Test of the proton-neutron random-phase approximation method within an extended Lipkin-type model

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An extended Lipkin-Meshkov-Glick model for testing the proton-neutron random-phase approximation (pnRPA) method is developed, taking into account explicitly proton and neutron degrees of freedom. Besides the proton and neutron single-particle terms two types of residual proton-neutron interactions, one simulating a particle-particle and the other a particle-hole interaction, are included in the model Hamiltonian so that the model is exactly solvable in an isospin $SU(2)\otimes SU(2)$ basis. The behavior of the first excited (collective) state obtained by (i) exact diagonalization of the Hamiltonian matrix and (ii) with the pnRPA is studied as a function of the model parameters and the two results are compared with each other. Furthermore, charge-changing operators simulating nuclear beta decay and their action on eigenfunctions of the model Hamiltonian are defined and transition amplitudes of them are calculated using exact, the Tamm-Dancoff, and pnRPA eigenfunctions.

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The proton-neutron random-phase approximation (pnRPA) method has been extensively employed in the literature to compute nuclear matrix elements (NME's) for charge-changing processes like, for instance, beta (β) and double-beta ($\beta\beta$) decay [1–4]. However, in these cases this method faces the problem of a large sensitivity of the NME's to the strength of the particle-particle (g_{pp}) interaction. Namely, they decrease rapidly in magnitude and cross through zero in the physically acceptable region of g_{pp}, making it difficult to fix this constant for accurate calculations. It should be mentioned that such a problem does not exist when the calculations are made with shell-model-based methods, but until now these methods have not been prepared to attack nuclei with a large number of nucleons outside closed shells, as is the case for most $\beta\beta$ emitters [14]. To overcome this problem extensions of the pnRPA method going beyond the boson approximation were developed in the recent past [5–9]. In these models like- and unlike-nucleon residual interactions are both taken into account at the RPA level. However, in the framework of those methods the competition between these two kinds of residual interaction, as well as their limits of applicability, are still not well understood. It thus appears useful to develop exactly solvable nuclear models distinguishing between proton (p) and neutron (n) degrees of freedom to test the pnRPA method and its recent extensions used in realistic nuclear-structure calculations for change-changing processes.

Recently, the Lipkin-Meshkov-Glick (LMG) model [10] was extended to take into account explicitly the proton and neutron degrees of freedom [11,12]. The proton and neutron parts of the Hamiltonian were taken to be of the LMG form and, in addition, a residual p-n interaction was included. This model is exactly solvable in a quasispin $SU(2)\otimes SU(2)$ basis and it was used to test the RPA method using the energy spectrum of the model Hamiltonian. Also (in [12]), charge-changing transitions simulating nuclear β decays were introduced and computed within this model and the results compared with RPA calculations. However, such a comparison is only a schematic one and has no connection with reality, since these transitions are calculated in the literature in the framework of pnRPA methods. The main conclusion was that the RPA gives results closer to the exact ones when the p-n interaction, besides the p-p and n-n interactions, is also introduced into the model Hamiltonian.

In this Brief Report we develop an extended Lipkin model suitable for testing pnRPA-type methods. The model Hamiltonian contains, besides single-particle terms of proton and neutron type, two types of p-n residual interactions, one attractive, simulating a particle–particle two-body interaction, the other repulsive, simulating a particle-hole–two-body interaction, chosen such that the model is exactly solvable in an isospin $SU(2)\otimes SU(2)$ basis. In this model we calculate the energy spectra and transition amplitudes of suitably defined charge-changing operators and compare the results with the ones of the pnRPA approximation to the model. The comparison between the exact and pnRPA calculations for charge-changing transitions allows us to extract useful conclusions about the limits of validity of pnRPA methods which may be now extrapolated for realistic situations. Charge-changing transitions have also been treated in exactly solvable models in [13] but their model, based on SO(5) symmetry, is basically different from our Lipkin-type one.

Consider an N-nucleon system composed of two subsystems: one containing N_p protons, the other N_n neutrons. In analogy with the LMG model, inside each subsystem the nucleons of the same kind are distributed into two levels, each having an N_p- (N_n-) fold degeneracy, and separated by an energy ϵ_p (ϵ_n), respectively. Furthermore, the states in each subsystem are characterized by two quantum numbers: σ distinguishing between the lower ($\sigma=-1$) and upper ($\sigma=1$) levels and p_i (n_i) denoting all the other quantum numbers characterizing the proton (neutron) states of the level.

For the model Hamiltonian we choose the following expression, written in terms of isospin operators:

$$H=\epsilon(T_z^{(1)}+T_z^{(2)})+V_{pn}(T_+^{(1)}T_+^{(2)}+T_-^{(2)}T_-^{(1)}) + W_{pn}(T_+^{(1)}T_-^{(1)}+T_+^{(2)}T_-^{(2)}), \quad (1)$$

where one considers $\epsilon_p=\epsilon_n=\epsilon$ as a resonable assumption of the model. The terms proportional to V_{pn} scatter a p-n pair from one level to the other one, while the terms proportional to W_{pn} scatter a p-n pair into an n-p pair within the same levels. The two kinds of isospin operator entering Eq. (1) are defined as

$$T_+^{(1)}=\sum_i a_{p_i+}^\dagger a_{n_i-}, \quad T_-^{(1)}=\sum_i a_{n_i-}^\dagger a_{p_i+},$$

$$T_z^{(1)}=\frac{1}{2}\sum_i (a_{p_i+}^\dagger a_{p_i+}-a_{n_i-}^\dagger a_{n_i-}), \tag{2}$$

$$T_+^{(2)}=\sum_i a_{n_i+}^\dagger a_{p_i-}, \quad T_-^{(2)}=\sum_i a_{p_i-}^\dagger a_{n_i+},$$

$$T_z^{(2)}=\frac{1}{2}\sum_i (a_{n_i+}^\dagger a_{n_i+}-a_{p_i-}^\dagger a_{p_i-}). \tag{3}$$

Each of the two sets of operators $(T_+^{(1)},T_-^{(1)},T_z^{(1)})$ and $(T_+^{(2)},T_-^{(2)},T_z^{(2)})$ fulfill the SU(2) commutator relations and the Hamiltonian matrix can be diagonalized exactly in the isospin SU(2)$\otimes$SU(2) basis:

$$\Phi_j=|T^{(1)}T_z^{(1)}\rangle\otimes|T^{(2)}T_z^{(2)}\rangle. \tag{4}$$

The (unperturbed) ground state (g.s.) and excited states $|k,l\rangle$ can be written formally as

$$|\text{g.s.}\rangle=|T^{(1)},-T^{(1)}\rangle|T^{(2)},-T^{(2)}\rangle,$$

$$|k,l\rangle=\sum_{k,l}(T_+^{(1)})^k(T_+^{(2)})^l|\text{g.s.}\rangle, \tag{5}$$

where $k=0,\ldots,N_p$ and $l=0,\ldots,N_n$. For instance, the first excited state of the system has one excited proton or neutron and is $N=(N_p+N_n)$ fold degenerate:

$$|1^{\text{st}}\rangle=\frac{1}{\sqrt{N}}(\sqrt{N_p}|T^{(1)},-T^{(1)}+1\rangle|T^{(2)},-T^{(2)}\rangle + \sqrt{N_n}|T^{(1)},-T^{(1)}\rangle|T^{(2)},-T^{(2)}+1\rangle), \tag{6}$$

where the norm factors in Eq. (6) were chosen such that $|1^{\text{st}}\rangle$ is normalized. One observes that the basis is formed from the g.s. of the system having N_p protons and N_n neutrons and from excited states belonging to systems with the same total number of nucleons (i.e., $N=N_p+N_n$) but with a different number of protons and neutrons. This brings us to the philosophy of the pnRPA method where one starts from the g.s. of a system with (N_p,N_n) and, by the action of the pnRPA phonon operator onto the g.s., one gets excited states of systems with a different number of protons and neutrons but such that the total number of nucleons is conserved.

The energy spectrum of Eq. (1) can be obtained exactly by diagonalizing the Hamiltonian matrix corresponding to the multiplet with $(T^{(1)}=\frac{1}{2}N_n,T^{(2)}=\frac{1}{2}N_p)$ which contains the g.s. and all the excited states of interest. After diagonalization the exact wave functions will be linear combinations of the SU(2) basis wave functions (4):

$$\Psi_i=\sum_{i,j} c_{ij}\Phi_j, \tag{7}$$

where c_{ij} are the mixing coefficients. Characteristic features of the obtained energy spectrum are as follows: the energies increase when the strengths of interactions increase and are generally nondegenerate when both types of residual interaction, one attractive (V_{pn}) and the other repulsive (W_{pn}), are present. A more detailed study of the Hamiltonian spectrum will be done elsewhere.

For treating our model Hamiltonian within the pnRPA we first define the pnRPA phonon operator as

$$\Gamma^\dagger=\frac{X\Theta^+-Y\Theta^-}{\sqrt{\langle\text{g.s.}|[\Theta^-,\Theta^+]|\text{g.s.}\rangle}}, \quad \Theta^+=T_+^{(1)}+T_+^{(2)}, \quad \Theta^-=T_-^{(1)}+T_-^{(2)}. \tag{8}$$

Above, X and Y are the pnRPA amplitudes, $|\text{g.s.}\rangle$ is the uncorrelated g.s. of Eq. (5), and the norm $\langle\text{g.s.}|[\Theta^-,\Theta^+]|\text{g.s.}\rangle=N_p+N_n=N$ appearing in the expression of $\Gamma^\dagger$ is taken such that the excited state is normalized to unity. The first excited (collective) state in the pnRPA framework is obtained by operating by the (collective) phonon operator onto the correlated g.s. $|pn\text{RPA}\rangle$:

$$|1^{\text{st}}\rangle=\Gamma^\dagger|pn\text{RPA}\rangle=\frac{X\Theta^+-Y\Theta^-}{\sqrt{N}}|pn\text{RPA}\rangle. \tag{9}$$

The pnRPA equation reads

$$\begin{pmatrix} A-\Omega & B \\ B & A+\Omega \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}=0, \tag{10}$$

and the excitation energy Ω and the X and Y amplitudes can be expressed in terms of the submatrices A and B as

$$\Omega=\sqrt{A^2-B^2}, \quad X=\sqrt{\frac{A+\Omega}{2\Omega}}, \quad Y=-\sqrt{\frac{A-\Omega}{2\Omega}}. \tag{11}$$

The submatrices (in this case just c numbers) have the expressions

$$A=\frac{\langle\text{g.s.}|[\Theta^-,H,\Theta^+]|\text{g.s.}\rangle}{\sqrt{\langle\text{g.s.}|[\Theta^-,\Theta^+]|\text{g.s.}\rangle}}, \quad B=-\frac{\langle\text{g.s.}|[\Theta^-,H,\Theta^-]|\text{g.s.}\rangle}{\sqrt{\langle\text{g.s.}|[\Theta^-,\Theta^+]|\text{g.s.}\rangle}}, \tag{12}$$

and after some algebra one obtains

$$A=\epsilon+\frac{(N_p^2+N_n^2)W_{pn}}{N}, \quad B=\frac{2N_pN_nV_{pn}}{N}. \tag{13}$$

As an example, in Fig. 1 we display the scaled excitation energy of the first excited state, $\omega=(E_1-E_0)/\epsilon$, as a function of the quantity NV_{pn}/ϵ and for a fixed value of the NW_{pn}/ϵ, in the cases $N=4+4$ and $N=20+20$. Since the strength parameters V_{pn} and W_{pn} represent different types of p-n residual interaction (i.e., attractive and repulsive, respectively), they are chosen with opposite sign. Comparing the exact and pnRPA results one observes that the pnRPA largely underestimates ω for low values of N, especially for large values of the model parameters NV_{pn}/ϵ. However, a strong improvement is observed when the number of nucleons increases ($N=20+20$), which is in accordance with the fact that the pnRPA works better for systems with a larger number of particles, where the collective features manifest stronger. These features demonstrate that the pnRPA treatment of our generalized LMG Hamiltonian produces meaningful interpretation of the excitation spectrum as a function of the model parameters.

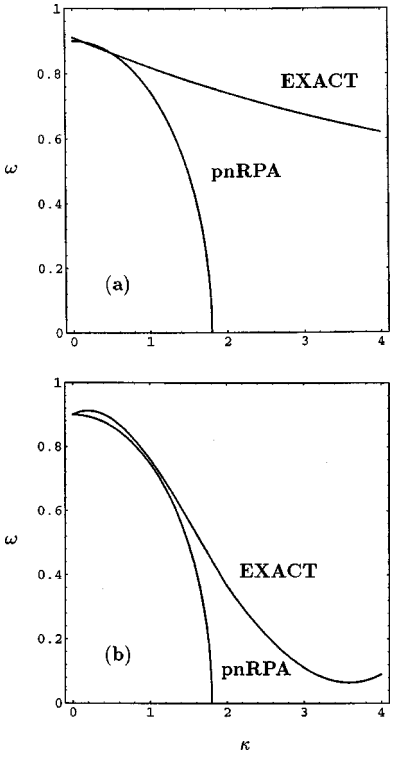

FIG. 1. First excited state versus $\kappa = NV_{pn}/\epsilon = -0.2$. (a) For $N = N_p + N_n = 4+4$. (b) For $N = N_p + N_n = 20+20$.

We consider now charge-changing transitions simulating, for instance, nuclear beta decay. In such a transition a neutron transforms into a proton (or vice versa), such that the total number of nucleons remains constant. Let us define the following charge-changing transition operators:

$$\hat{M}^+ = \chi^+ \sum_{i,j,\sigma,\sigma'} a^\dagger_{p_i\sigma} a_{n_j\sigma'}, \quad \hat{M}^- = \chi^- \sum_{i,j,\sigma,\sigma'} a^\dagger_{n_i\sigma} a_{p_j\sigma'}. \tag{14}$$

The $\hat{M}^+$ operator can destroy one neutron from any of the states of the two levels of the neutron subsystem and create one proton in any of the states of the two levels of the proton subsystem. Such an operator can connect states belonging to systems with the same total number of nucleons, N, but differing from each other in the number of protons and neutrons in their respective subsystems. For example, it can connect states of the system $|N_p\rangle|N_n\rangle$ with states of the system $|N_p+1\rangle|N_n-1\rangle$. This transition simulates nuclear β^- decay. The factor χ^+ in front can be interpreted as the strength of the transition. We imagine that the creation of the proton can occur only in a p state which is unoccupied (free), both in the lower and in the upper level of the p subsystem. Hence, we suppose that such free states exist in the lower level (with an accompanying free upper level) even in the g.s. of the system. The existence of free states in the subsystems of the N-nucleon system does not affect the consistency of the model. Moreover, their presence does not influence either the possible values of the total angular momentum (which only depend of the number of particles) or the Hamiltonian eigenvalues. The only quantity which is affected is the degeneracy of the g.s. and excited states of the system which now increases, but this effect can easily be accounted for. Allowing this degree of freedom will produce also a change in the values of the normalization factors of Eqs. (5) and (6), which have to be recalculated in order to keep the corresponding states normalized to unity. As an intuitive physical correspondence, one can imagine associating the p (or n) subsystem with a p (or n) closed or open nuclear shell.

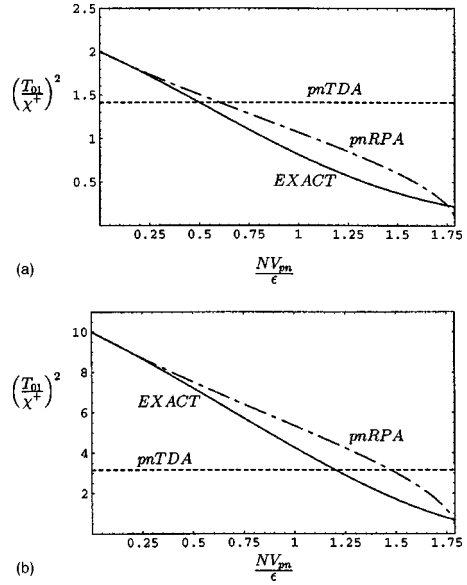

FIG. 2. (a) Transitions from (4;4) to (3;5) for $NW_{pn}/\epsilon = -0.2$. (b) Transitions from (20;20) to (19;21) for $NW_{pn}/\epsilon = -0.2$.

The $\hat{M}^-$ operates in a similar manner as $\hat{M}^+$, but now one proton is transformed into one neutron and the transition simulates nuclear β^+ decay.

We proceed now to the calculation of the g.s.$\rightarrow|1^{st}\rangle$ transition amplitudes (which correspond to a charge-changing transition in a realistic pnRPA calculation) using (i) exact wave functions and (ii) pnRPA wave functions. For the exact calculation we performed the diagonalization of the Hamiltonian and got the exact wave functions as linear combination of the unperturbed states in the full SU(2) basis. Denoting the exact g.s. of the (N_p, N_n) system as Ψ_0 and the first excited state belonging to the (N_p+1, N_n-1) system as Ψ_1 one obtains the following results.

(a) Exact transition amplitudes:

$$T_{01} \equiv \langle \Psi_1 | \hat{M}^+ | \Psi_0 \rangle = (\sqrt{N_n} c_{11} c_{21} + \sqrt{N_p} c_{22} c_{21}) \chi^+. \tag{15}$$

(b) The corresponding pnRPA results:

$$T_{01} \equiv \langle 1^{st} | \hat{M}^+ | pn\text{RPA} \rangle \cong \langle \text{g.s.} | [\Gamma, \hat{M}^+] | \text{g.s.} \rangle = \frac{1}{\sqrt{N}} (XN_n + YN_p) \chi^+. \tag{16}$$

The coefficients c_{ij} are the ones of Eq. (7) related to the decomposition of the g.s. and first excited state wave functions in terms of the unperturbed wave functions of the basis (4). In formula (16) the X term represents the proton-neutron Tamm-Dancoff approximation (pnTDA). Similar formulas are obtained for the $\hat{M}^-$ operator.

In Fig. 2 we display the corresponding transitions as a function of the model parameter NV_{pn}/ϵ and for a fixed

value of the NV/ϵ (-0.2), in the case of the systems with $N=8$ and $N=40$. For comparison, the exact results are drawn on each figure besides the pnTDA and pnRPA results. One notices that the results obtained with the pnTDA and pnRPA methods are quite different from each other, the pnTDA result being constant and deviating substantially from the exact result. Contrary to this, the pnRPA result closely follows the exact one both for the β^- as well as the β^+ type of transition. Another important conclusion emerging from our model is the following: the competition between the particle-particle– and particle-hole–type residual interactions leads to the characteristic behavior of the transition amplitudes of the charge-changing operators first discussed in [2] in the case of a realistic proton-neutron quantum (RPA) (pnQRPA) model using delta-force interaction. Similar behavior was observed in [3] in the case of the pnQRPA with realistic forces. This feature in the realistic cases refers to the fact that the suppression of the NME's is in agreement with the experimental value only if the both kinds of residual interactions (i.e., particle-particle and particle-hole type) are included. Their strengths do not differ very much in realistic calculations. Our model also shows that only when both types of such interactions are taken into account does one obtain the best agreement between exact and RPA result. Also, the results are very closed when the values of their strengths (i.e., NV_{pn}/ϵ and NW_{pn}/ϵ) do not differ much.

Figures 3 displays the same transitions (at the same values of the model parameters) as Fig. 2, but for the $\hat{M}^-$ operator, simulating β^+ decay. Similar conclusions as in the case of the $\hat{M}^+$ operator emerge.

Concluding, we have extended the LMG model to the case of an N-nucleon system composed of two subsystems, one consisting of N_p protons the other of N_n neutrons. The nucleons inside the two subsystems interact in the same manner as in the original LMG model but in proton-neutron space. The two terms simulate the particle-particle and the particle-hole forces of the more realistic models. Our model is exactly solvable in an $SU(2)\otimes SU(2)$ basis and the energy spectrum of the model Hamiltonian is obtained by an exact diagonalization. The first excited state of the spectrum was obtained also by the pnRPA method, and its behavior was studied as function of the model parameters. The pnRPA result deviates considerably from the exact one for small N, while for bigger N, when the collective effects manifest stronger, the pnRPA result comes close to the exact one.

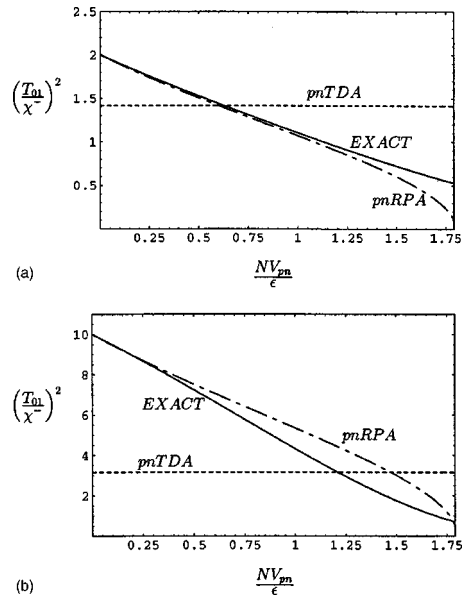

FIG. 3. (a) Transitions from (4;4) to (5;3) for $NW_{pn}/\epsilon = -0.2$. (b) Transitions from (20;20) to (21;19) for $NW_{pn}/\epsilon = -0.2$.

We also show how charge-changing transitions can be treated within our model by defining model charge-changing transition operators simulating nuclear $\beta^{\pm}$ decay and their action on eigenfunctions of the model Hamiltonian. Transition amplitudes of these operators were first calculated using exact wave functions and then using the pnTDA and pnRPA wave functions. The agreement between the pnRPA and exact results is good while for the pnTDA the agreement is poor. This demonstrates the importance of the presence of correlations in the ground state. Finally, our model can simulate the competition between the particle-hole and particle-particle interactions in the transition amplitudes of the charge-changing operators analogously to the realistic calculations [2,3]. All these features emerging from the study of charge-changing transitions in the framework of our exactly solvable Lipkin-type model encourage us to use it further, to test the limits of applicability of higher-order RPA-type approaches.

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