect of the relationship between the  $SO_8$  model and an IBM-like  $s$ - and  $d$ -boson counterpart based on boson-fermion mappings, where mapped Hamiltonians generally represent a restricted subset of the most general form (7).

Let us stress finally that the analysis would become much more complicated if we were to consider many-body Hamiltonians with interactions of arbitrary order. Equation (4) would then have to be applied in its general form and no obvious insight seems available to do so.

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