Degeneracies when $T=0$ two body matrix elements are set equal to zero and Regge's 6*j* symmetry relations

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The effects of setting all $T=0$ two body interaction matrix elements equal to a constant (or zero) in shell model calculations (designated as $\langle T=0\rangle=0$) are investigated. Despite the apparent severity of such a procedure, one gets fairly reasonable spectra. We find that using $\langle T=0\rangle=0$ in single-*j* shell calculations degeneracies appear, e.g., the $I=\frac{1}{2}^-$ and $\frac{13}{2}^-$ states in ^{43}Sc are at the same excitation energies; likewise the $I=3_2^+, 7_2^+, 9_1^+$, and 10_1^+ states in ^{44}Ti. The above degeneracies involve the vanishing of certain $6j$ and $9j$ symbols. The symmetry relations of Regge are used to explain why these vanishings are not accidental. Thus for these states the actual deviation from degeneracy are good indicators of the effects of the $T=0$ matrix elements. A further indicator of the effects of the $T=0$ interaction in an even-even nucleus is to compare the energies of states with odd angular momentum with those that are even.

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I. INTRODUCTION

In the early 1960's single-*j* shell calculations in the $f_{7/2}$ region were performed by McCullen, Bayman, and Zamick (MBZ) [1,2] and Ginocchio and French [3]. In these calculations the two body matrix elements were taken from experiment. However the $T=0$ neutron proton spectrum in ^{42}Sc was not well determined. Calculations with correct $T=0$ matrix elements were later performed by Kutschera, Brown, and Ogawa [4].

In order to see how neutron-proton two body matrix elements with isospin $T=0$ affect the low lying spectra of nuclei, we have set them to a constant in a single-*j* shell calculation in the $f_{7/2}$ region. We can then write $V^{T=0}=c(1/4-t_1\cdot t_2)$ where c is a constant. Hence $\Sigma_{i<j}V_{ij}^{T=0}=c/8(n(n-1)+6)-c/2T(T+1)$. This means that the spectrum of states of a given isospin, e.g., $T=0$ in ^{44}Ti is independent of what the constant is. It might as well be *zero*. What the constant is will affect only the energy splittings of states with different isospin. We shall denote this matrix element input as $\langle T=0\rangle=0$.

Although setting all $T=0$ matrix elements to a constant may seem like a severe approximation, it will be seen that one gets a fairly good representation of the spectrum. When the $T=0$ matrix elements are reintroduced, there is some fine tuning which improves the spectrum.

While the problem of $T=1$ pairing is better understood and studied, there exists a very extensive literature on the possibility of $T=0$ pairing, both pro and con. We here include some of the relevant references [5–13].

In a shell model calculation the effects of both $T=0$ and $T=1$ pairing are automatically included. The problem then is to sort out as much as possible the individual effects.

In the next sections we will consider calculations in the $f_{7/2}$ shell and in the full *f-p* space.

II. RESULTS OF SINGLE-*j* SHELL CALCULATIONS

In the following tables we show $T=T_{\rm min}$ calculated yrast spectra for ^{43}Ti (Table I) and ^{44}Ti (Table II) where we use two different sets of matrix elements. Following the idea first championed by Talmi [14] and others we take our matrix elements from experiment. In the first two columns we show $\langle T=0\rangle=0$ for the ^{42}Sc matrix elements. The last two columns consist of matrix elements from ^{42}Sc with the $T=0$ matrix elements now included. Also to gain some insight into how configuration mixing affects our results, we present full *f-p* space results for ^{43}Ti and ^{44}Ti in Tables III and IV, respectively.

In the single-*j* shell calculation for which the matrix elements were taken from the spectrum of ^{42}Sc the values of these matrix elements for $J=0$ to $J=7$ were 0.000 MeV, 0.6110 MeV, 1.5863 MeV, 1.4904 MeV, 2.8153 MeV, 1.5101 MeV, 3.242 MeV, and 0.6163 MeV, respectively. The yrast spectrum is also shown in Fig. 1. Note that with a j^2 configuration the even J states have T equal to one and the odd J T equal to zero. This is also true experimentally for these levels. Note that the $J=1^+$ and 7^+ are nearly degenerate near 0.6 MeV and the $J=3^+$ and 5^+ are nearly degenerate near 1.5 MeV. Thus the act of setting $T=0$ matrix

TABLE I. Spectra of ^{43}Ti.

^{42}Sc $\langle T=0\rangle=0$ interaction		^{42}Sc interaction	
I	E (MeV)	I	E (MeV)
7/2	0.0000	7/2	0.000
9/2	1.640	9/2	1.680
3/2	1.831	11/2	2.335
11/2	2.061	3/2	2.888
5/2	2.832	5/2	3.449
1/2	3.279	13/2	3.500
13/2	3.279	15/2	3.511
15/2	3.425	19/2	3.644
17/2	3.919	17/2	4.298
19/2	3.919	1/2	4.316

TABLE II. Spectra of ^{44}Ti.

^{42}Sc $\langle T=0\rangle=0$ interaction		^{42}Sc interaction	
I	E (MeV)	I	E (MeV)
0	0.000	0	0.000
2	1.303	2	1.163
4	2.741	4	2.790
6	3.500	6	4.062
3	4.716	3	5.786
5	4.998	5	5.871
7	5.356	7	6.043
8	5.656	8	6.084
9	7.200	10	7.384
10	7.200	12	7.702
12	7.840	9	7.984

TABLE III. ^{43}Ti full f-p calculation.

FPD6 $\langle T=0\rangle=0$		FPD6	
I	E (MeV)	I	E (MeV)
7/2	0.000	7/2	0.000
3/2	1.668	3/2	0.871
9/2	1.970	1/2	1.805
11/2	2.000	11/2	1.889
5/2	2.638	5/2	2.305
1/2	2.940	9/2	2.633
15/2	3.065	15/2	2.948
13/2	3.070	19/2	3.401
17/2	3.325	13/2	3.718
19/2	3.417	17/2	4.429

TABLE IV. ^{44}Ti full f-p calculation.

FPD6 $\langle T=0\rangle=0$		FPD6	
I	E (MeV)	I	E (MeV)
0	0.000	0	0.000
2	1.515	2	1.317
4	2.587	4	2.536
6	3.223	6	3.843
3	4.717	3	6.241
5	4.932	8	6.383
8	5.292	5	7.579
7	5.391	10	7.790
10	6.476	7	7.921
9	6.574	12	8.574
1	7.070	9	9.030
12	7.192	1	9.681
11	9.914	11	11.028

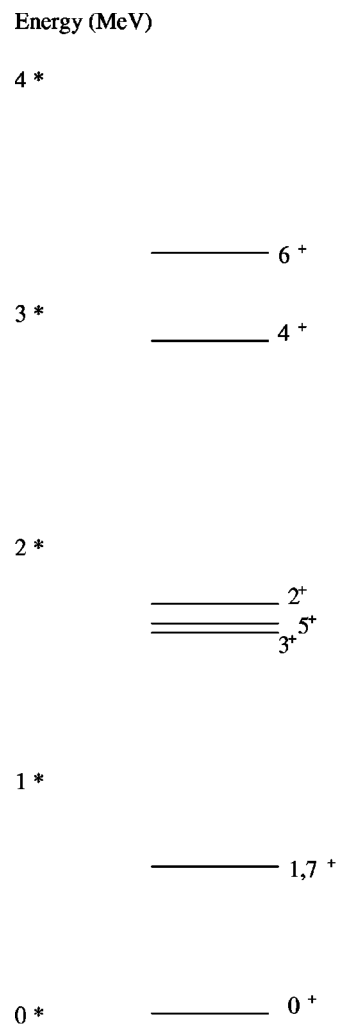

FIG. 1. Spectrum of ^{42}Sc.

elements to a constant is equivalent to moving the $J=1^+$ and 7^+ together up about 0.9 MeV. Or putting it another way, the act of removing the degeneracy is to lower the energies of the $J=1^+$ and 7^+ by about the same amount. This is in contrast to most studies in which only the effects of lowering the $J=1^+$ state are studied.

We will point out several features to be found in the tables. We observe many levels that were considerably separated in the ''normal'' interaction become degenerate when we go to $\langle T=0\rangle=0$. We explore this further in the next section. We find that in general with few exceptions that the odd I levels of ^{44}Ti are at a lower excitation energy when we go to the $\langle T=0\rangle=0$ version of the interactions and that the ^{43}Ti spectra is lowered in total.

III. THE DEGENERACIES THAT OCCUR IN $\langle T=0\rangle=0$ AND EXPLANATIONS

As can be seen from Tables I and II some energy levels are degenerate when the $T=0$ matrix elements are set equal to a constant. The degenerate pairs (I_1,I_2) include

^{43}Ti $(\frac{1}{2}^-,\frac{13}{2}^-)(\frac{17}{2}^-,\frac{19}{2}^-)$,
^{44}Ti $(9^+,10^+)$.

The wave functions for the titanium isotopes are written as

$$\psi=\Sigma D^{I\alpha}(J_p,J_n)[(j^2)^{J_p}(j^n)^{J_n}]^{I\alpha}, \tag{1}$$

where $D^{\pm}(J_p,J_n)$ is the probability amplitude that in a state of total angular momentum I the protons couple to J_p and the neutrons to J_n. The elements $D^I(J_p,J_n)$ form a column vector.

Let us first consider $(\frac{1}{2}^-,\frac{13}{2}^-)$ in ^{43}Ti. The basis states can be written as $[J_p,J_n]^I$ where J_p is the angular momentum of the two protons. The interaction matrix element $\langle[J'_p,J_n]^I V[J_p,J_n]^I\rangle=\delta_{J'_pJ_p}E_{J_p}+2\Sigma_J U(jjIj,J'_pJ)U(jjIj,J_pJ)E_J$ where E_J is the two particle matrix element $\langle[jj]^J V[jj]^J\rangle$. For even J, T is equal to one while for odd J, T is equal to zero.

We next consider ^{44}Ti The interaction matrix element $\langle[J'_pJ'_n]^I V[J_pJ_n]^I\rangle$ is given by

$$E_{J_p}\delta_{J_pJ'_p}\delta_{J_nJ'_n}+E_{J_n}\delta_{J_pJ'_p}\delta_{J_nJ'_n}+4\Sigma_{JJ_A}$$
$$\times\langle(jj)^{J'_p}(jj)^{J'_n}|(jj)^J(jj)^{J_A}\rangle^I\langle(jj)^{J_p}(jj)^{J_n}|(jj)^J(jj)^{J_A}\rangle$$
$$\times\,^IE_J,$$

where the unitary recouping coefficients are related to the Wigner $9j$ symbols

$$\langle(ab)^c(de)^f|(ad)^x(be)^y\rangle^I$$
$$=\sqrt{(2c+1)(2f+1)(2x+1)(2y+1)}\begin{Bmatrix} a & b & c\\ d & e & f\\ x & y & I\end{Bmatrix}. \tag{2}$$

For symmetry relations the $9j$ symbols are more convenient than the unitary coefficients.

It is instructive to look at the energies and wave functions (i.e., column vectors) for the $I=\frac{1}{2}^-$ and $I=\frac{13}{2}^-$ states that appear in the NYO Technical reports (which included $T=0$ matrix elements)

		$I=\frac{1}{2}$	$I=\frac{13}{2}$	
Energy (MeV)		5.4809	3.8477	5.8122
J_p	j_n			
4	7/2	1.000	0.9942	-0.1076
6	7/2	0.000	0.1076	0.9942

In the $f_{7/2}$ model the $I=\frac{1}{2}^-$ configuration is unique $[J_p=4\ \ J_n=\frac{7}{2}]^{1/2-}$. There are two configurations for the $I=\frac{13}{2}^-$ state $[4\frac{7}{2}]$ and $[6\frac{7}{2}]$.

When we go to $\langle T=0\rangle=0$ what basically happens is that the eigenvalues and eigenfunctions become

		$I=\frac{1}{2}$ E_1	$I=\frac{13}{2}$ E_1	E_2
J_p	j_n			
4	7/2	1.000	1.000	0.000
6	7/2	0.000	0.000	1.000

In order for this to happen the matrix element $\langle[J_p=4,\ j_n=\frac{7}{2}]^{I=13/2}V[J_p=6,\ j_n=\frac{7}{2}]^{I=13/2}\rangle$ must vanish. This vanishing is carried by the Racah coefficients $U(\frac{7}{2}\frac{7}{2}\frac{13}{2}\frac{7}{2};4J)U(\frac{7}{2}\frac{7}{2}\frac{13}{2}\frac{7}{2};6J)$ where J is the angular momentum of a neutron-proton pair.

In general J can be 4,5,6, or 7. However in $\langle T=0\rangle=0$, only the even J's contribute, i.e., $J=4$ or $J=6$. In either case one of the Racah coefficients will be $U(\frac{7}{2}\frac{7}{2}\frac{13}{2}\frac{7}{2};46)$. This Racah coefficient is zero. This guarantees a decoupling of $[4\frac{7}{2}]$ from $[6\frac{7}{2}]$ but does not in itself lead to a degeneracy of the $I=\frac{1}{2}^-$ and $I=\frac{13}{2}^-$ states. That happens because of this additional condition

$$U\left(\frac{7}{2}\,\frac{7}{2}\,\frac{13}{2}\,\frac{7}{2};44\right)=U\left(\frac{7}{2}\,\frac{7}{2}\,\frac{1}{2}\,\frac{7}{2};44\right)=\frac{1}{2}. \tag{3}$$

We next consider the degeneracy of $I=9^+_1$ and 10^+_1 in ^{44}Ti in $\langle T=0\rangle=0$. It is again instructive to write down the eigenfunctions as they appear in the NYO report

		$I=9$			$I=10$		
Energy		8.7799	8.8590	11.5951	7.8429	9.8814	10.5110
isospin			$T=1$	$T=1$			$T=1$
J_p	j_n						
4	6	-0.7071	0.5636	-0.4270	0.7037	-0.0696	0.7071
6	4	0.7071	0.5636	-0.4270	0.7037	-0.0696	-0.7071
6	6	0.0000	0.6039	0.7971	0.0984	0.9951	0.0000

Before proceeding, we remind the reader of a general rule that can clearly be seen in the wave functions above. For even total angular momentum I the wave functions of even T states of $N=Z$ nuclei do not change sign under the interchange of neutrons and protons but the $T=1$ wave functions do change sign. For odd I it is the opposite. This can be summarized by $D^{IT}(J_p,J_n)=(-1)^{I+T}D^{IT}(J_n,J_p)$.

We focus on the $T=0$ states. This makes life much simpler. Instead of three states each we need only worry about one $I=9^+$ and two $I=10^+$ states. Note that for $I=9^+$ $T=0$ the state has the simple wave function

$$\begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}.$$

What clearly happens for $I=10^+$ in $\langle T=0\rangle=0$ is that there is a decoupling of [6,4] and [4,6] from [6,6] so that the wave functions of the two $T=0$ states become

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and the eigenvalue of the first one becomes the same as that of the unique $I=9$ state.

We further note that aside from the yrast degeneracies there are other degeneracies. For example, the 7_2^+ and 3_2^+ are degenerate with the $I=9_1^+,10_1^+$ pair in ^{44}Ti. At first this is puzzling because the dimensions are different. There are seven basis states for $I=3^+$ and six for $I=7^+$, whereas for $I=9^+$ and 10^+ there are only three basis states. However, of the seven $I=3^+$ states, five have isospin *one*, and only two have isospin $T=0$. Of the six $I=7^+$ states, four have isospin one and only two have isospin zero. Since we are focusing on $T=0$ we only show only these wave functions in Table V. When the $T=0$ two particle matrix elements are set equal to zero the wave functions simplify as shown in the table.

We now begin to see a connection between $I=3_2^+, 7_2^+, 9_1^+$, and 10_1^+. For the 9_1^+ and 10_1^+ the only nonzero components of the wave function in the $\langle T=0\rangle=0$ are $D(4,6)$ and $D(6,4)$ both having magnitude $1/\sqrt{2}$. The 3_1^+ state has nonzero components $D(2,4)$ and $D(4,2)$. There is no connection with the 9_1^+ and 10_1^+ states. However for the 3_2^+ state the only nonvanishing matrix elements are $D(4,6)$ and $D(6,4)$ each with magnitude $1/\sqrt{2}$. This is the same as what occurs for the 9_1^+ and 10_1^+ states.

A similar story is written by $I=7^+$. The nonvanishing components for the 7_1^+ state in the $\langle T=0\rangle=0$ case are $D(2,6)$ and $D(6,2)$ however for the 7_2^+ state they are $D(4,6)$ and $D(6,4)$ each with magnitude $1/\sqrt{2}$.

Thus a common theme emerges for $I=3_2^+, 7_2^+, 9_1^+$, and 10_1^+ (all $T=0$) in that for the $\langle T=0\rangle=0$ case the only nonvanishing components of the wave functions are $D(4,6)$ and $D(6,4)$. Visually, the column vectors look the same. And it is precisely these states that are degenerate.

Let us now show why in the case of $\langle T=0\rangle=0$ the matrix element $\langle [J_p'=4J_N'=6]^{I=10}V[J_p=6J_N=6]^{I=10}\rangle$ vanishes. This is a necessary condition for the wave functions to have the simple form discussed in this section.

From the expression for the neutron-proton interaction previously given the above matrix element is ($j=\frac{7}{2}$)

$$(c)(13)(9)\begin{Bmatrix} j & j & 4 \\ j & j & 6 \\ 4 & 6 & 10 \end{Bmatrix}\begin{Bmatrix} j & j & 6 \\ j & j & 6 \\ 4 & 6 & 10 \end{Bmatrix}E^4 + (c)(13)\Sigma_{J_A}(2J_A+1)\begin{Bmatrix} j & j & 4 \\ j & j & 6 \\ 6 & J_A & 10 \end{Bmatrix} \times\begin{Bmatrix} j & j & 6 \\ j & j & 6 \\ 6 & J_A & 10 \end{Bmatrix}E^6, \tag{4}$$

where the proportionality constant c is $156\sqrt{13}$. (Note that E^5 and E^7 are equal to zero because all odd J have $T=0$.) Because the last $9j$ above has two rows identical it is necessary for J_A to be even ie $J_A=4$ or 6. Thus the coefficient of E^6 is

$$(c)(13)(9)\begin{Bmatrix} j & j & 4 \\ j & j & 6 \\ 6 & 4 & 10 \end{Bmatrix}\begin{Bmatrix} j & j & 6 \\ j & j & 6 \\ 6 & 4 & 10 \end{Bmatrix} + (c)(13)(13)\begin{Bmatrix} j & j & 4 \\ j & j & 6 \\ 6 & 6 & 10 \end{Bmatrix}\begin{Bmatrix} j & j & 6 \\ j & j & 6 \\ 6 & 6 & 10 \end{Bmatrix}. \tag{5}$$

Using symmetry properties of $9j$ symbols we note that every term in the above expression (both for E^4 and E^6) contains the $9j$ symbol

$$\begin{Bmatrix} j & j & 6 \\ j & j & 6 \\ 6 & 4 & 10 \end{Bmatrix}.$$

This $9j$ symbol is zero and hence we have shown why the above neutron-proton matrix element vanishes. It is by no means obvious why this $9j$ vanishes. There will be consid-

TABLE V. Comparison of wave functions of MBZ (from Technical Report No. NYO 9801 [2]) with those for which $\langle T=0\rangle=0$ matrix elements are set equal to zero.

$I=3$		MBZ	$\langle T=0\rangle=0$	MBZ	$\langle T=0\rangle=0$
Energy (MeV)		6.533		10.493	
J_P	J_N				
2	2	0.0000	0	0.0000	0
2	4	0.6968	$\frac{1}{\sqrt{2}}$	-0.1202	0
4	2	-0.6968	$\frac{-1}{\sqrt{2}}$	0.1202	0
4	4	0.0000	0	0.0000	0
4	6	0.1202	0	0.6968	$\frac{1}{\sqrt{2}}$
6	4	-0.1202	0	-0.6968	$\frac{-1}{\sqrt{2}}$
6	6	0.0000	0	0.0000	0
$I=7$		MBZ	$\langle T=0\rangle=0$	MBZ	$\langle T=0\rangle=0$
Energy (MeV)		6.5723		9.6570	
J_P	J_N				
2	6	0.6965	$\frac{1}{\sqrt{2}}$	0.1220	0
4	4	0.0000	0	0.0000	0
4	6	0.1220	0	-0.6965	$\frac{-1}{\sqrt{2}}$
6	2	-0.6965	$\frac{-1}{\sqrt{2}}$	-0.1220	0
6	4	-0.1220	0	0.6965	$\frac{1}{\sqrt{2}}$
6	6	0.0000	0	0.0000	0
$I=9$		MBZ	$\langle T=0\rangle=0$	MBZ	$\langle T=0\rangle=0$
Energy (MeV)		8.7799			
J_P	J_N				
4	6	-0.7071	$\frac{-1}{\sqrt{2}}$		
6	4	0.7071	$\frac{1}{\sqrt{2}}$		
6	6	0.0000	0		
$I=10$		MBZ	$\langle T=0\rangle=0$	MBZ	$\langle T=0\rangle=0$
Energy (MeV)		7.8429		9.8814	
J_P	J_N				
4	6	0.7037	$\frac{1}{\sqrt{2}}$	-0.0696	0
6	4	0.7037	$\frac{1}{\sqrt{2}}$	-0.0696	0
6	6	0.0084	0	0.9951	1

erable discussion in the next section of why some of the $6j$'s and $9j$'s we encounter vanish.

Although in Table V we have only shown $T=0$ wave functions there are several $T=1$ states interspaced amongst the $T=0$ states. For example, in the Technical Report No. NYO-9891 [2] for $I=3^+$ the lowest state calculated to be at 6.2357 MeV has $T=1$. The calculated energy for this state is about 300 keV lower than the lowest $T=0$ state shown in Table V. Other $T=1$ states are calculated to be at 9.2334, 10.0321, and 10.9022 MeV. For $I=7^+$ the lowest $T=1$ state is calculated to be at 6.7094 MeV, just above the other the lowest $T=0$ state shown in Table V. The other $T=1$ states for $I=7^+$ are calculated to be at 9.0744, 9.5141, and 12.1535 MeV. The closeness of $T=0$ and $T=1$ states was previously discussed by Goode and Zamick [15].

IV. WHY SOME RACAH COEFFICIENTS VANISH — REGGE SYMMETRIES

Thus far we have explained how degeneracies arise by matrices that certain Racah or $9j$ symbols vanish. In this section we look for a deeper meaning. We were aided in this by many insightful articles collected in Biedenharn and Van Dam [16].

For convenience we shall switch from unitary Racah coefficients to Wigner $6j$ symbols

$$U(abcd;ef)=(-1)^{a+b+c+d}\sqrt{(2e+1)(2f+1)}\times\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}. \tag{6}$$

In the previous section we noted that the $6j$ symbol

$$\begin{Bmatrix} \frac{7}{2} & \frac{7}{2} & 4 \\ \frac{7}{2} & \frac{13}{2} & 6 \end{Bmatrix}$$

vanished. We note that this is a particular case of a wider class of $6j$'s that vanish. All $6j$'s of the form

$$\begin{Bmatrix} j & j & (2j-3) \\ j & (3j-4) & (2j-1) \end{Bmatrix}$$

vanish for all j, both half integer and integer. Besides the six j above other examples are

$$\begin{Bmatrix} \frac{5}{2} & \frac{5}{2} & 2 \\ \frac{5}{2} & \frac{7}{2} & 4 \end{Bmatrix}, \quad \begin{Bmatrix} \frac{9}{2} & \frac{9}{2} & 6 \\ \frac{9}{2} & \frac{19}{2} & 8 \end{Bmatrix}, \quad \text{and} \quad \begin{Bmatrix} 4 & 4 & 5 \\ 4 & 8 & 7 \end{Bmatrix}.$$

We find we can relate the above $6j$ symbol to a simpler one using one of the six remarkable relations discovered by Regge in 1959 [17]. We follow the notation of Rotenberg *et al.* [18]

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} = \begin{Bmatrix} A & B & C \\ D & E & F \end{Bmatrix},$$

$$A=\frac{1}{2}(j_1+j_2+l_1-l_2),\quad B=\frac{1}{2}(j_2+j_3+l_2-l_3),$$

$$C=\frac{1}{2}(j_1+j_3-l_1+l_3),\quad D=\frac{1}{2}(j_1-j_2+l_1+l_2),$$

$$E=\frac{1}{2}(j_2-j_3+l_2+l_3),\quad F=\frac{1}{2}(-j_1+j_3+l_1+l_3).$$

From this Regge symmetry relation we find that

$$\begin{Bmatrix} j & j & (2j-3) \\ j & (3j-4) & (2j-1) \end{Bmatrix} = \begin{Bmatrix} 2 & (2j-3) & (2j-2) \\ (2j-2) & (2j-1) & (2j-2) \end{Bmatrix} = \begin{Bmatrix} (2j-2) & (2j-3) & 2 \\ (2j-2) & (2j-1) & (2j-2) \end{Bmatrix}. \tag{7}$$

We note that $6j$ symbols with a ''two'' in them have been worked out by Biedenharn, Blatt, and Rose [19]. Using their notation we find from their results that

$$\begin{Bmatrix} l_1 & J_1 & 2 \\ J_2 & l_2 & L \end{Bmatrix}$$

for $l_2=J_1+1$ and $l_1=J_1+1$ is proportional to X where

$$X=[(J_1+1)(J_1-J_2)-L(L+1)+J_2(J_2+2)]. \tag{8}$$

We have $L=2j-2$, $J_1=2j-3$, $l_1=2j-2$, $J_2=2j-2$, and $l_2=2j-1$. With these values we see that X vanishes.

In Regge's paper [17] he states ''although no direct connection has been established between these wider symmetries it seems very probably that it will be found in the future.'' He also states ''We see therefore that there are 144 identical Racah's coefficients... . It should be pointed out that this wider 144-group is isomorphic to the direct product of the permutation group of 3 and 4 objects.''

Following Regge's work Bargmann presented, amongst other things, his derivation of the Regge symmetries [20]. He there stated ''While the following analysis does not lead to a deeper understanding of the Regge symmetries it yields, at least a fairly transparent derivation of the symmetries.''

In Sec. III we pointed out that a certain $9j$ symbol ''unexpectedly'' vanished. Perhaps there are some symmetries involving the $9j$ symbols as well. The only comment by Bargmann on this [20] is ''Schwinger has computed the generating function for the $9j$ symbol. This does not reveal any new symmetries — at least none to be obtained by a permutation of the relevant quantities $k_{\alpha\beta}$.''

Nevertheless the Regge symmetries for $6j$ symbols do have some implications for $9j$'s. The $9j$ mentioned in the previous section

$$\begin{Bmatrix} \frac{7}{2} & \frac{7}{2} & 6 \\ \frac{7}{2} & \frac{7}{2} & 6 \\ 4 & 6 & 10 \end{Bmatrix}$$

is part of a wider class of identically zero $9j$ symbols. These are of the form

$$\begin{Bmatrix} j & j & (2j-1) \\ j & j & (2j-1) \\ (2j-1) & (2j-3) & (4j-4) \end{Bmatrix}.$$

Other examples are

$$\begin{Bmatrix} \frac{9}{2} & \frac{9}{2} & 8 \\ \frac{9}{2} & \frac{9}{2} & 8 \\ 8 & 6 & 14 \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} 4 & 4 & 7 \\ 4 & 4 & 7 \\ 7 & 5 & 12 \end{Bmatrix}.$$

Following the notation of Rotenberg *et al.* [18] we first use the well-known expression for a $9j$ as a sum over three $6j$ symbols:

$$\begin{Bmatrix} j & j & (2j-1) \\ j & j & (2j-1) \\ (2j-1) & (2j-3) & (4j-4) \end{Bmatrix} = \Sigma_\beta(-1)^{2\beta}(2\beta+1) \times \begin{Bmatrix} j & j & (2j-1) \\ (2j-3) & (4j-4) & \beta \end{Bmatrix} \begin{Bmatrix} j & j & (2j-3) \\ j & \beta & (2j-1) \end{Bmatrix} \times \begin{Bmatrix} (2j-1) & (2j-1) & (4j-4) \\ \beta & j & j \end{Bmatrix}. \tag{9}$$

The parameter β is constrained by triangle relations in each of the $6j$ symbols. In particular first $6j$ symbol constrains β as follows:

$$\beta \geqslant (3j-4), \tag{10}$$

$$\beta \leqslant (3j-3). \tag{11}$$

From these constraints $\beta=(3j-3)$ or $(3j-4)$ or the first $6j$ symbol is zero. If $\beta=(3j-4)$ the second $6j$ symbol becomes the one previously discussed above in Eq. (6) and was there shown to be zero. This leaves $\beta=(3j-3)$.

In this case the last $6j$ symbol becomes

$$\begin{Bmatrix} (2j-1) & (2j-1) & (4j-4) \\ (3j-3) & j & j \end{Bmatrix}$$

which we now show vanishes.

We will use the Regge symmetry [18]

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} = \begin{Bmatrix} A & B & C \\ D & E & F \end{Bmatrix},$$

$$A=j_1, \quad B=\frac{1}{2}(j_2+j_3-l_2+l_3),$$

$$C=\frac{1}{2}(j_2+j_3+l_2-l_3), \quad D=l_1,$$

$$E=\frac{1}{2}(-j_2+j_3+l_2+l_3), \quad F=\frac{1}{2}(j_2-j_3+l_2+l_3),$$

so that we can now write

$$\begin{Bmatrix} (2j-1) & (2j-1) & (4j-4) \\ (3j-3) & j & j \end{Bmatrix} = \begin{Bmatrix} (2j-1) & \left(3j-\frac{5}{2}\right) & \left(3j-\frac{5}{2}\right) \\ (3j-3) & \left(2j-\frac{3}{2}\right) & \frac{3}{2} \end{Bmatrix} = \begin{Bmatrix} \left(3j-\frac{5}{2}\right) & \left(3j-\frac{5}{2}\right) & (2j-1) \\ \frac{3}{2} & \left(2j-\frac{3}{2}\right) & (3j-3) \end{Bmatrix}. \tag{12}$$

The results of $6j$ symbols with a "$\frac{3}{2}$" are found in Varshalovich, Moskalev, and Khersonski [21]

$$\begin{Bmatrix} a & b & c \\ \frac{3}{2} & e & f \end{Bmatrix}$$

for $e=c-\frac{1}{2}$ and $f=b-\frac{1}{2}$, as we have here, is proportional to

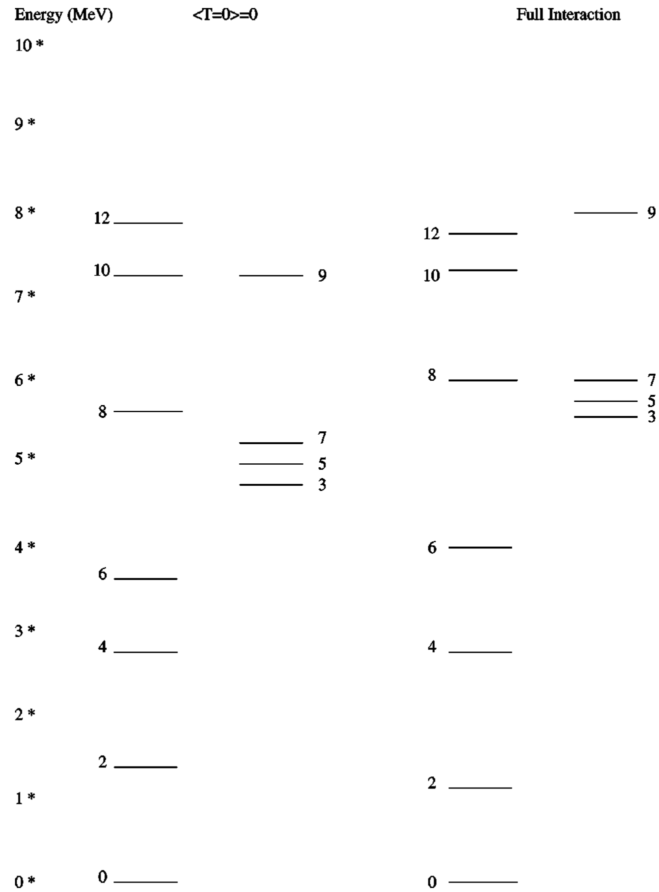

FIG. 2. Single-j $T=0$ ^{44}Ti with matrix elements from ^{42}Sc.

$$3[-a(a+1)+b(b+1)+c(c+1)]-2(b+1)(c+1), \tag{13}$$

which for $a=(3j-\frac{5}{2})$, $b=(3j-\frac{5}{2})$, and $c=(2j-1)$ is zero. Thus in the lone remaining case of $\beta=(3j\text{-}3)$ the final $6j$ symbol in the sum is zero. So for any allowed value of β one of the $6j$ symbols is zero implying that the $9j$ symbol above is zero.

V. FULL f-p CALCULATION FOR ^{43}Ti AND ^{44}Ti

We have performed full f-p calculations for ^{44}Ti and ^{43}Ti with the FPD6 interaction [22]. We shall show these and also compare the ^{44}Ti calculations with single-j results using the spectrum of ^{42}Sc as input. The latter is shown in Fig. 1.

We first discuss $T=0$ states in the even-even nucleus ^{44}Ti. In Table II and Fig. 2 we show the single-j results. The first two columns show the results when the $T=0$ two body matrix elements are set to zero, i.e., $\langle T=0\rangle=0$. In Fig. 2 we show the even I states of ^{44}Ti in the first column and the odd I in the second column. Note that the $I=9_1^+$ and $I=10_1^+$ states are degenerate as has been previously discussed.

In the last two columns we have the single-j shell results when the full spectrum of ^{42}Sc is introduced including the $T=0$ matrix elements. We note that there is much more change in the odd I spectrum than in the even I. The odd I spectrum raises considerably. The even I spectrum gets

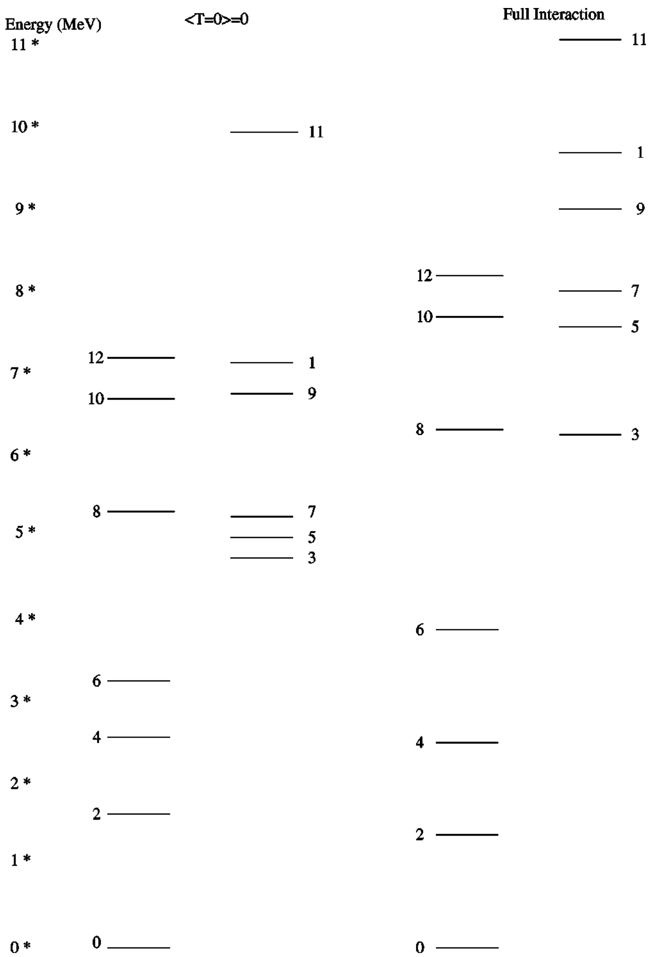

FIG. 3. Full f-p $T=0$ ^{44}Ti with FPD6 interaction.

spread out a bit but this is tame in comparison to the alteration in the odd I spectrum.

In Table IV and Fig. 3 we show results for a full f-p calculation using FPD6. We use the same format as for Table II. When the two body $T=0$ matrix elements are set equal to zero (first two columns), we find surprisingly that there is not much difference with the single-j shell calculation shown in Table II and Fig. 2. The $I=9^+$ and 10^+ state which were exactly degenerate in the single-j shell calculation are still nearly degenerate in the full f-p calculation. The overall spectra do not look very different (see first two columns in Tables II and IV and Figs. 2 and 3).

There is one difference however, the appearance in Table IV and Fig. 3 of $I=1^+$ and 11^+ $T=0$ states. In a single-j shell calculation the $I=1^+$ and 11^+ states all have isospin $T=1$.

We now come to the full f-p calculation in which all the two body matrix elements of the FPD6 interaction are in play— both $T=0$ and $T=1$. Now we see major differences for both the even I and odd I states of ^{44}Ti. (See Table IV and Fig. 3 right hand columns.)

If we look at the low spin states, $I=0^+$, 2^+, and 4^+ they are largely unaffected when the $T=0$ two body matrix elements are put back in. The main difference comes from the higher spin states. With the full FPD6 the spectrum of the even I gets spread out more looking somewhat rotational. For example the $I=10^+$ state increases in energy from 6.476 MeV to 7.790 MeV. In the corresponding single-j shell calculation there was hardly any change in the $I=10^+$ energy. Likewise the $I=12^+$ energy goes up from 7.192 MeV to 8.574 MeV when the $T=0$ two body matrix elements are put back into FPD6.

The odd I states experience a substantial upward shift in the spectrum. Now the $I=9^+$ state is considerably higher than the $I=10^+$ state (9.030 vs 7.790 MeV).

The full FPD6 fits, as it was designed to do, the even I levels quite well. The even I levels 2^+, 4^+, 6^+, 8^+, 10^+, and 12^+ are measured at 1.083 MeV, 2.454 MeV, 4.015 MeV, 6.509 MeV, 7.671 MeV, and 8.040 MeV, respectively. What is missing from the experimental picture is odd I positive parity information. It would be useful for the purpose of clarifying the importance of $T=0$ matrix elements to have more odd I, $T=0$ positive parity states.

In the single-j shell calculation with matrix elements from ^{42}Sc the even I columns corresponding to $\langle T=0\rangle=0$ and full spectrum (the first and third columns of energy levels) are not that different. It appears that the reintroduction of the $T=0$ two body matrix elements does not make much difference. In Fig. 3 however the third column, again even I, gets more spread out relative to the first column going a bit in the direction of giving a more rotational spectrum. Thus it would appear that for even I the $T=0$ two body matrix elements will affect the spectrum in a significant way only when configuration mixing is present.

We now consider the odd-even spectrum ^{43}Ti(^{43}Sc). The results are shown in Tables I and III and in Fig. 4. In the figure we only show a full calculation with FPD6 and compare results when the $T=0$ two body matrix elements are set equal to zero (first column) with those where the full FPD6 interaction is included (second column).

The results at first look a bit complicated but a careful examination shows systematic behavior.

For I less than $\frac{7}{2}^-$ the states come down in energy (relative to the $I=\frac{7}{2}^-$ ground state). For I greater than $\frac{7}{2}^-$ there is another systematic. When the $T=0$ two body matrix elements are set to zero there are three nearly degenerate doublets $(\frac{9}{2}^-,\frac{11}{2}^-)$ $(\frac{13}{2}^-,\frac{15}{2}^-)$ and $(\frac{17}{2}^-,\frac{19}{2}^-)$. The effect of putting $T=0$ two body matrix elements back in is to cause the lower spin member of each doublet to rise in energy by a substantial amount, while the higher spin member lowers itself a small amount, i.e., $I=\frac{9}{2}^-$, $\frac{13}{2}^-$, and $\frac{17}{2}^-$, rise noticeably but $I=\frac{11}{2}^-$, $\frac{15}{2}^-$, and $\frac{19}{2}^-$ drop slightly. The difference in energy with and without the two body $T=0$ matrix elements is shown in Fig. 5. This spectral staggering should be good evidence of the importance of $T=0$ two body matrix elements. The results with FPD6 for the low spin states are not so good. Experientially the excitation energies of the $I=\frac{3}{2}^-$ and $\frac{5}{2}^-$ are 472 and 845 keV while the calculated ones are 0.871 and 2.305 MeV. This is no fault of the interaction and is expected due to the presence of intruder states in the lower part of the f-p shell. Further evidence for this comes from the fact that in the cross conjugate nucleus ^{53}Fe the $I=\frac{3}{2}^-$ and $\frac{5}{2}$ excitation energies are higher than in ^{43}Sc. They

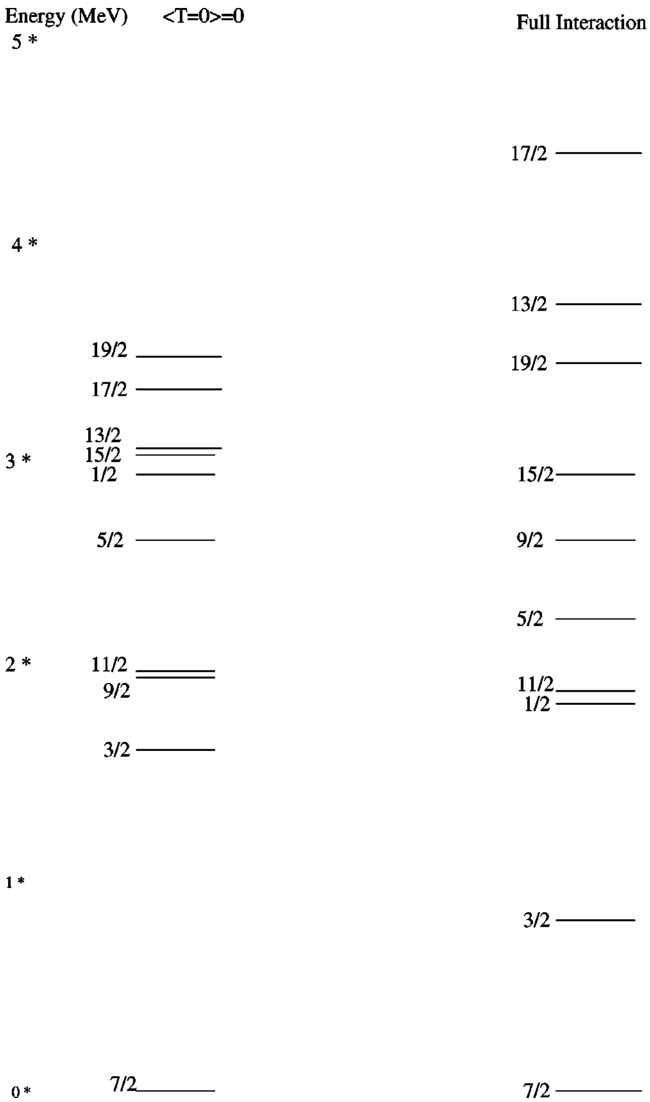

FIG. 4. Full f-p ^{43}Ti (^{43}Sc) with FPD6 interaction.

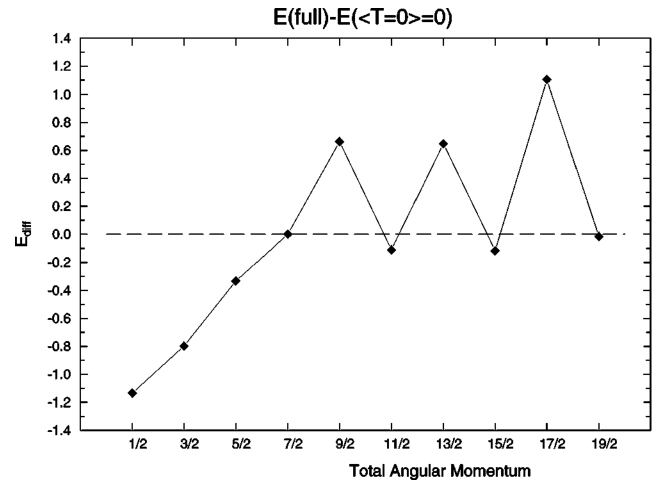

FIG. 5. E (full)-$E\langle T=0\rangle=0$ (MeV) vs total angular momentum ($\hbar$) in ^{43}Ti.

are 741 and 1433 keV, respectively. Indeed the interaction was built with the assumption that the lowest $I=\frac{3}{2}^-$ and $\frac{5}{2}^-$ states in ^{43}Sc are intruder influenced $(5p2h)$. The full FPD6 interaction fits the higher levels $I=\frac{11}{2}^-$ and $\frac{13}{2}^-$ (experimentally at 1830 keV and 2987 keV, respectively) quite well.

Work on the effect of $L=0$, $T=1$ and $L=1$, $T=0$ pairing in the f-p shell has already been performed by Poves and Martinez-Pinedo [23]. They start with a realistic interaction, KB3, and study the effects of removing the $T=1$ pairing from the $T=0$, $S=1$ pairing. They focused on binding energies and on the even spin states of ^{48}Cr. Relative to their work, whose conclusions we certainly agree with, we have made a more severe approximation of setting all $T=0$ matrix elements equal to zero. The payoff for us is that certain degeneracies appear between states, the deviation of which in the physical spectrum can largely be attributed to $T=0$ two body matrix elements. Also, we focused on odd I excited states. The deviation in the physical spectrum of the energies of odd I states from even I is also a good indication of the effects of $T=0$ matrix elements. We hope our work will provide stimulation and motivation for the study of odd spin even parity states.

Let us end by addressing the question of why the $T=1$ two body matrix elements are more important than the $T=0$ ones for the spectra of ^{43,44}Ti. First of all it should be noted that we are not considering binding energies. Their effects have been subtracted out by setting the ground states to be at zero energy. The $T=0$ two body matrix elements *are* important for binding energies. Once however we limit ourselves to the spectra, Fig. 1 (the spectrum of ^{42}Sc from which the empirical two body matrix reaction is deduced for a single-j shell calculation) provides us a partial answer to our query. Note that the spread of the $T=1$ states is much greater than that of the $T=0$ states. The energy difference of the highest energy $T=0$ state and the lowest one is 0.9 MeV. For $T=1$ the corresponding difference is 3.24 MeV. This greater spread makes the $T=1$ matrix elements much more important for setting up the general framework for the spectra.

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