Relativistic quantum mechanics and the *S* matrix

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(Received 17 January 2001; revised manuscript received 27 April 2001; published 23 July 2001)

In the standard development of scattering theory the Hamiltonian of the system plays a central role; however when the Bakamjian-Thomas method is used to construct a relativistic model of a few-particle system, the mass operator plays the essential role. Here a simple procedure for translating the Hamiltonian formulation of scattering theory to a mass-operator formulation is given. A simple proof of the Poincaré invariance of the *S* operator that is obtained from a Bakamjian-Thomas model is also given.

DOI: 10.1103/PhysRevC.64.027001 PACS number(s): 21.45.+v, 24.10.Jv, 11.80.−m

The central issue in relativistic quantum mechanics is the construction of a set of unitary operators that represent the elements of the Poincaré group. This group is the set of all inhomogeneous Lorentz transformations (a,b) that map the space-time variables of one inertial frame to those of another inertial frame according to $x'=ax+b$. Unitary operators $U(a,b)$ that represent these transformations are used to map the quantum mechanical state vectors of a system from one inertial frame to another. The unitary operators that represent the proper group can be constructed by exponentiation from a set of ten generators $\{H,\mathbf{P},\mathbf{J},\mathbf{K}\}$. Here H is the Hamiltonian of the system, $\mathbf{P}$ is the three-momentum operator, $\mathbf{J}$ is the angular momentum operator, and $\mathbf{K}$ is the boost operator. These generators satisfy a rather complicated set of commutation rules known as the Poincaré algebra. It is quite straightforward to construct the generators for a fixed number of noninteracting particles. In a seminal paper, Dirac [1] discussed three different schemes for incorporating interactions into the noninteracting generators. These schemes are known as the instant form, front form, and point form. Here we will focus on the instant form, since it is the most familiar. In this form the Hamiltonian H and boost operator $\mathbf{K}$ contain interactions, while the three-momentum operator $\mathbf{P}$ and the angular momentum operator $\mathbf{J}$ are noninteracting. A practical method for constructing the generators was developed some time ago by Bakamjian and Thomas [2]. In their approach a set of operators is introduced that satisfies a much simpler set of commutation rules than those of the Poincaré algebra. In the instant form these operators are a mass operator M, the three-momentum operator $\mathbf{P}$, a spin operator $\mathcal{J}$, and the Newton-Wigner position operator $\mathbf{X}$ [3,4]. The only nonzero commutators of the set $\{M,\mathbf{P},\mathcal{J},\mathbf{X}\}$ are $[P^m,X_n]=-i\delta_{mn}$ and $[\mathcal{J}_l,\mathcal{J}_m]=i\varepsilon_{lmn}\mathcal{J}_n$. In the Bakamjian-Thomas approach only the mass operator M contains an interaction; the other operators are taken to be the same as those of the noninteracting system. As a result of this, it is only necessary to ensure that M commutes with $\mathbf{P}$, $\mathcal{J}$, and $\mathbf{X}$ in order to satisfy the commutation rules for the set $\{M,\mathbf{P},\mathcal{J},\mathbf{X}\}$. The generators in turn are obtained from M, $\mathbf{P}$, $\mathcal{J}$, and $\mathbf{X}$ by means of a set of nonlinear relations.

The Bakamjian-Thomas construction guarantees that the mass operator M is Poincaré invariant, which in turn implies that its eigenvalues are invariants. Another important set of observables for a system that possesses scattering states is given by the elements of the S matrix. In nonrelativistic models the S-matrix elements are usually obtained from the T matrix, which in turn is the solution of a Lippmann-Schwinger equation [5]. The operators that go into the construction of this equation are the free Hamiltonian H_0 and the interaction $H_1=H-H_0$. This formalism can be carried over to the relativistic domain; however with models constructed using the Bakamjian-Thomas approach it is much more convenient to work with the mass operator M rather than the Hamiltonian H. The development of scattering theory within the framework of the Bakamjian-Thomas construction has been considered previously by other authors [4,6–8]. These earlier developments are quite general; however they rely heavily on mathematical theorems. Here we show in a very elementary way how to transform the well-known equations of Hamiltonian-based scattering theory into equations that involve the mass operator directly. Admittedly, the development here is not as general or as mathematically rigorous as that given by other authors [4,6–8], but it has the virtue of being transparent.

In order for a Bakamjian-Thomas construction to be satisfactory, the S-matrix elements must transform properly in going from one inertial frame to another. In particular, they must transform in such a way that the probability of a scattering event is relativistically invariant. In quantum field theory where S-matrix elements are usually calculated from Feynman diagrams, this is not a major concern since the contribution of a single Feynman diagram is manifestly covariant. In contrast the Lippmann-Schwinger equations that are solved to obtain S-matrix elements from a Bakamjian-Thomas mass operator are three dimensional and hence not manifestly covariant. So in this case it is necessary to prove that the S-matrix elements transform properly. This issue has also been considered previously [4,6–9]. Quite general proofs for the invariance of the S operator have been given that again rely heavily on mathematical theorems. Here we will develop a less rigorous, but hopefully more transparent, proof of the invariance of the S operator. There is probably no simpler way than the development given here to demonstrate that a formulation of relativistic quantum mechanics that is not manifestly covariant can lead to a relativistically invariant S matrix. It is important to appreciate the fact that manifest covariance is not a necessary requirement of a satisfactory relativistic quantum mechanics. It is only required that observables transform correctly in passing from one inertial frame to another.

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We assume a two-particle system and begin by introducing the basic operators. The interacting Hamiltonian, the noninteracting Hamiltonian, and the interaction Hamiltonian are denoted by H, H_0, and H_1, respectively. They are defined in terms of the interacting mass operator M, the noninteracting mass operator M_0, and the three-momentum operator $\mathbf{P}$ by

$$H=(\mathbf{P}^2+M^2)^{1/2},\ \ H_0=(\mathbf{P}^2+M_0^2)^{1/2},\ \ H_1=H-H_0. \quad (1)$$

Our noninteracting basis states are denoted by $|\mathbf{pq}m_1m_2\rangle$ where $\mathbf{p}=\mathbf{p}_1+\mathbf{p}_2$, $\mathbf{q}=(\mathbf{p}_1)_{\mathrm{c.m.}}$, and m_1 and m_2 are the z components of the particles' spins. These states are eigenstates of H_0, $\mathbf{P}$, and M_0, i.e.,

$$H_0|\mathbf{pq}m_1m_2\rangle=E(\mathbf{p},\mathbf{q})|\mathbf{pq}m_1m_2\rangle,$$

$$\mathbf{P}|\mathbf{pq}m_1m_2\rangle=\mathbf{p}|\mathbf{pq}m_1m_2\rangle,$$

$$M_0|\mathbf{pq}m_1m_2\rangle=W(\mathbf{q})|\mathbf{pq}m_1m_2\rangle, \quad (2)$$

where the energy E and the c.m. energy W are given by

$$E(\mathbf{p},\mathbf{q})=[\mathbf{p}^2+W^2(\mathbf{q})]^{1/2},\ \ W(\mathbf{q})=\varepsilon_1(\mathbf{q})+\varepsilon_2(\mathbf{q}), \quad (3)$$

with ε_1 and ε_2 the single particle energies. We define in $(+)$ and out $(-)$ states for the interacting system in the usual way [5], i.e.,

$$|\mathbf{pq}m_1m_2\rangle^{(\pm)}$$

$$=\Omega^{(\pm)}|\mathbf{pq}m_1m_2\rangle \quad (4a)$$

$$=\left[1+\frac{1}{E(\mathbf{p},\mathbf{q})\pm i\varepsilon-H}H_1\right]|\mathbf{pq}m_1m_2\rangle \quad (4b)$$

$$=\left\{1+\frac{1}{E(\mathbf{p},\mathbf{q})\pm i\varepsilon-H_0}T[E(\mathbf{p},\mathbf{q})\pm i\varepsilon]\right\}|\mathbf{pq}m_1m_2\rangle, \quad (4c)$$

where $\Omega^{(\pm)}$ are the Möller wave operators and the T operator is given by

$$T(z)=H_1+H_1\frac{1}{z-H}H_1. \quad (5)$$

We now reexpress Eq. (4b) in terms of the interacting mass operator M, the noninteracting mass operator M_0, and the mass-operator interaction V, which we assume are related by

$$M^2=M_0^2+V. \quad (6)$$

Using Eqs. (1), (2), (3), and (6) we can show that

$$[H+E(\mathbf{p},\mathbf{q})]H_1|\mathbf{pq}m_1m_2\rangle=V|\mathbf{pq}m_1m_2\rangle, \quad (7)$$

$$[E(\mathbf{p},\mathbf{q})\pm i\varepsilon-H][E(\mathbf{p},\mathbf{q})+H]$$
$$=E^2(\mathbf{p},\mathbf{q})-H^2\pm i\varepsilon[E(\mathbf{p},\mathbf{q})+H]$$
$$\rightarrow\mathbf{p}^2+W^2(\mathbf{q})\pm i\eta-\mathbf{P}^2-M^2. \quad (8)$$

Along with the fact that $\mathbf{P}$ and V commute, these relations allow us to rewrite Eq. (4b) as

$$|\mathbf{pq}m_1m_2\rangle^{(\pm)}=\left[1+\frac{1}{W^2(\mathbf{q})\pm i\eta-M^2}V\right]|\mathbf{pq}m_1m_2\rangle. \quad (9)$$

This result suggests that we define an alternative T operator by

$$t(s)=V+V\frac{1}{s-M^2}V. \quad (10)$$

Just as with the standard Hamiltonian-based scattering formalism, we can easily derive the identities

$$\frac{1}{s-M^2}V=\frac{1}{s-M_0^2}t(s),\qquad V\frac{1}{s-M^2}=t(s)\frac{1}{s-M_0^2}, \quad (11)$$

$$t(s)=V+V\frac{1}{s-M_0^2}t(s)=V+t(s)\frac{1}{s-M_0^2}V. \quad (12)$$

Using Eq. (11) we can rewrite Eq. (9) as

$$|\mathbf{pq}m_1m_2\rangle^{(\pm)}=\left\{1+\frac{1}{W^2(\mathbf{q})\pm i\eta-M_0^2}\times t[W^2(\mathbf{q})\pm i\eta]\right\}|\mathbf{pq}m_1m_2\rangle. \quad (13)$$

Since in the Bakamjian-Thomas construction, $\mathbf{P}$, $\mathbf{X}$, and $\mathcal{J}$ commute with M and V [2,4], it follows from Eq. (10) that our T operator satisfies the commutation relations

$$[\mathbf{P},t(s)]=[\mathbf{X},t(s)]=[\mathcal{J},t(s)]=0, \quad (14)$$

which in turn implies that $t(s)$ has the representation [10]

$$\langle\mathbf{p}'\mathbf{q}'m_1'm_2'|t(s)|\mathbf{pq}m_1m_2\rangle$$
$$=(2\pi)^3 2[E(\mathbf{p}',\mathbf{q}')E(\mathbf{p},\mathbf{q})]^{1/2}\delta^3(\mathbf{p}'-\mathbf{p})$$
$$\times t_{m_1'm_2',m_1m_2}(\mathbf{q}',\mathbf{q};s). \quad (15)$$

The commutivity of $\mathbf{P}$ and $t(s)$ leads to the delta function in Eq. (15), while the commutivity of $\mathbf{X}$ and $t(s)$ implies that the function $t_{m_1'm_2',m_1m_2}(\mathbf{q}',\mathbf{q};s)$ does not depend on $\mathbf{p}'=\mathbf{p}$.

If we compare Eqs. (4c) and (13) and use Eqs. (1), (3), and (14), we find that the two T operators are related by the identity

$$T[E(\mathbf{p},\mathbf{q})\pm i\varepsilon]|\mathbf{pq}m_1m_2\rangle$$

$$=\frac{1}{E(\mathbf{p},\mathbf{q})+H_0}t[W^2(\mathbf{q})\pm i\eta]|\mathbf{pq}m_1m_2\rangle. \quad (16)$$

This relation shows that matrix elements of the standard T operator (5) can be obtained by solving the Lippmann-Schwinger equation (12) whose ingredients are the noninteracting mass operator M_0 and the mass-operator interaction V.

We now show that the S operator is invariant with respect to the action of the unitary operators $U_0(a,b)$ that transform our noninteracting basis states from one inertial frame to another. This property of S ensures that the probability of a scattering event is the same in all inertial frames.

We begin by establishing *intertwining relations* for the set of operators used in the Bakamjian-Thomas construction, i.e., $\{M,\mathbf{P},\mathcal{J},\mathbf{X}\}$, as well as for the generators $\{H,\mathbf{P},\mathbf{J},\mathbf{K}\}$. From Eqs. (4b), (1), and (2), and the fact that $\mathbf{P}$ commutes with both H and H_1, we can easily derive the relations

$$H\Omega^{(\pm)}|\mathbf{pq}m_1m_2\rangle = E(\mathbf{p},\mathbf{q})\Omega^{(\pm)}|\mathbf{pq}m_1m_2\rangle = \Omega^{(\pm)}H_0|\mathbf{pq}m_1m_2\rangle, \tag{17}$$

$$\mathbf{P}\Omega^{(\pm)}|\mathbf{pq}m_1m_2\rangle = \mathbf{p}\Omega^{(\pm)}|\mathbf{pq}m_1m_2\rangle = \Omega^{(\pm)}\mathbf{P}|\mathbf{pq}m_1m_2\rangle. \tag{18}$$

It then follows from Eqs. (1), (2), and (3) that

$$M\Omega^{(\pm)}|\mathbf{pq}m_1m_2\rangle = W(\mathbf{q})\Omega^{(\pm)}|\mathbf{pq}m_1m_2\rangle = \Omega^{(\pm)}M_0|\mathbf{pq}m_1m_2\rangle. \tag{19}$$

Since $\mathcal{J}$ and $\mathbf{X}$ commute with M and V we can move them past the square bracket in Eq. (9), and then using the fact that the noninteracting states provide representations for $\mathcal{J}$ and $\mathbf{X}$ [10], we can write

$$\mathcal{J}\Omega^{(\pm)}|\mathbf{pq}m_1m_2\rangle = \sum_{m_1'm_2'}[\mathcal{J}^*(\mathbf{q})]_{m_1m_2,m_1'm_2'}\Omega^{(\pm)}|\mathbf{pq}m_1'm_2'\rangle = \Omega^{(\pm)}\mathcal{J}|\mathbf{pq}m_1m_2\rangle, \tag{20}$$

$$\mathbf{X}\Omega^{(\pm)}|\mathbf{pq}m_1m_2\rangle = \mathbf{X}^*(\mathbf{p},\mathbf{q})\Omega^{(\pm)}|\mathbf{pq}m_1m_2\rangle = \Omega^{(\pm)}\mathbf{X}|\mathbf{pq}m_1m_2\rangle. \tag{21}$$

Assuming the completeness of the basis states $|\mathbf{pq}m_1m_2\rangle$, we arrive at the intertwining relations

$$M\Omega^{(\pm)} = \Omega^{(\pm)}M_0, \quad \mathbf{P}\Omega^{(\pm)} = \Omega^{(\pm)}\mathbf{P},$$

$$\mathcal{J}\Omega^{(\pm)} = \Omega^{(\pm)}\mathcal{J}, \quad \mathbf{X}\Omega^{(\pm)} = \Omega^{(\pm)}\mathbf{X}. \tag{22}$$

The total angular momentum operator $\mathbf{J}$ and the boost operator $\mathbf{K}$ are given by [4,10]

$$\mathbf{J} = \mathbf{X}\times\mathbf{P} + \mathcal{J}, \quad \mathbf{K} = -\frac{1}{2}(\mathbf{X}H + H\mathbf{X}) - \frac{\mathbf{P}\times\mathcal{J}}{M+H}. \tag{23}$$

We see that the angular momentum operator $\mathbf{J}$ commutes with $\Omega^{(\pm)}$; while $\mathbf{K}$ intertwines with $\Omega^{(\pm)}$. Summarizing, we have the intertwining relations for the generators, i.e.,

$$H\Omega^{(\pm)} = \Omega^{(\pm)}H_0, \quad \mathbf{P}\Omega^{(\pm)} = \Omega^{(\pm)}\mathbf{P},$$

$$\mathbf{J}\Omega^{(\pm)} = \Omega^{(\pm)}\mathbf{J}, \quad \mathbf{K}\Omega^{(\pm)} = \Omega^{(\pm)}\mathbf{K}_0. \tag{24}$$

It follows trivially from these relations that

$$(\boldsymbol{\omega}\cdot\mathbf{K} + \boldsymbol{\theta}\cdot\mathbf{J})\Omega^{(\pm)} = \Omega^{(\pm)}(\boldsymbol{\omega}\cdot\mathbf{K}_0 + \boldsymbol{\theta}\cdot\mathbf{J}),$$

$$b(H,\mathbf{P})\Omega^{(\pm)} = \Omega^{(\pm)}b(H_0,\mathbf{P}). \tag{25}$$

Since the unitary operators that map state vectors from one inertial frame to another can be expressed in the form

$$U(a,b) = \exp(ib\cdot P)\exp[i(\boldsymbol{\omega}\cdot\mathbf{K} + \boldsymbol{\theta}\cdot\mathbf{J})], \quad P = (H,\mathbf{P}), \tag{26}$$

we see that these operators satisfy the intertwining relation

$$U(a,b)\Omega^{(\pm)} = \Omega^{(\pm)}U_0(a,b). \tag{27}$$

The S operator is given by [5]

$$S = \Omega^{(-)\dagger}\Omega^{(+)}, \tag{28}$$

which when combined with Eq. (27) leads to

$$S = U_0^{-1}(a,b)SU_0(a,b). \tag{29}$$

That this establishes the relativistic invariance of the S-matrix elements is easy to see. If we associate the noninteracting states $|\mathbf{pq}m_1m_2\rangle$ and $|\mathbf{p}'\mathbf{q}'m_1'm_2'\rangle$ with the x frame, then the corresponding states in the x' frame, where $x' = ax+b$, are given by $U_0(a,b)|\mathbf{pq}m_1m_2\rangle$ and $U_0(a,b)|\mathbf{p}'\mathbf{q}'m_1'm_2'\rangle$, respectively. Obviously

$$[\langle\mathbf{p}'\mathbf{q}'m_1'm_2'|U_0^{-1}(a,b)]S[U_0(a,b)|\mathbf{pq}m_1m_2\rangle] = \langle\mathbf{p}'\mathbf{q}'m_1'm_2'|S|\mathbf{pq}m_1m_2\rangle, \tag{30}$$

so the scattering amplitudes associated with the two different inertial frames are identical and the probability of the scattering event is relativistically invariant.

In the development given here it has been assumed that the mass operator has the form (6) in which the interaction is added to the square of the noninteracting mass operator. It is also common to construct the mass operator in the form $M = M_0 + U$ where U is the interaction. This case can be easily treated with the methods used here. It is found that Eq. (9) gets replaced with

$$|\mathbf{pq}m_1m_2\rangle^{(\pm)} = \left[1 + \frac{1}{W(\mathbf{q}) \pm i\eta - M}U\right]|\mathbf{pq}m_1m_2\rangle, \tag{31}$$

which instead of Eq. (10) suggests the T operator

$$\tau(z) = U + U\frac{1}{z-M}U. \tag{32}$$

The relation between this T operator and the traditional one is given by

$$T[E(\mathbf{p},\mathbf{q})\pm i\varepsilon]|\mathbf{pq}m_1m_2\rangle = \frac{W(\mathbf{q})+M_0}{E(\mathbf{p},\mathbf{q})+H_0}\tau[W^2(\mathbf{q})\pm i\eta]|\mathbf{pq}m_1m_2\rangle, \tag{33}$$

rather than Eq. (16). The proof of the invariance of the S operator that arises from assuming $M=M_0+U$ is essentially the same as the proof given above.

Here we have assumed single-channel scattering. The extension of the development given here to Bakamjian-Thomas models for systems involving several one- and two-particle channels [11] is quite straightforward. It essentially amounts to assuming that the interactions V or U couple these various channels.

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