Importance of single-boson and single-fermion mappings in the thermal boson expansion

Z. Aouissat,[1] A. Storozhenko,[2] A. Vdovin,[2] and J. Wambach[1]
[1]*Institut für Kernphysik, Technische Universität Darmstadt, Schlossgartenstrasse 9, D-64289 Darmstadt, Germany*
[2]*Bogoliubov Laboratory of Theoretical Physics, JINR, Dubna RU-141980 Russia*

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In the context of the boson expansion theory, it is usually the case that the bosonization of single-boson (fermion) states is ignored. Although this is tolerable to some extent in cold systems, it causes serious difficulties in the treatment of thermal ensembles where the single-boson (fermion) density of states plays an important role. In the framework of the thermo-field dynamics it is shown how extended forms of the Holstein-Primakoff mapping for both bosons and fermions can lead to consistent thermal-boson expansions. Applications to the $O(N)$ anharmonic oscillator and the Lipkin model are presented.

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I. INTRODUCTION

The thermo-field dynamics (TFD) method with its appealing simplicity is a powerful tool in the study of thermal many-body problems [1] as is evidenced from a number of applications in condensed matter, nuclear, and high energy physics [2,3].

As originally conceived by Umezawa *et al.*, the idea behind the TFD relates to a revealing similarity between the thermal noise and the noise induced by a simple two-mode squeezed state [4]. It thus became possible to express the statistical ensemble average for a given operator as the quantum average of the same operator on a thermal ''ground state.'' This was the prelude that led to the formulation of a thermal theory through a formal doubling of all dynamical degrees of freedom. In practice this is achieved by introducing an auxiliary Fock space, the ''tilde'' transform of the original dynamical Fock space, with the requirement that the expectation value of any operator in the TFD thermal vacuum is exactly equal to the statistical average of the same operator [4].

After several successful tests on systems of interacting particles, the TFD has proven to be a reliable substitute for the standard temperature-dependent Green-function methods in perturbative [3] as well as in nonperturbative [5,6] applications. Recently the TFD has also been combined with the boson expansion theory with the claim of providing a consistent thermal-boson expansion (TBE) approach very much needed in the description of collective phenomena in hot nuclear systems [7]. This question was considered by Providencia and Fiolhais [8], Hatsuda [9], Walet and Klein [10], and also recently by Civitarese and Reboiro [11] within the TFD formalism.

In this regard, Hatsuda considered two points of view regarding the TBE [9]. The first, here, called path I, consists of a bosonization of the original degrees of freedom of the system, substituting for these ideal boson images. Thereafter the thermalization of the system is undertaken by doubling those newly introduced bosons. The second possibility (path II) proceeds on the other hand via a thermalization of the system by doubling the original degrees of freedom and then a bosonization of the entire new system. As concluded by the author the two paths do not lead to the same results in the lowest order when applied to the Lipkin model as a test case. Moreover, a closer look shows that a choice of path I implies that all thermal density of states in the system are of bosonic type (Bose occupation numbers) although the original system, the Lipkin model in this case, is pure fermionic. To circumvent this problem one may consider choosing path II. The thermal density of states are then of fermionic type (Fermi occupation numbers) since the thermalization is performed on the original fermionic degrees of freedom. Therefore with regard to the statistics of the system, path II seems to be the correct answer.

However, there remains one hurdle that is still plaguing this path with inconsistencies. This relates to the quasiparticle energies defining the thermal density of states. As usual, these are taken as solutions of a Hartree-Fock-Bogoliubov (HFB) mean-field approximation, which in most systems leads to a dynamical mass generation. In the case of massless modes, such as the Goldstone modes for instance, this turns out to be the wrong mean-field solution. Indeed it was shown a while ago in the context of the $O(N)$ vector model [12,13] that the Hartree-Bogoliubov (HB) approximation is preferable to the HFB approximation because the first is symmetry conserving while the latter is not. Therefore, the choice of the mean-field approximation is of prime importance for a subsequent correct thermal treatment of the system. On the other hand, it was shown in Refs. [12,13] that the HB solution is obtained after a bosonization of the $O(N)$ vector model by means of the Holstein-Primakoff mapping (HPM). Thus a HB mean-field solution is only obtained after bosonization. Therefore when considering the symmetry constraints, it is rather path I that is favored.[1] This is also the point of view we wish to adopt here. However, it is clear that amendments are needed in order to reconcile path I with the requirements of the statistics as explained earlier.

Therefore, in the present paper, we intend to revisit once more the problem of the TBE. It is our understanding that the

[1]For fermionic systems and, in particular, in the case of the Lipkin model used in [9,10], this problem does not show up in all generality and has therefore escaped attention. In fact one can clearly see that in this case no substantial difference exists between the HFB and HB solutions.

previous works although pioneering and very thorough in their study of the problem, failed to recognize one particular aspect of the boson expansion, namely, the single-particle mapping. It is formally important because it allows the mapping of all original degrees of freedom of the system. It was pointed out a while ago for fermions [14] and recently for bosons [15] that, in order to build a meaningful ideal Fock space that contains at least images for each state of the original Fock space, the usual bosonization of pairs of operators *à la* Holstein-Primakoff needs to be extended such that it accommodates single-operator mapping. We want to show here that this offers a simple solution to the problems outlined above. In the following we first develop the idea for a pure bosonic system, the $O(N)$ anharmonic oscillator, where we explicitly show that the HFB solution is not viable for symmetry reasons. Afterwards we show how to use path I together with and extended HPM to provide a consistent finite-temperature $1/N$ expansion. This prescription is then applied to the purely fermionic Lipkin model.

II. BOSONIC SYSTEM

To illustrate the idea advocated earlier, let us here consider the anharmonic oscillator with an $O(N+1)$ symmetry. The (properly scaled) model Hamiltonian reads

$$H=\frac{\vec{P}_\pi^2}{2}+\frac{P_\sigma^2}{2}+\frac{\omega^2}{2}[\vec{X}_\pi^2+X_\sigma^2]+\frac{g}{N}[\vec{X}_\pi^2+X_\sigma^2]^2-\sqrt{N}\eta X_\sigma . \tag{2.1}$$

Furthermore, we want to take into account an explicit ($\eta \neq 0$) and a spontaneous ($\langle X_\sigma\rangle \neq 0$) symmetry breaking along the X_σ mode.[2] The variables $\vec{X}_\pi, X_\sigma$ and their respective conjugate momenta will be defined below. The subscripts π and σ are used in analogy with the linear σ model in quantum field theory, where these modes represent the pion and the sigma fields, respectively.

In the framework of the TFD formalism, the thermal treatment of the model proceeds as usual by introducing the "tilde" transform of each mode. As such the total Hilbert space of a thermal system is then spanned by the direct product of the eigenstates of the Hamiltonian $H|n\rangle = E_n|n\rangle$ and those of the "tilde" Hamiltonian with the same eigenvalues $\tilde{H}|\tilde{n}\rangle = E_n|\tilde{n}\rangle$. The thermal Hamiltonian $\mathcal{H}$ defined by $\mathcal{H} = H-\tilde{H}$ is the time-translation operator. Its diagonalization leads finally to the excitation energies of the thermal system. However, as is generally the case one is not able to extract the exact eigenvalues of the Hamiltonian. A common approximation scheme consists of splitting the Hamiltonian into a diagonalizable mean-field part and a residual interaction. The two commonly used approximations, the HFB and HB, lead to two qualitatively conflicting results in the interesting situation of a spontaneously broken symmetry. The HFB approximation results in a dynamical mass generation and as such the Goldstone bosons are not massless in the exact symmetry limit. The HB approximation, on the other hand, leads to massless Goldstone excitations. In fact the latter corresponds to the leading-order solution of the $1/N$ expansion. This applies to the model at hand and we want to briefly review both of them.

A. HFB mean-field

Since this question has been studied earlier we only recall here the results needed for our discussion. According to the work [12,13] on the linear sigma model, the HFB mean-field solution can be extracted using a Bogoliubov transformation, that is, a pure rotation for the pions, and an inhomogeneous transformation for the sigma that accounts for the condensation of this mode in the vacuum. Therefore, leaving aside all details of the calculation (which can be found in Refs. [12,13]) one can show that the HFB quasiparticle energies as well as the vacuum condensate are given by the following coupled BCS gap equations

$$\mathcal{E}_\pi^2=\omega^2+\frac{4g}{N}\left[(N+2)\frac{1}{2\mathcal{E}_\pi}+\frac{1}{2\mathcal{E}_\sigma}+\langle X_\sigma\rangle^2\right],$$

$$\mathcal{E}_\sigma^2=\omega^2+\frac{4g}{N}\left[N\frac{1}{2\mathcal{E}_\pi}+3\frac{1}{2\mathcal{E}_\sigma}+3\langle X_\sigma\rangle^2\right], \tag{2.2}$$

$$\langle X_\sigma\rangle=\frac{\eta}{\omega^2}-\frac{1}{\omega^2}\frac{4g}{N}\left[N\frac{1}{2\mathcal{E}_\pi}+3\frac{1}{2\mathcal{E}_\sigma}+\langle X_\sigma\rangle^2\right].$$

It is easy to see from the above that, in the exact symmetry limit (η=0), there are no massless Goldstone bosons in the mass spectrum

$$\mathcal{E}_\pi^2=\frac{\sqrt{N}\eta}{\langle X_\sigma\rangle}+\frac{4g}{N}\left[\frac{1}{\mathcal{E}_\pi}-\frac{1}{\mathcal{E}_\sigma}\right]. \tag{2.3}$$

Indeed, the difference between the quasiparticle energies is finite and this leads to a finite mass for the pion in the limit of vanishing η. Therefore it is clear that the HFB vacuum can by no means serve as a viable mean-field vacuum for any perturbative treatment of the full Hamiltonian. However, as is apparent from Eq. (2.3), in the large N limit ($N\to\infty$), the Goldstone mode can be recovered.

B. HB mean field

To fix the notations needed later on, let us consider here the quantized forms for the variables $\vec{X}_\pi$ and X_σ in terms of which the Hamiltonian in Eq. (2.1) is written as

$$\vec{X}_\pi=\frac{1}{\sqrt{2\omega}}(\vec{a}+\vec{a}^+), \quad X_\sigma=\frac{1}{\sqrt{2\mathcal{E}_\sigma}}(b+b^+), \tag{2.4}$$

$\vec{P}_\pi$ and P_σ denote their respective conjugate momenta.

As mentioned, in the large N limit, the pion exhibits the desired behavior of the Goldstone mode in the broken phase [12]. In this limit the pion is a "Hartree" particle, a semi-

[2] When discussing the question of the Goldstone mode we disregard in the following all problems related to the infrared divergences.

classical mode that does not obey anymore the quantum statistics. Indeed in this limit ($N\rightarrow\infty$) the Fock contributions induced by its wave function vanishes.

The state of a ''Hartree pion'' is therefore suitable for a perturbative treatment of the residual interaction as well as the thermal treatment of the system in TFD. However, the delicate problem that remains is how to actually build a Hartree pion state. To answer this question a mapping was designed in Ref. [15] that allows a consistent bosonization of pairs of pions and a simultaneous rebosonization of the single pions. We recall here the extended Holstein-Primakoff mapping for bosons derived for that purpose

$$(\vec{a}\vec{a})_I=\sqrt{2N+4(n+m)}A,$$

$$(\vec{a}^+\vec{a})_I=2n+m,$$

$$(\vec{a}^+\vec{a}^+)_I=(\vec{a}\vec{a})_I^+,$$

$$(a_i)_I=\sqrt{2N+4(n+m)}\Gamma_N(m)\alpha_i+2\alpha_i^+A\Gamma_N(m),$$

$$(a_i^+)_I=(a_i)_I^+. \tag{2.5}$$

Here N is an integer, $n=A^+A$, $m=\Sigma_i\alpha_i^+\alpha_i$, and Γ_N is given by

$$\Gamma_N(m)=\left[\frac{m+N-2}{2(2m+N)(2m+N-2)}\right]^{1/2}. \tag{2.6}$$

Thus instead of the original ''Hartree-Fock pions'' a_i we have now an ideal boson α_i which, as was shown in Ref. [15], is a Hartree pion. This is at the expense of introducing a power series in a new boson A that plays an auxiliary role. Now that the Hartree state is constructed we can proceed to the thermalization of the system using the TFD formalism.

The time-translation operator or thermal Hamiltonian, $\mathcal{H}=H-\tilde{H}$, of the system is obtained as usual by considering the tilde conjugate of all operators A, α_i and b. Using the mapping in Eqs. (2.5) and (2.6), the thermal Hamiltonian $\mathcal{H}$ is expanded in powers of the parameter N

$$\mathcal{H}=N\mathcal{H}_0+\sqrt{N}\mathcal{H}_1+\mathcal{H}_2+\frac{1}{\sqrt{N}}\mathcal{H}_3+\frac{1}{N}\mathcal{H}_4+\cdots, \tag{2.7}$$

where $\mathcal{H}_i=H_i-\tilde{H}_i$.

In order to determine the asymptotic field, or the independent thermal quasiparticle representation, we introduce a unitary thermal Bogoliubov transformation that rotates the initial single-boson creation and annihilation operators $\alpha_i,\alpha_i^+,b,b^+$ and their tilde conjugate (t.c.) $\tilde{\alpha}_i$, $\tilde{\alpha}_i^+$, $\tilde{b}$, $\tilde{b}^+$ into the thermal quasiboson operators γ_i, γ_i^+, β, β^+ and their t.c.

$$\alpha_i^+=u(T)\gamma_i^++v(T)\tilde{\gamma}_i,$$

$$\tilde{\alpha}_i^+=u(T)\tilde{\gamma}_i^++v(T)\gamma_i,$$

$$b^+=x(T)\beta^++y(T)\tilde{\beta},$$

$$\tilde{b}^+=x(T)\tilde{\beta}^++y(T)\beta. \tag{2.8}$$

We insist here on the fact that the bosons $A,\tilde{A}$ need not be transformed since they are just auxiliary modes. This point of view is different from those adopted in all earlier works [9–11]. The unitarity of the Bogoliubov transformations leads to the following constraints on the thermal amplitudes:

$$u^2(T)-v^2(T)=1,\quad x^2(T)-y^2(T)=1.$$

It was established in Ref. [15] that a vacuum state that is compatible with the $1/N$ expansion as given in Eq. (2.7) is a coherent state $|\Psi\rangle$ that accommodates condensates for the A and b modes. The tilde conjugate Fock space will also have a coherent state $|\tilde{\Psi}\rangle$ as the vacuum state buildin analogy with $|\Psi\rangle$, such that

$$|\Psi\rangle=\exp[\langle A\rangle A^++\langle b\rangle b^+]|0\rangle,$$

$$|\tilde{\Psi}\rangle=\exp[\langle A\rangle\tilde{A}^++\langle b\rangle\tilde{b}^+]|\tilde{0}\rangle, \tag{2.9}$$

where the condensates $\langle A\rangle$ and $\langle b\rangle$ are taken as real numbers for simplicity. The states $|0\rangle$ and $|\tilde{0}\rangle$ denote the vacua for the original basis (i.e., $\vec{\alpha}|0\rangle=\tilde{\vec{\alpha}}|\tilde{0}\rangle=b|0\rangle=\tilde{b}|\tilde{0}\rangle=A|0\rangle=\tilde{A}|\tilde{0}\rangle=0$). The states $|\Psi\rangle$ and $|\tilde{\Psi}\rangle$, on the other hand, are annihilated by the shifted operators $\mathcal{A}$, $\tilde{\mathcal{A}}$, $\mathcal{B}$, and $\tilde{\mathcal{B}}$ defined by

$$\mathcal{A}=A-\langle A\rangle,\quad \tilde{\mathcal{A}}=\tilde{A}-\langle A\rangle,\quad \mathcal{B}=\beta-\langle\beta\rangle,\quad \tilde{\mathcal{B}}=\tilde{\beta}-\langle\beta\rangle. \tag{2.10}$$

The condensates $\langle A\rangle$ and $\langle b\rangle$ scale with N according to

$$\langle A\rangle=d\sqrt{\frac{N}{2}},$$

$$\langle X_\sigma\rangle=\frac{\langle b\rangle+\langle b^+\rangle}{\sqrt{2\mathcal{E}_\sigma}}=\frac{2[x(T)+y(T)]\langle\beta\rangle}{\sqrt{2\mathcal{E}_\sigma}}\equiv\sqrt{N}s. \tag{2.11}$$

Finally the thermal vacuum $|0(T)\rangle$ for the γ_i, $\tilde{\gamma}_i$, $\mathcal{A}$, $\tilde{\mathcal{A}}$, $\mathcal{B}$, and $\tilde{\mathcal{B}}$ operators is given by the following two-mode squeezed state:

$$|0(T)\rangle\sim\exp\left[\sum_i z_\alpha(T)\alpha_i^+\tilde{\alpha}_i^++z_b(T)b^+\tilde{b}^+\right]|\Psi\rangle|\tilde{\Psi}\rangle, \tag{2.12}$$

where $z_\alpha(T)=[v(T)/u(T)]$ and $z_b(T)=[y(T)/x(T)]$. To set up the expansion of the thermal Hamiltonian in Eq. (2.7) we need to fix the values of the two condensates. This can be readily done by minimizing the free energy

$$F=NH_0-TS, \tag{2.13}$$

where the entropy S, in the approximation of independent thermal quasiparticles, is given by

$$S=-N[v^2\ln(v^2)-(1+v^2)\ln(1+v^2)] \\ -y^2\ln(y^2)+(1+y^2)\ln(1+y^2), \tag{2.14}$$

and the average energy of the thermal vacuum state H_0 is given by

$$H_0=\frac{1}{N}\frac{\langle 0(T)|H|0(T)\rangle}{\langle 0(T)|0(T)\rangle} \\ =\frac{\omega(1+2v^2)}{2}(1+2D^2)+\frac{gs^2(1+2v^2)}{\omega}(D+\sqrt{1+D^2})^2 \\ +\frac{g(1+2v^2)^2}{4\omega^2}(D+\sqrt{1+D^2})^4+\frac{\omega^2 s}{2}+gs^4-\eta s. \tag{2.15}$$

Here, for convenience, we introduce the parameter D,

$$D=\frac{d}{\sqrt{1+2v^2}}. \tag{2.16}$$

The minimization procedure of the free energy F with respect to v, s, and d leads to the following three coupled equations:

$$v^2=\frac{1}{\exp[\mathcal{E}_\pi/T]-1}, \\ \frac{\eta}{s}=\omega^2+4gs^2+\frac{2g(1+2v^2)}{\omega}(D+\sqrt{1+D^2})^2, \\ 0=2\omega D\sqrt{1+D^2}+\Delta(D+\sqrt{1+D^2})^2, \tag{2.17}$$

where the gap parameter Δ is given by

$$\Delta=\frac{2gs^2}{\omega}+\frac{g(1+2v^2)}{\omega^2}(D+\sqrt{1+D^2})^2 \tag{2.18}$$

and the Hartree pion mass as

$$\mathcal{E}_\pi=\left[\omega+\Delta+\frac{\Delta D}{\sqrt{1+D^2}}\right]=\omega(\sqrt{1+D^2}-D)^2. \tag{2.19}$$

To exhibit the full dynamics of the leading order in the $1/N$ expansion one needs to generate the terms H_1 and H_2 of the Hamiltonian and construct the corresponding terms of the thermal Hamiltonian: $\mathcal{H}_1=H_1-\tilde{H}_1$ and $\mathcal{H}_2=H_2-\tilde{H}_2$. This can be realized by using parameter differentiation techniques (see Ref. [12] for details). The net results for both $\mathcal{H}_1$ and $\mathcal{H}_2$ are

$$\mathcal{H}_1=\sqrt{\frac{1+2v^2}{2}}\left[2\omega D+\frac{(D+\sqrt{1+D^2})^2}{\sqrt{1+D^2}}\Delta\right] \\ \times[\mathcal{A}+\mathcal{A}^+-(\text{t.c.})] \\ +\left[\frac{2gs(1+2v^2)}{\omega}(D+\sqrt{1+D^2})^2+\omega^2 s+4gs^3-\eta\right] \\ \times(x-y)[\mathcal{B}+\mathcal{B}^+-(\text{t.c.})], \tag{2.20}$$

$$\mathcal{H}_2=\sum_i \mathcal{E}_\pi[\gamma_i^+\gamma_i-(t.c.)]+\mathcal{E}_\sigma[\mathcal{B}^+\mathcal{B}-(\text{t.c.})] \\ +2\mathcal{E}_\pi[(G_+\mathcal{A}^+-G_-\mathcal{A})(G_+\mathcal{A}-G_-\mathcal{A}^+)-(\text{t.c.})] \\ +\frac{g(1+2v^2)}{2\mathcal{E}_\pi^2}[(G_+-G_-)^2(\mathcal{A}+\mathcal{A}^+)^2-(\text{t.c.})] \\ +\frac{2gs\sqrt{1+2v^2}}{\mathcal{E}_\pi\sqrt{\mathcal{E}_\sigma}}[(G_+-G_-)(\mathcal{A}+\mathcal{A}^+) \\ \times[x(\mathcal{B}+\mathcal{B}^+)+y(\tilde{\mathcal{B}}+\tilde{\mathcal{B}}^+)]-(\text{t.c.})], \tag{2.21}$$

where the functions $G_\pm$ are given by

$$G_\pm=\frac{1}{2}\left[\sqrt{1+D^2}\pm\frac{1}{\sqrt{1+D^2}}\right]. \tag{2.22}$$

From the gap equations (2.17) it is easy to verify that, at the minimum of the free energy, the term $\mathcal{H}_1$ vanishes. From the expression of $\mathcal{H}_2$ and more precisely from the coefficient of the bilinear part in γ_i and $\tilde{\gamma}_i$, one can deduce the existence of N uncoupled modes that are nothing but thermal Hartree pions with the mass $\mathcal{E}_\pi$ given by

$$\mathcal{E}_\pi^2=\omega(\omega+2\Delta)=\omega^2+4gs^2+2g\frac{(1+2v^2)}{\mathcal{E}_\pi}=\frac{\eta}{s}. \tag{2.23}$$

The sigma quasimass, $\mathcal{E}_\sigma$ that has purely perturbative character, is fixed by demanding that the bilinear part of $\mathcal{H}_2$ in the $\mathcal{B}$ and $\tilde{\mathcal{B}}$ operators is diagonal. It reads

$$\mathcal{E}_\sigma^2=\omega^2+12gs^2+2g\frac{(1+2v^2)}{\mathcal{E}_\pi}. \tag{2.24}$$

Besides the thermal Hartree pion modes there exist non-Goldstone excitations corresponding to the σ mode. They can be found by diagonalizing the remaining nondiagonal part of $\mathcal{H}_2$. This can be realized by means of the thermal random phase approximation (TRPA) with the following ansatz for the excitation operator:

$$Q_\nu^+=X_\nu\mathcal{B}^+-Y_\nu\mathcal{B}+\tilde{X}_\nu\tilde{\mathcal{B}}^+-\tilde{Y}_\nu\tilde{\mathcal{B}}+U_\nu\mathcal{A}^+ \\ -V_\nu\mathcal{A}+\tilde{U}_\nu\tilde{\mathcal{A}}^+-\tilde{V}_\nu\tilde{\mathcal{A}}. \tag{2.25}$$

Using the usual Rowe equation of motion

$$\langle 0(T)|[\delta Q_\nu,(\mathcal{H}_2,Q_\nu^+)]|0(T)\rangle$$
$$=\Omega_\nu\langle 0(T)|[\delta Q_\nu,Q_\nu^+]|0(T)\rangle, \tag{2.26}$$

one can deduce readily the characteristic equation for the TRPA eigenvalues

$$\Omega_\nu^2=\frac{\eta}{s}+\frac{8gs^2}{1-4g\dfrac{(1+2v^2)}{\mathcal{E}_\pi(\Omega_\nu^2-4\mathcal{E}_\pi^2)}}. \tag{2.27}$$

Thus to order $O(1/\sqrt{N})$ the Hamiltonian in Eq. (2.7) can then be written in the TRPA phonon basis as

$$\mathcal{H}_2=\mathcal{E}_\pi\sum_{i=1}^{N}[\gamma_i^+\gamma_i-(\text{t.c.})]+\sum_{\nu=1,2}\Omega_\nu[Q_\nu^+Q_\nu-(\text{t.c.})], \tag{2.28}$$

It appears from Eqs. (2.23), (2.24), and (2.27) that the results obtained in this paper are in perfect agreement with those of the finite temperature large N limit obtained diagrammatically in four space-time dimensions. In other words the contribution $(1+2v^2/2\mathcal{E}_\pi)$ stands for the thermal tadpole while the term $[1+2v^2/\mathcal{E}_\pi(\Omega_\nu^2-4\mathcal{E}_\pi^2)]$ is nothing but the loop that corresponds to two convoluted thermal propagators (see Refs. [12,13]).

In the next section we wish to show how the extended Holstein-Primakoff mapping can be used for fermions to yield a thermal-boson expansion for the Lipkin model consistent with known results.

III. FERMIONIC SYSTEM

To show that the idea of an extended HP mapping leads, as in the case of pure bosonic systems, to a consistent thermal treatment of fermionic systems, we consider now the two-level Lipkin model.[3] It consists of N fermions distributed over two levels with degeneracy Ω $(\Omega=N)$. The energy of the lower and upper level is $-\varepsilon/2$ and $+\varepsilon/2$, respectively. The model Hamiltonian, written in terms of the generators of an SU(2) algebra [16], reads

$$H=\varepsilon J_z-\frac{1}{2}V(J_+J_++J_-J_-), \tag{3.1}$$

where the quasispin operator J and its components J_+,J_-,J_z are defined as

$$J^2=\frac{1}{2}(J_+J_++J_-J_-)+J_z^2,$$

$$J_z=\frac{1}{2}\sum_{p=1}^{N}(c_{2p}^+c_{2p}-c_{1p}^+c_{1p}),\quad J_+=\sum_{p=1}^{N}c_{2p}^+c_{1p},$$

$$J_-=(\hat{J}_+)^+.$$

Here c_{ip}^+,c_{ip} $(i=1,2;p=1,\ldots,N)$ are fermionic creation and annihilation operators. The indices ‘‘1’’ and ‘‘2’’ label the lower and upper level, respectively, while the index ‘‘p’’ enumerates the sublevels.

Although, from the start, the model only involves fermions it is conceivable that collective modes may induce also bosonic degrees of freedom. Usually this fact is technically accounted for by the bosonization of pairs of fermions. A very convenient dynamical tool is the Holstein-Primakoff mapping for fermions. The original HPM for fermions, however, leads to an ideal Fock space in which there is no state representing a single fermion. Marshalek has proposed in Ref. [14] an extended form for the HPM that allows a consistent mapping of pairs and single fermion states. We recall the fermion-boson images of the quasispin and single fermionic operators according to this mapping

$$(J_z)_I=\frac{1}{2}n_f+B_0^+B_0,$$

$$(J_+)_I=\sqrt{N}B_0^+\sqrt{1-\frac{B_0^+B_0+n_f}{N}};\quad (J_-)_I=(J_+)_I^+,$$

$$(c_{2p})_I=\sqrt{1-\frac{B_0^+B_0}{N}}a_{2p}+\frac{B_0}{N}a_{1p}, \tag{3.2}$$

$$(c_{1p})_I=\sqrt{1-\frac{B_0^+B_0}{N}}a_{1p}-\frac{B_0}{N}a_{2p},$$

$$n_f=\sum_{p=1}^{N}(a_{2p}^+a_{2p}-a_{1p}^+a_{1p}).$$

Here B_0^+ and a_{ip}^+ as well as their Hermitian conjugates are ideal boson and fermion operators, respectively.

Substituting the images (3.2) into Eq. (3.1) we obtain the fermion-boson image H_I of the Hamiltonian of the Lipkin model. The thermal fermion-boson Hamiltonian $\mathcal{H}_I$ then becomes $\mathcal{H}_I=H_I-\tilde{H}_I$. Thus we again follow path I by thermalizing the system obtained after mapping. The diagonalization of $\mathcal{H}_I$ gives as usual the thermal excitations of the system. To determine the thermal vacuum state $|0(T)\rangle$, we introduce as previously the thermal Bogoliubov transformation that rotates only the ideal fermions and their tilde transform, such that

$$a_{ip}^+=x_i\beta_{ip}^++y_i\tilde{\beta}_{ip},$$
$$\tilde{a}_{ip}^+=x_i\tilde{\beta}_{ip}^+-y_i\beta_{ip}. \tag{3.3}$$

The auxiliary bosons B_0 on the other hand are not rotated since they, again, represent auxiliary degrees of freedom. Obviously our procedure cures the problems of path I that

[3]To avoid any confusion we want to attract the attention of the reader to the fact that the notations used in this section are completely independent from those of the previous section.

were encountered in Ref. [9] regarding the fermion statistics. As a consequence of Eq. (3.3) one has only fermionic occupation numbers.

Depending on the value of effective coupling constant $\chi_0=(VN/\varepsilon)$ the Lipkin system may be in one of two phases: ''normal'' ($\chi_0\leqslant 1$) or ''deformed'' ($\chi_0>1$). In terms of ideal fermions and bosons the deformed phase is characterized by the appearance of a boson condensate. To take this possibility into account we introduce a properly normalized shift of the initial boson operator B_0 and define new bosons B and their tilde transform such that

$$B=B_0-d\sqrt{N},\quad \tilde{B}=\tilde{B}_0-d\sqrt{N}. \tag{3.4}$$

Finally the thermal vacuum is written as a direct product of a fermionic thermal vacuum state $|0(T)\rangle_f$ that has the BCS-like form and of a bosonic vacuum state that does not depend on temperature and in turn is a direct product of two vacua—for normal and tilde B_0 bosons, $|0\rangle_{B_0}$ and $|\tilde{0}\rangle_{B_0}$, respectively. Thus,

$$\begin{aligned}|0(T)\rangle&=|0(T)\rangle_f|0\rangle_{B_0}|\tilde{0}\rangle_{B_0}\\&=\exp\left[\sum_{ip}\frac{y_i(T)}{x_i(T)}a^+_{ip}\tilde{a}^+_{ip}+d(B^+_0+\tilde{B}^+_0)\right]\\&\quad\times|0\rangle_a|\tilde{0}\rangle_a|0\rangle_B|\tilde{0}\rangle_B,\end{aligned} \tag{3.5}$$

where

$$a|0\rangle_a=B|0\rangle_B=0,\quad \tilde{a}|\tilde{0}\rangle_a=\tilde{B}|\tilde{0}\rangle_B=0.$$

To proceed with the evaluation of the dynamics we follow the steps of the previous section by first expanding the quasispin operators in powers of N. This yields to order $O(1/\sqrt{N})$

$$\begin{aligned}(J_-)_I&=Nd\sqrt{K}+\sqrt{N}\left[\sqrt{K}B-\frac{d^2}{2\sqrt{K}}(B^++B)\right]\\&\quad-\frac{d}{2\sqrt{K}}\left[2B^+B+BB+\frac{d^2}{4K}(B^++B)+n_f^\beta\right],\end{aligned} \tag{3.6}$$

where $n_f^\beta=\Sigma_{p=1}^N(\beta^+_{2p}\beta_{2p}-\beta^+_{1p}\beta_{1p})$, $K=z-d^2$, and $z=y_1^2-y_2^2$.

With similar expansions for $(J_+)_I$ and $(J_z)_I$ one can readily write down the three first terms of the expansion of the Hamiltonian relevant in the large N limit

$$H_0=\varepsilon\left(-\frac{z}{2}+d^2-\chi_0d^2K\right),$$

$$H_1=\varepsilon[d-\chi_0d(K-d^2)](B^++B), \tag{3.7}$$

$$\begin{aligned}H_2&=\varepsilon\left(\frac{1}{2}+\chi_0d^2\right)n_f^\beta+\varepsilon(1+3\chi_0d^2)B^+B\\&\quad-\varepsilon\frac{\chi_0}{2}(K-2d^2)(B^+B^++BB).\end{aligned}$$

The value of the condensate ''shift'' d and the coefficients of the thermal Bogoliubov transformation are fixed at the minimum of the grand canonical potential (free energy) F of the system, which reads

$$F=\langle 0(T)|H_I|0(T)\rangle-TS-\mu\langle 0(T)|\hat{N}|0(T)\rangle, \tag{3.8}$$

where the entropy S and the number operator $\hat{N}$ for the ideal fermions a^+ and a are given by

$$S=-N\sum_{i=1,2}[y_i^2\ln(y_i^2)-x_i^2\ln(x_i^2)],$$

$$\hat{N}=\sum_{p=1}^N(a^+_{1p}a_{1p}+a^+_{2p}a_{2p}). \tag{3.9}$$

The averages in Eq. (3.8) refer to the thermal vacuum $|0(T)\rangle$ in accordance with TFD. It is obvious from Eqs. (3.7) that the only contributing to the free energy is H_0. One obtains

$$\begin{aligned}F&=N\varepsilon\left(-\frac{n_1-n_2}{2}+d^2-\chi_0d^2K\right)\\&\quad+TN\sum_{i=1,2}[n_i\ln(n_i)-(1-n_i)\ln(1-n_i)]-\mu N[n_1+n_2],\end{aligned} \tag{3.10}$$

where the notations n_1,n_2 are used instead of y_1^2,y_2^2. Minimizing F with respect to d, n_1, and n_2, one obtains, for vanishing chemical potential (μ=0), the following solutions for the shift d, the occupation numbers n_i, and the quasiparticle energies E_i:

$$d^2=\begin{cases}0: & \text{(normal phase)}\\ \dfrac{\chi_0(n_2-n_1)-1}{2\chi_0}: & \text{(deformed phase)}\end{cases} \tag{3.11}$$

$$n_i=\frac{1}{1+\exp(E_i/T)}, \tag{3.12}$$

$$E_{1,2}=\mp\varepsilon\left[\frac{1}{2}+\chi_0d^2\right]. \tag{3.13}$$

Here one sees that the single-particle energies become temperature dependent in the deformed phase. Their T dependence is exactly the same as given by the thermal Hartree-Fock method (see, e.g., [17]).

From Eq. (3.11) one deduces that $\mathcal{H}_1=0$ at the minimum. The part $\mathcal{H}_2$ of the thermal Hamiltonian is already diagonal in the fermionic sector. There remains, however, the bosonic sector that is at the most bilinear in the bosons and is diagonalized by means of two separate Bogoliubov rotations on the bosons B and their tilde conjugate $\tilde{B}$, respectively:

$$B^{+}=uC^{+}+vC,\quad \tilde{B}^{+}=u\tilde{C}^{+}+v\tilde{C}. \tag{3.14}$$

In fact this is nothing but the RPA for pairs of fermions and the RPA frequencies ω can be extracted after few simple manipulations. One gets

$$\omega=\begin{cases}\varepsilon\sqrt{1-\chi_0^2(n_1-n_2)^2}: & \text{for the normal phase}\\ \varepsilon\sqrt{2(\chi_0^2(n_1-n_2)^2-1)}: & \text{for the deformed phase.}\end{cases} \tag{3.15}$$

The term $\mathcal{H}_2$ of the thermal Hamiltonian of the system takes the form

$$\mathcal{H}_2=\sum_{i=1,2;p}E_i(\beta_{ip}^{+}\beta_{ip}-\tilde{\beta}_{ip}^{+}\tilde{\beta}_{ip})+\omega[C^{+}C-(\text{t.c.})]$$

in agreement with the result of the thermal HFB-RPA [17].

Finally it is worth noting that the boson-fermion HP mapping in Eqs. (3.2) produces an interesting physical picture for the thermal behavior of the Lipkin model. After the mapping one obtains a system of ideal fermions interacting via the exchange of an ideal boson [see Eq. (3.7)]. Although we did not introduce a direct T dependence for the auxiliary boson[4] the correct temperature dependence of the boson excitation energies are recovered. It appears that the bosons are thermalized due to the collisions with the ‘‘heated’’ fermions. The same is true for the A boson in case of the $O(N)$ anharmonic oscillator.

[4]No boson occupation numbers were introduced.

IV. CONCLUSION

In the present paper we have addressed the question of the thermal-boson expansion within the thermo-field dynamics formalism. We have stressed the importance of the bosonization of single-boson (fermion) operators in addition to the usual boson (fermion)-pair bosonization. To our knowledge this point has been overlooked previously. We have shown on two models, the bosonic $O(N)$ anharmonic oscillator and the fermionic Lipkin model, how this can lead to a consistent thermal-boson expansion. The leading-order results of the expansion in both cases were derived and the thermal excitations were shown to comply with both the symmetry and the particle statistics requirements. It was also shown, in the case of the anharmonic oscillator, that without the single-particle mapping the thermal Goldstone excitations were missing from the spectrum of the theory.

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