Selective bosonization of the many-fermion problem in a model framework

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The higher-order terms of a novel boson-fermion expansion formalism previously introduced by one of the authors are discussed within the framework of the soluble SU(2) LMG model. A new formulation is provided to rectify certain undesirable features of the original expansion. In particular, the subsidiary condition that defines the physical subspace is obtained in a much simpler form. With the new formulation, the infinite expansion is summed in closed form. A comparison is made with earlier extended Holstein-Primakoff boson-fermion expansions.

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I. INTRODUCTION

A long-standing problem of conventional shell-model calculations, especially in heavy nuclei, is the difficulty of describing collective excitations because of the astronomically large basis required. The boson-mapping approach has been proposed as a possible solution [1]. Taking a cue from the physical idea that collective excitations are coherent superpositions of fermion pairs that behave approximately like bosons, the boson-expansion approach employs exact mappings from the many-fermion to a many-boson Hilbert space in such a way as to preserve the Pauli principle. In the typical treatment of an even-even nucleus, the fermion-pair operators, which generate a Lie algebra, are mapped onto functions of boson creation and destruction operators in an algebra-preserving way. Correspondingly, the many-fermion Fock space is injectively mapped onto a subspace of a many-boson Fock space called the physical subspace. This procedure formally replicates the fermion system, including the full effects of the Pauli principle, within the confines of the physical subspace, whose orthogonal complement, the unphysical subspace, is entirely irrelevant to the many-fermion problem. The bosonization of the nuclear many-body problem affords a number of advantages (see Ref. [1]), the chief one being that anharmonic collective states become elementary boson excitations, which can be automatically incorporated in a boson shell-model of reasonable dimensionality. A price to be paid, however, is that noncollective excitations also become boson states. While this is nonproblematic from a formal mathematical viewpoint (even a noninteracting-fermion Hamiltonian can be expressed exactly in terms of bosons), it is possible, as argued below, that the coupling between the many noncollective and the few collective modes slows down the convergence in computations of the latter. Other problems also occur in realistic applications of boson expansions. In some treatments, the fermion space is drastically truncated to operators generating an approximate closed subalgebra prior to bosonization. The validity of such approximations is difficult to gauge. Another problem is the difficulty of distinguishing physical and unphysical eigenvectors, which may become entangled because of truncations.

For these reasons, it is important to study methods that bosonize only the collective degrees of freedom while allowing the noncollective ones to retain their fermion nature. We refer to such methods as *selective bosonization*, in contrast to the traditional *pan-bosonization*. Selective bosonization offers the possibility of a shell model capable of fully describing collective as well as noncollective states. In this paper, we explore such a formalism within the framework of a simple soluble SU(2) model, namely, the model of Lipkin, Meshkov, and Glick (LMG) [2], although it can be applied just as well to other SU(2) models. The springboard for the formalism is the boson-fermion expansion theory (BFET) developed by Miyanishi and colleagues [3–5]. The formalism presented here significantly improves upon the original BFET in the handling of higher-order corrections and the subsidiary constraints that define the physical subspace. If one regards the BFET as version 1.0, then this is version 2.0, which is sufficiently different to merit a renaming—the *selective unitary bosonization* method, with associated acronyms SUB or SUBM. Here, *unitary* refers to the use of unitary transformations to effect the mappings. The BFET and the SUBM are equivalent in the sense that they generate the same $1/N$ expansions of physical quantities, but the SUBM is much more convenient in practice, especially because of the simplification of the subsidiary conditions.

Before proceeding to the new developments, it is worthwhile to point out that boson-fermion expansions have some early precedents. For example, in 1965, Yamamura [6] mapped the fermion Fock space into a tensor product of the fermion space with a boson space in order to describe the degrees of freedom of an odd nucleon added to an even-even system. Marshalek [7,8], with the same purpose in mind, introduced a *boson-quasifermion* mapping. The quasifermions, which describe the states of an odd nucleon, obey anticommutation rules that deviate somewhat from ordinary fermion anticommutation rules. The deviation is required to satisfy the underlying Lie algebra exactly without introducing redundant degrees of freedom. In both of these examples, the paired fermion degrees of freedom are completely bosonized. A better example of selective bosonization is provided by a formalism, developed independently by Marshalek [9,10] on the one hand and by Geyer and Hahne [11] on the other, in which valence particles and holes added to a closed shell are treated as quasifermions while particle-hole excitations of the closed shell itself are treated as bosons. A special case of this is the extended Holstein-Primakoff (EHP) mapping of the LMG model in Ref. [9], which is mathematically isomorphic to the EHP mapping of Suzuki and Matsuyanagi [12] that had been applied earlier to the degenerate pairing model. The close relationship of this earlier work with the present work is discussed later. Another closely related work is that of Kuriyama *et al.* [13], who also use auxiliary variables to map collective coordinates.

Except for the early effort of Yamamura, all of the above examples correspond to boson-quasifermion mappings, in contrast to the SUBM discussed here, which is a true boson-fermion mapping. It is argued later that quasifermions are required to satisfy the underlying Lie algebra only when represented on Hilbert spaces without redundant degrees of freedom, whereas representations on spaces with redundant degrees of freedom permit the luxury of true fermions. The price for this, the need for constraints, turns out to be very minor.

This paper is organized as follows. In Sec. II, the general concepts of the BFET are reviewed, the LMG model is reviewed, and the BFET is applied to the LMG model. In this context, the strengths and weaknesses of the BFET are made transparent. In Sec. III, the SUBM is introduced and formulated as an expansion theory in the context of the LMG model. In Sec. IV, closed forms, representing the summation of the expansions, are derived for the elementary operators of the LMG model. These depend critically on the concept of the *principal subspace*, defined in Sec. IV.

II. THE BFET AND THE LMG MODEL

We begin by reviewing the main ideas of the original BFET [3–5], which will then be applied to the LMG model.

A. Review of the general BFET

Given a fermion Fock space generated by the fermion creation and annihilation operators $c_k^\dagger$ and c_k, respectively, one may define a complete set of pair operators $X_\mu^\dagger$, X_μ, B_μ in the form

$$X_\mu^\dagger = \frac{1}{2}\sum_{kl} \Psi_\mu(kl) c_k^\dagger c_l^\dagger, \quad X_\mu = (X_\mu^\dagger)^\dagger,$$

$$B_\mu = \sum_{kl} \Phi_\mu(kl) c_k^\dagger c_l. \tag{2.1}$$

The pair-excitation operators $X_\mu^\dagger$ are commonly called *phonon* creation operators, in particular when the coefficients $\Psi_\mu(kl) = -\Psi_\mu(lk)$ are obtained from a Tamm-Dancoff (TD) [14] calculation, while the B_μ are called *scattering* operators, with the coefficients $\Phi_\mu(kl)$ scaled so that all generators have the same order of magnitude. The fermion Fock space can correspond to particles or quasiparticles, including particle-holes as a special case. As is well known, the operators of Eq. (2.1) generate a closed Lie algebra under commutation that is equivalent to the algebra of the group SO$(2n)$.

In particular, consider the commutator of phonon destruction and creation operators, which takes the form

$$[X_\mu, X_\nu^\dagger] = \delta_{\mu\nu} + \sum_\lambda \Gamma_\lambda(\mu\nu) B_\lambda. \tag{2.2}$$

The sum on the right, which represents the deviation of the phonons from ideal boson behavior, is, of course, a manifestation of the Pauli principle. In the traditional pan-bosonized mappings for even-even systems, all the generators are mapped onto polynomial functions of perfect bosons so as to satisfy the Lie algebra. Infinite boson expansions, in which each phonon $X_\mu^\dagger$ is replaced by a corresponding boson $b_\mu^\dagger$ in leading order, are predicated on the assumption that the coefficients $\Gamma_\lambda(\mu\nu)$ are all small. In particular, one attempts to identify an expansion parameter $\varepsilon \equiv 1/\sqrt{\Omega}$, the exact definition of which depends on the system. For example, if the generators in Eq. (2.1) all carry good angular momentum then one can make the identification $\Omega = 2j+1$, where j is an average value for the j-subshells of the system [15]. From the Racah and Clebsch-Gordan coefficients, it then follows that the coefficients $\Gamma_\lambda(\mu\nu) \sim O(\varepsilon^2)$ if the fermion-pair operators are all scaled as $O(1)$. In reality, however, the presence of a formal small parameter is insufficient to guarantee the smallness of *all* the coefficients $\Gamma_\lambda(\mu\nu)$, unless all the phonons are collective, which is never the case. In general, then, the deviation from boson behavior in Eq. (2.2) is small only for the subset of phonons that are truly collective. For an ideally collective phonon, the coefficients $\Psi_\mu(kl)$ would all be of the same magnitude, so that for a superposition of Ω pairs one would have $|\Psi_\mu(kl)| = 1/\sqrt{\Omega} = \varepsilon$ and $[X_\mu, X_\mu^\dagger] = 1 + O(\varepsilon^2)$. These conditions are well approximated to the extent that the phonon approaches the ideal collective limit.

The philosophy of the BFET is to employ expansions in which in leading order only collective phonons are bosonized while noncollective excitations remain fermionic. The underlying Hilbert space $\mathcal{H}$ is taken to be the tensor product of the Fock space of collective bosons $\mathcal{H}_\mathrm{B}$ with the many-fermion Fock space $\mathcal{H}_\mathrm{F}$, or $\mathcal{H} = \mathcal{H}_\mathrm{B} \otimes \mathcal{H}_\mathrm{F}$. Thus, the BFET is a formalism employing redundant degrees of freedom, which, of course, requires subsidiary conditions. Since the physics of the many-fermion problem originally resides in the subspace $\mathcal{H}_\mathrm{F}$, the physical states of $\mathcal{H}$, denoted generically by $|\mathrm{phys}\rangle$, satisfy the set of conditions $b_c|\mathrm{phys}\rangle = 0$, where b_c is any collective boson annihilation operator. The next step in the BFET is the introduction of a unitary transformation on $\mathcal{H}$ given by

$$U_0(\theta) = \exp(-\theta S_0), \tag{2.3}$$

where

$$S_0 \equiv \sum_c (X_c^\dagger b_c - b_c^\dagger X_c), \tag{2.4}$$

the sum running over all collective operators, denoted by the index c. Any operator $\mathcal{O}$ defined on $\mathcal{H}$ can be transformed as $\mathcal{O} \to \mathcal{O}(\theta) \equiv U_0(\theta) \mathcal{O} U_0^\dagger(\theta)$ and expanded in powers of a small parameter ε whose scale is set by the coefficients

$\Gamma_\lambda(\mu\nu)$ in Eq. (2.2) applied to collective phonons only. In particular, in lowest order the transforms of the collective phonon and boson annihilation operators are given by

$$X_c(\theta)=X_c\cos\theta+b_c\sin\theta+O(\varepsilon),$$
$$b_c(\theta)=-X_c\sin\theta+b_c\cos\theta+O(\varepsilon). \tag{2.5}$$

It then clear that for the choice $\theta=\pi/2$ the collective phonons and bosons essentially exchange roles in lowest order, i.e.,

$$X_c(\pi/2)=b_c+O(\varepsilon),\quad b_c(\pi/2)=-X_c+O(\varepsilon), \tag{2.6}$$

while noncollective phonon operators remain totally unchanged to all orders. The higher orders in Eq. (2.6) involve both boson and fermion operators.

The next essential step is the transformation of operators $\mathcal{O}\rightarrow\overline{\mathcal{O}}\equiv\mathcal{O}(\pi/2)=U_0(\pi/2)\mathcal{O}U_0^\dagger(\pi/2)$, which are obtained as a power series in ε. Since in general the transforms couple boson and fermion states, $U_0(\pi/2)$ maps the fermion subspace $\mathcal{H}_{\rm F}$ onto a certain subspace of the boson-fermion space, which is the physical subspace. The subsidiary condition defining the physical states $\overline{|\mathrm{phys}\rangle}$ after the transformation is just given by $\overline{b_c}\overline{|\mathrm{phys}\rangle}=0$. In the original BFET, this condition is a complicated one since $\overline{b_c}$ mixes bosons and fermions.

In summary, the BFET, by bosonizing only collective degrees of freedom, provides a true expansion in a small parameter, which may converge faster than conventional panbosonized expansions. The BFET also provides a clear criterion for distinguishing the physical subspace, although the condition is rather complicated. In order to clearly bring out the advantages and shortcomings of the BFET, we employ the LMG model in the remainder of the paper. This model also points the way to overcome the shortcomings, which is the main topic.

B. The LMG model

The simplest exactly soluble models involve Hamiltonians constructed from SU(2) quasispin operators. This includes that *sine qua non* of nuclear physics—the LMG model, which we adopt here as a paradigm. However, the boson-fermion mappings of the generators can also be applied to other SU(2) models such as the single-shell pairing model. The LMG model describes particles distributed among two single-particle levels of equal degeneracy split by an amount ϵ_0. Relative to the closed-shell system with Ω particles occupying the lower level, one may define creation and destruction operators for particles in the upper level, $\alpha_m^\dagger$, α_m, and for holes in the lower level $\beta_m^\dagger$, β_m, $m=1,\ldots,\Omega$. These operators obey the standard fermion anticommutation rules

$$\{\alpha_m,\alpha_{m'}^\dagger\}=\delta_{mm'},\quad \{\beta_m,\beta_{m'}^\dagger\}=\delta_{mm'},$$
$$\{\alpha_m,\alpha_{m'}\}=0,\quad \{\beta_m,\beta_{m'}\}=0,\quad \text{and H.c. eqs.},$$
$$\{\alpha_m,\beta_{m'}^\dagger\}=0,\quad \{\alpha_m,\beta_{m'}\}=0,\quad \text{and H.c. eqs.} \tag{2.7}$$

The SU(2) generators $J_\pm$ and J_0 obeying the commutation rules

$$[J_+,J_-]=2J_0,\quad [J_0,J_\pm]=\pm J_\pm \tag{2.8}$$

are then defined by

$$J_+=\sum_{m=1}^{\Omega}\alpha_m^\dagger\beta_m^\dagger,\quad J_-=(J_+)^\dagger,$$
$$J_0=-\frac{1}{2}\Omega+\frac{1}{2}\sum_{m=1}^{\Omega}(\alpha_m^\dagger\alpha_m+\beta_m^\dagger\beta_m). \tag{2.9}$$

The general Hamiltonian for the LMG model is given by

$$H_{\rm LMG}=\epsilon_0J_0+\frac{1}{2}V_0J_+J_-+\frac{1}{2}V_1(J_+^2+J_-^2), \tag{2.10}$$

with coupling constants V_0 and V_1. We note in passing that if $\epsilon_0=V_1=0$, then this Hamiltonian reduces to that of the degenerate pairing model, providing that $V_0<0$ and $\alpha_m^\dagger$and $\beta_m^\dagger$ are reinterpreted as creation operators for time-reversal conjugate orbitals.

For later reference, we also briefly discuss the nature of the eigenvectors of $H_{\rm LMG}$. Since this Hamiltonian lies in the enveloping algebra of SU(2), the eigenvectors are, of course, labeled by the total quasispin quantum number J. These eigenvectors are linear combinations of the basis vectors for an irreducible representation given by $|\gamma;J,M\rangle$, where M is the eigenvalue of J_0, and γ is a generic marker for additional quantum numbers defining the many-body configuration. Most discussions of the LMG model focus on the so-called *collective subspace* of the closed-shell system with Ω particles, which has $J=\frac{1}{2}\Omega$, spanned by the normalized vectors

$$\left|0;\frac{\Omega}{2},-\frac{\Omega}{2}+\nu\right\rangle=\left[\frac{(\Omega-\nu)!}{\Omega!\,\nu!}\right]^{1/2}J_+^\nu|0\rangle_{\rm F},\quad \nu=0,1,\ldots,\Omega, \tag{2.11}$$

where the fermion vacuum $|0\rangle_{\rm F}\equiv|0;\Omega/2,\text{-}\Omega/2\rangle$, which is the (uncorrelated) closed-shell ground state, satisfying $\alpha_m|0\rangle_{\rm F}=0$ and $\beta_m|0\rangle_{\rm F}=0$. However, our treatment is not limited to the collective subspace, but, in principle, encompasses all the eigenstates of $H_{\rm LMG}$. The basis vectors for an SU(2) irrep for such states are obtained by repeated application of the operator J_+ to a ‘‘base state’’ with a total of n particles plus holes and have $J=(\Omega-n)/2$. For example, for the Ω-particle system, one has the following SU(2) multiplets with $J=\frac{1}{2}\Omega-1$ built on a particle-hole excitation:

$$\left|m_1m_2^{-1};\frac{1}{2}\Omega-1,\text{-}\frac{1}{2}\Omega+1+\nu\right\rangle=\left[\frac{(\Omega-2-\nu)!}{(\Omega-2)!\,\nu!}\right]^{1/2}J_+^\nu\alpha_{m_1}^\dagger\beta_{m_2}^\dagger|0\rangle_{\rm F},$$
$$\nu=0,\ldots,\Omega-2,\ m_1\neq m_2. \tag{2.12}$$

In addition to the closed-shell system, one can also describe systems with one or more particles or holes added to the closed shell.

C. The BFET applied to the LMG model

In order to assess the advantages and disadvantages of the original BFET, we apply it now to the Lipkin model. In this model, the smallness parameter is given by $\varepsilon=(1/\Omega)^{1/2}$ and the natural choice for the phonon creation operator is one proportional to J_+. If one defines the scaled operators $X^\dagger$, X and B by

$$X^\dagger\equiv\varepsilon J_+,\quad X\equiv\varepsilon J_-,\quad B\equiv 2\left(\frac{1}{2\varepsilon^2}+J_0\right),\tag{2.13}$$

then the SU(2) commutation rules of Eq. (2.8) take the form

$$[X,X^\dagger]=1-\varepsilon^2 B,\quad [X,B]=2X\quad\text{and H.c. eq..}\tag{2.14}$$

It is then obvious that in the limit $\varepsilon\to 0$ $(\Omega\to\infty)$, X and $X^\dagger$ become boson operators, which, in the BFET description, are preempted in lowest order by the collective boson operators b and $b^\dagger$, respectively. It is also important to note that the scaling of the SU(2) generators in Eq. (2.13) is chosen to guarantee that the three operators $X^\dagger$, X and B are all of the same order of magnitude, i.e., $O(1)$. For later reference, we also note that

$$B=\hat{n}\equiv\sum_{m=1}^{\Omega}(\alpha_m^\dagger\alpha_m+\beta_m^\dagger\beta_m),\tag{2.15}$$

i.e., B is operator for the total number $\hat{n}$ of particles and holes.

In line with the discussion in Sec. II A, the Lipkin model is assumed to be defined on the extended Hilbert space $\mathcal{H}=\mathcal{H}_{\mathrm{B}}\otimes\mathcal{H}_{\mathrm{F}}$, where $\mathcal{H}_{\mathrm{B}}$ is the boson Hilbert space generated by the familiar oscillator vectors $|n\rangle_{\mathrm{B}}=(n!)^{-1/2}(b^\dagger)^n|0\rangle_{\mathrm{B}}$, and $\mathcal{H}_{\mathrm{F}}$ is the fermion Hilbert space of the LMG model. The operators b, $b^\dagger$ obey the usual boson commutation rule $[b,b^\dagger]=1$. It will be implicitly assumed forthwith that all of the LMG operators are extended to the full Hilbert space $\mathcal{H}$; thus the Hamiltonian is to be understood as $H_{\mathrm{LMG}}\otimes I_{\mathrm{B}}$ and any purely bosonic function $f(b^\dagger,b)$ as $I_{\mathrm{F}}\otimes f(b^\dagger,b)$, where I_{B} and I_F are the respective identity operators to the purely bosonic and fermionic subspaces. It is also convenient to introduce the common boson-fermion vacuum $|0\rangle\equiv|0\rangle_{\mathrm{B}}\otimes|0\rangle_{\mathrm{F}}$ ($|0\rangle_{\mathrm{F}}$ being the closed-shell configuration), which satisfies $X|0\rangle=0$ and $b|0\rangle=0$. Prior to any transformation, the physical subspace of $\mathcal{H}$ is, of course, the fermion subspace $\mathcal{H}_{\mathrm{F}}$ itself, and the physical vectors $|\mathrm{phys}\rangle$ are the fermion vectors, which must satisfy the condition $b|\mathrm{phys}\rangle=0$.

Given a θ-dependent unitary operator $U(\theta)=\exp(-\theta S)$, the transform of an arbitrary operator $\mathcal{O}$ on $\mathcal{H}$ will be written as $\mathcal{O}(\theta)\equiv U(\theta)\mathcal{O}U^\dagger(\theta)$, which must obey a differential equation of the Heisenberg type; specifically,

$$\frac{d\mathcal{O}(\theta)}{d\theta}=[\mathcal{O}(\theta),S(\theta)]=[\mathcal{O}(\theta),S]\tag{2.16}$$

[note: $S(\theta)=S$] subject to the ''initial'' condition $\mathcal{O}(0)=\mathcal{O}$. This equation will play an essential role in the new developments of the next section, but for now we apply it to the BFET case when $S=S_0$, which, in analogy with Eq. (2.4), is given by

$$S_0\equiv X^\dagger b-b^\dagger X.\tag{2.17}$$

The Heisenberg equation for $b(\theta)$ and $X(\theta)$ is then readily evaluated with the aid of the commutators (2.14), with the result

$$\frac{db(\theta)}{d\theta}=-X(\theta),$$

$$\frac{dX(\theta)}{d\theta}=b(\theta)-\varepsilon^2 B(\theta)b(\theta),\quad\text{and H.c. eqs.}\tag{2.18}$$

Now, the function $B(\theta)$ on the right can be evaluated in one of two ways. First, integration of the differential, Eq. (2.16), with $\mathcal{O}=B$ leads to

$$B(\theta)=B+2\int_0^\theta[X^\dagger(\theta')b(\theta')+b^\dagger(\theta')X(\theta')]d\theta',\tag{2.19}$$

which makes Eqs. (2.18) integro-differential equations. However, there is a simpler method to evaluate $B(\theta)$ following from the observation that

$$[B+2b^\dagger b,S_0]=0.\tag{2.20}$$

That is, the quantity $B+2b^\dagger b$ is invariant under the transformation $U(\theta)$. This immediately leads to the simpler result

$$B(\theta)=B+2[b^\dagger b-b^\dagger(\theta)b(\theta)].\tag{2.21}$$

In the limit $\varepsilon\to 0$, Eqs. (2.18) have the harmonic solution $b(\theta)=b_0(\theta)$, $X(\theta)=X_0(\theta)$, where

$$b_0(\theta)=b\cos\theta-X\sin\theta,\quad X_0(\theta)=b\sin\theta+X\cos\theta.\tag{2.22}$$

The differential, Eqs. (2.18), can then be solved perturbatively as a power series in ε^2 as follows:

$$b(\theta)=b_0(\theta)+\sum_{n=1}^{\infty}\varepsilon^{2n}b_n(\theta),$$

$$X(\theta)=X_0(\theta)+\sum_{n=1}^{\infty}\varepsilon^{2n}X_n(\theta),$$

$$B(\theta)=\sum_{n=0}^{\infty}\varepsilon^{2n}B_n(\theta),\tag{2.23}$$

the expansion of $B(\theta)$ being induced by the expansions of $b(\theta)$ and $X(\theta)$ through Eq. (2.19) or (2.21). In practice, it is convenient to employ the following second-order differential equation derived from Eqs. (2.18):

$$\frac{d^2 b(\theta)}{d\theta^2}+b(\theta)=\varepsilon^2 B(\theta)b(\theta)\equiv\varepsilon^2 R(\theta), \tag{2.24}$$

with $B(\theta)$ given by Eq. (2.21). Separation of orders then leads to the following driven-oscillator equations for $b_n(\theta)$:

$$\frac{d^2 b_n(\theta)}{d\theta^2}+b_n(\theta)=R_{n-1}(\theta),\quad n=1,2,\ldots,\infty, \tag{2.25}$$

where

$$R_n(\theta)=\sum_{m=0}^{n} B_m(\theta)b_{n-m}(\theta). \tag{2.26}$$

It should be noted that the right-hand side (rhs) of Eq. (2.25) depends only on previously determined lower orders. Thus, starting with the zeroth-order solution (2.22), Eq. (2.25) can be integrated in each successive order using the well-known solution (for example, see Ref. [16]):

$$\begin{aligned} b_n(\theta)&=\mathrm{Im}\left[\exp(i\theta)\int_0^\theta d\theta'\exp(-i\theta')R_{n-1}(\theta')\right],\\ X_n(\theta)&=-db_n(\theta)/d\theta\\ &=-\mathrm{Re}\left[\exp(i\theta)\int_0^\theta d\theta'\exp(-i\theta')R_{n-1}(\theta')\right]. \end{aligned} \tag{2.27}$$

Through order ε^2, the explicit expressions for the quantities in Eq. (2.23) are as follows:

$$\begin{aligned} b(\theta)=&-X\sin\theta+b\cos\theta+\frac{1}{16}\varepsilon^2[(\cos\theta-\cos 3\theta-4\theta\sin\theta)(2X^\dagger Xb+b^\dagger X^2-b^\dagger b^2)\\ &+(9\sin\theta+\sin 3\theta-12\theta\cos\theta)X^\dagger X^2+(7\sin\theta-\sin 3\theta-4\theta\cos\theta)X^\dagger b^2\\ &-8(\sin\theta-\theta\cos\theta)BX+8\theta\sin\theta\;\;Bb-2(\sin\theta+\sin 3\theta-4\theta\cos\theta)b^\dagger Xb]+O(\varepsilon^4), \end{aligned} \tag{2.28}$$

$$\begin{aligned} X(\theta)=&X\cos\theta+b\sin\theta+\frac{1}{16}\varepsilon^2[(5\sin\theta-3\sin 3\theta+4\theta\cos\theta)(2X^\dagger Xb+b^\dagger X^2-b^\dagger b^2)\\ &+3(\cos\theta-\cos 3\theta-4\theta\sin\theta)X^\dagger X^2-(3\cos\theta-3\cos 3\theta+4\theta\sin\theta)X^\dagger b^2\\ &-8(\sin\theta+\theta\cos\theta)Bb+8\theta\sin\theta\;\;BX-2(3\cos\theta-3\cos 3\theta-4\theta\sin\theta)b^\dagger Xb]+O(\varepsilon^4), \end{aligned} \tag{2.29}$$

$$\begin{aligned} B(\theta)=&B+(1-\cos 2\theta)(b^\dagger b-X^\dagger X)+\sin 2\theta(X^\dagger b+b^\dagger X)\\ &+\varepsilon^2\bigg[(-1+\cos 2\theta+\theta\sin 2\theta)X^\dagger BX+\frac{1}{2}(\sin 2\theta-2\theta\cos 2\theta)(X^\dagger Bb+b^\dagger BX)-\theta\sin 2\theta b^\dagger Bb\\ &+\frac{1}{8}(1-\cos 4\theta-4\theta\sin 2\theta)b^{\dagger 2}b^2+\frac{1}{8}(9-8\cos 2\theta-\cos 4\theta-12\theta\sin 2\theta)X^{\dagger 2}X^2\\ &-\frac{1}{2}(1-\cos 4\theta-4\theta\sin 2\theta)X^\dagger b^\dagger bX-\frac{1}{4}(\sin 2\theta+\sin 4\theta-6\theta\cos 2\theta)(X^\dagger b^\dagger X^2+X^{\dagger 2}bX)\\ &-\frac{1}{4}(\sin 2\theta-\sin 4\theta+2\theta\cos 2\theta)(X^\dagger b^\dagger b^2+b^{\dagger 2}bX)+\frac{1}{8}(3-4\cos 2\theta+\cos 4\theta)(X^{\dagger 2}b^2+b^{\dagger 2}X^2)\bigg]+O(\varepsilon^4). \end{aligned} \tag{2.30}$$

The operators $b^\dagger(\theta)$ and $X^\dagger(\theta)$ are, of course, obtained by Hermitian conjugation of Eqs. (2.28) and (2.29), respectively [$B(\theta)$ is Hermitian]. It should be noted that these expressions do *not* correspond to a Fourier expansion because of the presence of aperiodic terms of the form $\theta^m\sin n\theta$ or $\theta^m\cos n\theta$. This becomes important when operators are evaluated for the physical value $\theta=\pi/2$. From now on, we use the "bar notation" in which the physical unitary transform of any operator $\mathcal{O}$ is denoted by $\overline{\mathcal{O}}$ as follows:

$$\overline{\mathcal{O}}\equiv\mathcal{O}(\pi/2)=U(\pi/2)\mathcal{O}U^\dagger(\pi/2). \tag{2.31}$$

Thus the operators of Eqs. (2.30) evaluated at $\theta=\pi/2$ are given by

$$\overline{b} = -X + \varepsilon^2\left[\frac{1}{2}(-BX + X^\dagger X^2 + X^\dagger b^2) + \frac{\pi}{8}(2Bb - b^\dagger X^2 - 2X^\dagger Xb + b^\dagger b^2)\right] + O(\varepsilon^4), \tag{2.32}$$

$$\overline{X} = b + \varepsilon^2\left[\frac{1}{2}(b^\dagger X^2 - b^\dagger b^2 - Bb + 2X^\dagger Xb) + \frac{\pi}{8}(2BX - X^\dagger b^2 - 3X^\dagger X^2 + 2b^\dagger bX)\right] + O(\varepsilon^4), \tag{2.33}$$

$$\overline{B} = B + 2b^\dagger b - 2X^\dagger X + \varepsilon^2\bigg(-2X^\dagger BX + 2X^{\dagger 2}X^2 + X^{\dagger 2}b^2 + b^{\dagger 2}X^2 + \frac{\pi}{2}(X^\dagger Bb + b^\dagger BX) + \frac{\pi}{4}(X^\dagger b^\dagger b^2 + b^{\dagger 2}bX) - \frac{3\pi}{4}(b^\dagger X^\dagger X^2 + X^{\dagger 2}Xb)\bigg) + O(\varepsilon^4). \tag{2.34}$$

The numerous terms in Eqs. (2.32)–(2.34) are required to satisfy the commutation relations

$$[\overline{X},\overline{X^\dagger}] = 1 - \varepsilon^2\overline{B}, [\overline{X},\overline{B}] = 2\overline{X} \quad \text{and H.c. eqs.},$$
$$[\overline{b},\overline{b^\dagger}] = 1, [\overline{b},\overline{X}] = [\overline{b},\overline{X^\dagger}] = [\overline{b},B] = 0 \quad \text{and H.c. eqs.} \tag{2.35}$$

through order ε^2. One observes that some terms are proportional to π. In higher orders, one finds powers of π which arise from the aperiodic terms $\theta^m \sin n\theta$ for odd values of n and $\theta^m \cos n\theta$ for even values of n. These "π terms" must then occur in all transformed operators, including the Hamiltonian, which at first sight may appear a little disturbing, as if π itself were a fundamental coupling constant. However, the physical basis vectors also have π terms that conspire with the transformed Hamiltonian to guarantee that the eigenvalues and other physical observables are independent of π. This must obviously be the case since a unitary transformation cannot change the eigenvalues.

If one denotes the physical vectors after the transformation by $\overline{|\mathrm{phys}\rangle} \equiv U(\pi/2)|\mathrm{phys}\rangle$, then the condition defining the physical subspace after the transformation is just

$$\overline{b}\overline{|\mathrm{phys}\rangle} = 0. \tag{2.36}$$

It is clear from Eq. (2.32) that this condition, which intertwines bosons and fermions and depends on ε^2, is actually rather complicated. However, we can proceed as follows.

Let P_{phys} be the projection operator to the transformed physical subspace, which is spanned by the states $\overline{|\mathrm{phys}\rangle}$. Let $\overline{\mathcal{O}}_{P_{\mathrm{phys}}}$ denote the projection of an arbitrary operator onto this subspace, i.e.,

$$\overline{\mathcal{O}}_{P_{\mathrm{phys}}} \equiv P_{\mathrm{phys}}\overline{\mathcal{O}}P_{\mathrm{phys}}. \tag{2.37}$$

From Eqs. (2.32) and (2.36), it is readily seen that

$$X\overline{|\mathrm{phys}\rangle} = \varepsilon^2\left(\frac{\pi}{8}b^\dagger b^2 + \frac{1}{2}X^\dagger b^2 + \frac{\pi}{4}Bb\right)\overline{|\mathrm{phys}\rangle} + O(\varepsilon^4). \tag{2.38}$$

Thus, the operators X annihilate all physical states through order ε, but not higher orders. From Eq. (2.38) and its H.c., it follows that in the projection of the operators $\overline{X}$ and $\overline{B}$ given above, all terms involving X and $X^\dagger$ may be dropped through order ε^2. Therefore,

$$\overline{X}_{P_{\mathrm{phys}}} = P_{\mathrm{phys}}\left[b - \frac{1}{2}\varepsilon^2(b^\dagger b^2 + Bb)\right]P_{\mathrm{phys}} + O(\varepsilon^4),$$
$$\overline{B}_{P_{\mathrm{phys}}} = P_{\mathrm{phys}}(B + 2b^\dagger b)P_{\mathrm{phys}} + O(\varepsilon^4). \tag{2.39}$$

We note that the "π terms" are automatically eliminated in the physical subspace through $O(\varepsilon^2)$. Explicitly including the next order gives the result

$$\overline{X}_{P_{\mathrm{phys}}} = P_{\mathrm{phys}}\Bigg[b - \frac{1}{2}\varepsilon^2(b^\dagger b^2 + Bb) - \frac{1}{8}\varepsilon^4\bigg\{\left(1 - \frac{\pi^2}{16}\right)b^{\dagger 2}b^3 - \frac{\pi^2}{8}b^\dagger b^2 + \left(1 - \frac{\pi^2}{4}\right)bB^2 + \left(2 - \frac{\pi^2}{4}\right)b^\dagger b^2 B\bigg\}\Bigg]P_{\mathrm{phys}} + O(\varepsilon^6),$$
$$\overline{B}_{P_{\mathrm{phys}}} = P_{\mathrm{phys}}(B + 2b^\dagger b)P_{\mathrm{phys}} + O(\varepsilon^6). \tag{2.40}$$

One observes that the "π terms" arise again in $O(\varepsilon^4)$ of $\overline{X}_{\mathrm{phys}}$, the boson part of which is obviously different from the corresponding expansion of the Holstein-Primakoff (HP) boson map [17] beyond $O(\varepsilon^2)$:

$$X_{\mathrm{HP}} = (1 - \varepsilon^2 b^\dagger b)^{1/2}b = b - \frac{1}{2}\varepsilon^2 b^\dagger b^2 - \frac{1}{8}\varepsilon^4(b^{\dagger 2}b^3 + b^\dagger b^2) + O(\varepsilon^6),$$
$$B_{\mathrm{HP}} = 2b^\dagger b. \tag{2.41}$$

Note that the boson part of $\overline{B}_{\mathrm{phys}}$, however, happens to coincide with B_{HP} .The discrepancy between $\overline{X}_{\mathrm{phys}}$ and X_{HP}, of course, does not mean Eq. (2.40) is incorrect. Instead, it should be attributed to the fact that the physical vectors in the BFET deviate from pure boson states in higher order. Indeed, the physical states in general are complicated superpositions of boson-fermion states.

Now, we can proceed as follows. Let $|n\rangle \equiv (n!)^{-1/2}b^{\dagger n}|0\rangle = |n\rangle_{\mathrm{B}} \otimes |0\rangle_{\mathrm{F}}$ be the orthonormal set ofpure boson states defined on $\mathcal{H}$. With the aid of Eq. (2.32), one then obtains

$$\overline{b}|n\rangle = \frac{1}{2}\varepsilon^2\left(X^\dagger b^2 + \frac{\pi}{4}b^\dagger b^2\right)|n\rangle + O(\varepsilon^4), \quad (2.42)$$

where the rhs expresses the deviation of the vector $|n\rangle$ from physicality. Although $|n\rangle$ is not a purely physical state, it can be employed to construct a corresponding physical state $\overline{|n\rangle}$ by means of a unitary transformation. Thus, $\overline{|n\rangle}$ is defined through order ε^2 by an infinitesimal unitary transformation as follows:

$$\overline{|n\rangle} \equiv \exp(\varepsilon^2 F)|n\rangle, \overline{b}\overline{|n\rangle} = O(\varepsilon^4), \quad (2.43)$$

where F is a suitable anti-Hermitian operator of $O(\varepsilon^0)$. From Eqs. (2.32) and (2.43), the explicit form of F is readily found to be

$$F = \frac{1}{4}(X^{\dagger 2}b^2 - b^{\dagger 2}X^2) + \frac{\pi}{8}(X^\dagger b^\dagger b^2 - b^{\dagger 2}bX). \quad (2.44)$$

The calculation of matrix elements between the physical states is straightforward with the help of the relations $X|n\rangle = B|n\rangle = 0$ and their Hermitian conjugates, yielding the results

$$\overline{\langle n'|}\overline{X}_{P_{\text{phys}}}\overline{|n\rangle} = \langle n'|b - \frac{1}{2}\varepsilon^2 b^\dagger b^2 - \frac{1}{8}\varepsilon^4(b^{\dagger 2}b^3 + b^\dagger b^2)|n\rangle + O(\varepsilon^6),$$

$$\overline{\langle n'|}\overline{B}_{P_{\text{phys}}}\overline{|n\rangle} = \langle n'|2b^\dagger b|n\rangle + O(\varepsilon^6), \quad (2.45)$$

which are identical to the HP matrix elements between pure boson states to the given order. This demonstrates for a certain subspace of states the possibility of going from the BFET to the HP expansion via a unitary transformation. We also note that the second of Eqs. (2.45) follows from the invariance implied by $[B + 2b^\dagger b, F] = 0$.

In summary, the BFET does provide a well-defined expansion in the small parameter ε, as long as the subsidiary condition defining the physical states is properly taken into account. However, there are some significant drawbacks. First, the constraint defining the physical subspace is given in the form of an expansion in ε^2, and the pure boson states are not automatically physical states except for the lowest order. According to the general argument given in Ref. [3], if the states are physical in $O(\varepsilon^n)$ then the matrix elements of any transformed fermion operator between these states are correctly given in $O(\varepsilon^{n+2})$. Therefore BFET gives the correct matrix elements for any transformed fermion operator automatically up to $O(\varepsilon^2)$. However if one goes on to higher order than $O(\varepsilon^2)$ it is necessary to manipulate both the operator and the subsidiary condition in expansion form. To avoid this complexity, we have to seek an expansion in which the physical states are ε independent. Another related drawback is the existence of the π terms in the expansion. Since these are canceled by the associated π terms in the physical basis vectors, one must deal with a surfeit of seemingly unnecessary terms. Since these π terms arise from the aperiodic coefficients, they would disappear if a purely periodic expansion could be found. The argument in the preceding paragraph points the way to overcoming both drawbacks. Namely, by using successive infinitesimal unitary transformation, one may attempt to generate a modified BFET transformation in which the physical states are ε independent and the π terms are absent. However, since a product of unitary transformations is equivalent to a single unitary transformation, it is reasonable to seek an extended unitary transformation generated by an operator of the form

$$S = S_0 + \Delta S, \quad (2.46)$$

where ΔS is $O(\varepsilon^2)$. Can ΔS be chosen to remove all the drawbacks of the original BFET? Furthermore, can one find an operator S that allows a closed summation of the expansions? As demonstrated in the next two sections, the answers to these rhetorical questions are, of course, both affirmative (otherwise, we would not have much of a paper).

III. THE SUBM: ε EXPANSIONS

In this section we present the solution to the problem discussed at the end of the previous section. This modification of the BFET will be called the *selective unitary bosonization method* or SUBM. The SUBM can be developed along two lines: either as an expansion formalism in powers of ε as was done for the BFET, or in terms of closed forms, which represent the summations of the expansions. In this section, we develop the expansion formalism. In applications to realistic cases, it is the expansions that are of most practical use while closed forms may not always be achievable. However, since in the SU(2) model it is possible to obtain exact closed forms, these are discussed in the subsequent section. The closed forms are important for assessing the global validity of the mappings on the Hilbert space in question and also to make contact with previous work on SU(2) models.

A. General formulation of the SUBM

The SUBM is formulated on the same boson-fermion Hilbert space as the BFET and the notations are all the same. It is not our intention to provide the most general solution to the ''π problem'' but rather the simplest solution. To this end, we introduce the unitary transformation

$$U(\theta) = \exp(-\theta S), \quad (3.1)$$

with the generator S having the form

$$S = Y^\dagger b - b^\dagger Y, \quad (3.2)$$

where Y and its Hermitian conjugate $Y^\dagger$ are assumed to be pure *fermionic* operators, which, of course, commute with the boson operators b and $b^\dagger$. Furthermore, it is assumed that in zeroth order Y and X coincide, i.e.,

$$Y = X + \Delta Y, \tag{3.3}$$

where ΔY is $O(\varepsilon)$. Therefore,

$$S = S_0 + \Delta S, \tag{3.4}$$

where S_0 is given by Eq. (2.17) while $\Delta S = (\Delta Y)^\dagger b - b^\dagger \Delta Y$ is $O(\varepsilon)$. Thus the BFET and the SUBM coincide in zeroth order.

We continue to use the notation $\mathcal{O}(\theta) \equiv U(\theta)\mathcal{O}U^\dagger(\theta)$ for the unitary transform of a generic operator $\mathcal{O}$. From Eq. (2.16) applied to the operators b and Y, one readily obtains the pair of differential equations

$$\frac{db(\theta)}{d\theta} = -Y(\theta),$$

$$\frac{dY(\theta)}{d\theta} = [Y(\theta), Y^\dagger(\theta)]b(\theta) \quad \text{(and H.c. eqs.)} \tag{3.5}$$

which also implies the second-order equation

$$\frac{d^2 b(\theta)}{d\theta^2} = -[Y(\theta), Y(\theta)^\dagger]b(\theta). \tag{3.6}$$

It is immediately obvious that $b(\theta)$ would be a simple harmonic function of θ if the commutator $[Y(\theta), Y(\theta)^\dagger]$ were a positive constant, which would also make $Y(\theta)$ a harmonic function. If one takes into account the zero-order requirements

$$\lim_{\varepsilon\to 0} b(\theta) = b_0(\theta) = b\cos\theta - X\sin\theta,$$

$$\lim_{\varepsilon\to 0} X(\theta) = X_0(\theta) = b\sin\theta + X\cos\theta, \tag{3.7}$$

as well as the initial values $b(0) = b$, $Y(0) = Y$, one finds in fact that the condition making these functions harmonic is $[Y(\theta), Y(\theta)^\dagger] = 1$, which is equivalent to the θ-independent condition

$$[Y, Y^\dagger] = 1. \tag{3.8}$$

But this means that Y and $Y^\dagger$ would have to be, respectively, boson destruction and creation operators.

Is it possible to construct boson operators on a many-fermion space? Since it is certainly possible to construct fermionic operators on a boson space, which is just the usual boson mapping, why not the reverse? There is just one general obstacle we are aware of, namely, the dimensionality of the Hilbert space. As is well known (see, for example, Ref. [18]), a pair of boson operators satisfying Eq. (3.8) can only exist on an infinite-dimensional space. The fermion space in the present model is clearly of finite dimension. A closer examination of the proof, which depends on a closure argument, reveals that it depends on the assumption that the boson operators are well defined over the entire space, which is then invariant under these operators.[1] However, the proof does not rule out the possibility that a pair of operators can obey the boson commutation rule on a *noninvariant subspace* of the many-fermion space but not obey the rule on the whole space. If the subspace includes the ground state and is sufficiently large, a physically meaningful theory can be developed. In the next subsection, it is shown that, in fact, there is no problem in finding fermion operators satisfying Eq. (3.8) as expansions in powers of ε. Subsequently, a global operator corresponding to this series is constructed, which clearly demarcates its domain. The operators Y and $Y^\dagger$ will be referred to as *fermionic boson* (FB) operators.

Before leaving this subsection, we note that with Eq. (3.8) taken into account, the solution of Eq. (3.5) satisfying the conditions (3.7) as well as the initial-value conditions is just

$$b(\theta) = b\cos\theta - Y\sin\theta,$$

$$Y(\theta) = b\sin\theta + Y\cos\theta, \quad \text{and H.c. eqs.} \tag{3.9}$$

B. Series expansions of pair operators in the SUBM

It is natural to seek expansions of Y and $Y^\dagger$ as polynomials in the generators X, $X^\dagger$, and B. In fact, one finds that the series can be written in the form

$$Y = \Gamma X, \quad Y^\dagger = X^\dagger \Gamma, \tag{3.10}$$

where Γ is the Hermitian operator given by the expansion

$$\Gamma \equiv 1 - \sum_{n=1}^{\infty} \sum_{m=0}^{n} \varepsilon^{2n} c(n,m) (X^\dagger)^m B^{n-m} X^m, \tag{3.11}$$

where the coefficients $c(n,m)$, which are all $O(1)$, are expanded in powers of ε^2 and determined by the requirement that the boson commutator (3.8) be fulfilled order by order in ε^2. In each order, this condition leads to a set of linear equations for the undetermined coefficients. We found it very convenient to carry out this procedure using the computer-algebra system REDUCE 3.6, which easily handles noncommuting operators. As an example, the solution for Y through $O(\varepsilon^6)$ is given by[2]

[1]An elementary proof by contradiction is provided by observing that on a finite-dimensional vector space, the trace of a commutator always vanishes, while the trace of the identity is the dimension of the vector space. We note that the proof given in Ref. [18] shows instead that boson operators are unbounded.

[2]We actually successfully obtained the solution through order ε^{14} in about 3 min on a 333 MHz Pentium II machine running Windows NT 4.0.

$$Y=X-\frac{1}{2}\varepsilon^2(X^\dagger X^2-BX)-\frac{1}{8}\varepsilon^4(9X^\dagger X^2-7X^{\dagger 2}X^3+10X^\dagger BX^2-3B^2X)$$
$$-\frac{1}{32}\varepsilon^6(70X^\dagger X^2-234X^{\dagger 2}X^3+66X^{\dagger 3}X^4+126X^\dagger BX^2-126X^{\dagger 2}BX^3+70X^\dagger B^2X^2-10B^3X)+O(\varepsilon^8). \quad (3.12)$$

This form is in normal order with respect to the fermion creation and destruction operators. However, one can also rewrite it so that Γ is a function of $X^\dagger X$ and B. To facilitate a later comparison it is convenient to write $Y=X\Gamma'$ with Γ' a function of $X^\dagger X$ and B. Then Eq. (3.12) corresponds to

$$Y=X\bigg[1-\frac{1}{2}\varepsilon^2(1+X^\dagger X-B)+\frac{1}{8}\varepsilon^4(3+10X^\dagger X+7(X^\dagger X)^2-6B+3B^2-10X^\dagger XB)-\frac{1}{32}\varepsilon^6(10+70X^\dagger X+126(X^\dagger X)^2$$
$$+66(X^\dagger X)^3-30B+30B^2-10B^3-140X^\dagger XB+70X^\dagger XB^2-126(X^\dagger X)^2B)\bigg]+O(\varepsilon^8). \quad (3.13)$$

Equations (3.12) and (3.13) are rearrangements representing one and the same operator. Therefore, while one might be tempted to impose the Pauli constraints $(X^\dagger)^{\Omega+1}=0$ and $(X)^{\Omega+1}=0$ on the expansion (3.12), this would be incorrect since it is impossible to impose these constraints on Eq. (3.13). The later discussion in Sec. IV A makes it clear that the Pauli principle arises from the cutoff on the fermion vector space rather than from constraints placed on the operators.

Having obtained the solution for Y and $Y^\dagger$, one is in a position to obtain the ε expansions of θ-dependent operators and, in particular, the values for $\theta=\pi/2$. We first consider the derivation of $B(\theta)$. One observes first that the operator Γ defined by Eq. (3.11) does not change the eigenvalue of J_0 or, equivalently, B. Therefore, it follows that $[\Gamma,B]=0$ and from Eqs. (2.14) and (3.10)

$$[Y,B]=2Y. \quad (3.14)$$

From Eq. (3.2) for S, it is then easily calculated that

$$[S,B+2b^\dagger b]=0, \quad (3.15)$$

or, in other words, the operator $B+2\varepsilon b^\dagger b$ is invariant under the general unitary transformation $U(\theta)$ just as in the BFET. Consequently, $B(\theta)$ is given by Eq. (2.21), which, together with the first of Eqs. (3.9), provides the following exact result:

$$B(\theta)=B+(1-\cos 2\theta)(b^\dagger b-Y^\dagger Y)+\sin 2\theta(Y^\dagger b+b^\dagger Y). \quad (3.16)$$

The ε expansion of $B(\theta)$ can then be obtained from the expansions of Y and $Y^\dagger$.

To obtain the expansion of $X(\theta)$ requires more work. From the unitary transformation of Eq. (3.3) written as $X(\theta)=Y(\theta)-\Delta Y(\theta)$, with $\Delta Y(\theta)=[\Gamma(\theta)-1]X(\theta)$, together with Eqs. (3.9) and (3.11), one obtains

$$X(\theta)=Y\cos\theta+b\sin\theta+\sum_{n=1}^{\infty}\sum_{m=0}^{n}\varepsilon^{2n}c(n,m)\times[X^\dagger(\theta)]^m B(\theta)^{n-m}[X(\theta)]^{m+1}. \quad (3.17)$$

Since on the right-hand side, the first two terms have a zeroth-order part equal to $X_0(\theta)$, while the sum is $O(\varepsilon)$, the ε expansion of $X(\theta)$ can be obtained through successive approximations beginning with $X_0(\theta)$. Of course, the known expansions of Y and $B(\theta)$ to the appropriate order must be substituted first, together with the previously determined coefficients $c(n,m)$. Substitution of the expansion $X(\theta)=\sum_{k=0}^{n-1}\varepsilon^{2k}X_k$ to order $2n-2$ then determines $X(\theta)$ to order $2n$. Since $B(\theta)$ is periodic and the zeroth-order value $X_0(\theta)$ is periodic, the expansion generated must be periodic. Therefore, the aperiodic terms that give rise to the π problem in the BFET cannot occur in the SUBM. As an example, $X(\theta)$ through order ε^4 is given by

$$X(\theta)=X\cos\theta+b\sin\theta+\frac{1}{8}\varepsilon^2[(3\sin\theta-\sin 3\theta)(2X^\dagger Xb-b^\dagger b^2)+(\cos\theta-\cos 3\theta)(X^\dagger X^2-X^\dagger b^2-2b^\dagger Xb)$$
$$-(\sin\theta+\sin 3\theta)b^\dagger X^2-4\sin\theta Bb]+\frac{1}{16}\varepsilon^4[(3\sin\theta-\sin 3\theta)(6X^\dagger BXb-b^\dagger Bb^2)$$
$$+2(\cos\theta-\cos 3\theta)(2X^\dagger BX^2-X^\dagger Bb^2-2b^\dagger BXb)-3(\sin\theta+\sin 3\theta)b^\dagger BX^2-2\sin\theta B^2b]+O(\varepsilon^6). \quad (3.18)$$

Not only are the aperiodic terms absent here, but beyond $O(\varepsilon)$ the periodic terms have different coefficients than in the BFET counterpart, Eq. (2.29).

The transformed operators of physical importance correspond to the value $\theta=\pi/2$. We continue to use the notation $\overline{\mathcal{O}}\equiv\mathcal{O}(\pi/2)$ for an arbitrary operator $\mathcal{O}$. Then from Eqs. (3.9) one immediately obtains

$$\overline{b}=-Y,\quad \overline{Y}=b,\quad \text{and H.c. eqs.,} \tag{3.19}$$

and from Eq. (3.16),

$$\overline{B}=B+2(b^\dagger b-Y^\dagger Y), \tag{3.20}$$

which has the expansion

$$\overline{B}=B+2(-X^\dagger X+b^\dagger b)+2\varepsilon^2(2X^{\dagger 2}X^2-X^\dagger BX)+2\varepsilon^4(3X^{\dagger 2}BX^2-X^\dagger B^2X)+O(\varepsilon^6). \tag{3.21}$$

Now, $\overline{X}$ can, of course, be obtained by setting $\theta=\pi/2$ in the solution [Eq. (3.18)] of Eq. (3.17). However, it is much faster to directly solve the counterpart of Eq. (3.17) for $\theta=\pi/2$, namely,

$$\overline{X}=b+\sum_{n=1}^{\infty}\sum_{m=0}^{n}\varepsilon^{2n}c(n,m)\overline{X}^{\dagger m}\overline{B}^{n-m}\overline{X}^{m+1}, \tag{3.22}$$

beginning with the zeroth-order value $\overline{X_0}=b$. This results in the expansion

$$\begin{aligned}\overline{X}=&\,b+\varepsilon^2\left(X^\dagger Xb-\frac{1}{2}Bb-\frac{1}{2}b^\dagger b^2\right)\\&+\varepsilon^4\left(\frac{1}{2}X^\dagger Xb-\frac{3}{2}X^{\dagger 2}X^2b+\frac{1}{2}b^\dagger X^\dagger Xb^2-\frac{1}{8}B^2b\right.\\&\left.+\frac{3}{2}X^\dagger BXb-\frac{1}{4}b^\dagger Bb^2-\frac{1}{8}b^\dagger b^2-\frac{1}{8}b^{\dagger 2}b^3\right)\\&+O(\varepsilon^6),\quad\text{and H.c. eq.}\end{aligned} \tag{3.23}$$

Finally in this subsection, we briefly discuss the auxiliary condition in the SUBM. In accord with the earlier discussion, the general condition is given by $\overline{b}\,\overline{|\text{phys}\rangle}=0$, where $\overline{|\text{phys}\rangle}\equiv U(\pi/2)|\text{phys}\rangle$. However, from Eq. (3.19), this implies that

$$Y\overline{|\text{phys}\rangle}=0. \tag{3.24}$$

If the operator Γ has an inverse on the domain of Y, and we show later (Sec. IV A) that it does, then from Eq. (3.10) the auxiliary condition simplifies to

$$X\overline{|\text{phys}\rangle}=0. \tag{3.25}$$

In fact, using the ε expansion of Γ, one can derive a concomitant expansion of Γ^{-1}, but this is insufficient to establish the domain.

C. Expansion of single-fermion operators

A complete theory requires not only the transformation of fermion-pair operators, but also the transformation of single-fermion operators, which are required for the description of one-nucleon transfer processes and chains of nuclei with different particle numbers. In this subsection we describe a straightforward technique for obtaining the ε expansions of single-fermion operators using the ''equations of motion.'' In the next subsection, we present an alternative formalism that permits the transformation of both single-fermion and pair operators within the same algorithm.

We begin by noting that the single-fermion operators $\alpha_m^\dagger$ and β_m are components of a rank-1/2 SU(2) spherical tensor, while $\beta_m^\dagger$ and α_m are the corresponding components of the H.c. tensor. Indeed, the commutators with the SU(2) generators of Eqs. (2.9) are given by

$$\begin{gathered}[J_0,\alpha_m^\dagger]=\frac{1}{2}\alpha_m^\dagger,\quad [J_0,\beta_m]=-\frac{1}{2}\beta_m,\\ [J_+,\alpha_m^\dagger]=0,\quad [J_-,\alpha_m^\dagger]=\beta_m,\\ [J_+,\beta_m]=\alpha_m^\dagger,\quad [J_-,\beta_m]=0,\quad\text{and H.c. eqs.}\end{gathered} \tag{3.26}$$

In terms of the generators X, $X^\dagger$ and B, the equivalent commutators are

$$\begin{gathered}[B,\alpha_m^\dagger]=\alpha_m^\dagger,\quad [B,\beta_m]=-\beta_m,\\ [X^\dagger,\alpha_m^\dagger]=0,\quad [X,\alpha_m^\dagger]=\varepsilon\beta_m,\\ [X^\dagger,\beta_m]=\varepsilon\alpha_m^\dagger,\quad [X,\beta_m]=0,\quad\text{and H.c. eqs.}\end{gathered} \tag{3.27}$$

The ''equations of motion'' for the single-fermion operators follow from Eq. (2.16), namely,

$$\frac{d\alpha_m^\dagger(\theta)}{d\theta}=[\alpha_m^\dagger(\theta),S(\theta)],$$

$$\frac{d\beta_m(\theta)}{d\theta}=[\beta_m(\theta),S(\theta)],\qquad\text{and H.c. eqs.,} \tag{3.28}$$

where the expansion of $S(\theta)=Y(\theta)^\dagger b(\theta)-b^\dagger(\theta)Y(\theta)$ has been previously determined. The formal solution of these equations satisfying the initial values $\alpha_m^\dagger(0)=\alpha_m^\dagger$ and $\beta_m(0)=\beta_m$ is

$$\alpha_m^\dagger(\theta)=\alpha_m^\dagger+\int_0^\theta[\alpha_m^\dagger(\theta'),S(\theta')]d\theta',$$

$$\beta_m(\theta)=\beta_m+\int_0^\theta[\beta_m(\theta'),S(\theta')]d\theta',\quad\text{and H.c. eqs.} \tag{3.29}$$

From the commutators (3.27) and the expansions of $Y(\theta)$ and $Y^\dagger(\theta)$ it is obvious that the integrals in Eqs. (3.29) are $O(\varepsilon)$. Therefore, the ε expansions of $\alpha_m^\dagger(\theta)$ and $\beta_m(\theta)$ can be obtained by successive approximations beginning with the zeroth-order values $\alpha_m^\dagger(0)=\alpha_m^\dagger$ and $\beta_m(0)=\beta_m$. This approach was easily set up using REDUCE 3.6. Once the solutions are obtained to a given order, they may be evaluated at $\theta=\pi/2$. Since higher-order terms become very numerous, we give the explicit results for the θ-dependent operators only through order ε^2:

$$\begin{aligned}\alpha_m^\dagger(\theta)=&\alpha_m^\dagger-\varepsilon[(1-\cos\theta)X^\dagger-\sin\theta b^\dagger]\beta_m\\&-\frac{1}{4}\varepsilon^2[(3-4\cos\theta+\cos 2\theta)X^\dagger X+\sin 2\theta X^\dagger b\\&-(4\sin\theta-\sin 2\theta)b^\dagger X+(1-\cos 2\theta)b^\dagger b]\alpha_m^\dagger\\&+O(\varepsilon^3),\quad \text{and H.c. eq.,}\\ \beta_m(\theta)=&\beta_m+\varepsilon[(1-\cos\theta)X-\sin\theta b]\alpha_m^\dagger\\&-\frac{1}{4}\varepsilon^2[(3-4\cos\theta+\cos 2\theta)X^\dagger X+\sin 2\theta b^\dagger X\\&-(4\sin\theta-\sin 2\theta)X^\dagger b+(1-\cos 2\theta)b^\dagger b\\&+4(1-\cos\theta)]\beta_m+O(\varepsilon^3),\quad \text{and H.c. eq.}\end{aligned}\tag{3.30}$$

The final transformed fermions $\overline{\alpha_m^\dagger}\equiv\alpha_m^\dagger(\pi/2)$, etc., through order ε^3 are given by

$$\begin{aligned}\overline{\alpha_m^\dagger}=&\alpha_m^\dagger+\varepsilon(-X^\dagger+b^\dagger)\beta_m\\&+\varepsilon^2\left(-\frac{1}{2}X^\dagger X+b^\dagger X-\frac{1}{2}b^\dagger b\right)\alpha_m^\dagger\\&+\varepsilon^3\left(\frac{1}{2}X^\dagger+\frac{3}{2}X^{\dagger 2}X-X^\dagger B-\frac{3}{2}b^\dagger X^\dagger X\right.\\&\left.+\frac{1}{2}b^\dagger B+\frac{1}{2}b^\dagger X^\dagger b-b^\dagger\right)\beta_m+O(\varepsilon^4),\\ \overline{\beta_m}=&\beta_m+\varepsilon(X-b)\alpha_m^\dagger\\&+\varepsilon^2\left(-\frac{1}{2}X^\dagger X+X^\dagger b-\frac{1}{2}b^\dagger b-1\right)\beta_m\\&+\varepsilon^3\left(\frac{3}{2}X^\dagger Xb-\frac{3}{2}X^\dagger X^2-X+BX\right.\\&\left.-\frac{1}{2}Bb-\frac{1}{2}b^\dagger Xb+\frac{1}{2}b\right)\alpha_m^\dagger+O(\varepsilon^4),\\ \overline{\beta_m^\dagger}=&\beta_m^\dagger+\varepsilon(X^\dagger-b^\dagger)\alpha_m\\&+\varepsilon^2\left(-\frac{1}{2}X^\dagger X+b^\dagger X-\frac{1}{2}b^\dagger b\right)\beta_m^\dagger\\&+\varepsilon^3\left(-\frac{1}{2}X^\dagger-\frac{3}{2}X^{\dagger 2}X+X^\dagger B+\frac{3}{2}b^\dagger X^\dagger X\right.\\&\left.-\frac{1}{2}b^\dagger B-\frac{1}{2}b^\dagger X^\dagger b+b^\dagger\right)\alpha_m+O(\varepsilon^4),\\ \overline{\alpha_m}=&\alpha_m+\varepsilon(-X+b)\beta_m^\dagger\\&+\varepsilon^2\left(-\frac{1}{2}X^\dagger X+X^\dagger b-\frac{1}{2}b^\dagger b-1\right)\alpha_m\\&+\varepsilon^3\left(-\frac{3}{2}X^\dagger Xb+\frac{3}{2}X^\dagger X^2+X-BX\right.\\&\left.+\frac{1}{2}Bb+\frac{1}{2}b^\dagger Xb-\frac{1}{2}b\right)\beta_m^\dagger+O(\varepsilon^4).\end{aligned}\tag{3.31}$$

Here, the H.c. of each operator is explicitly listed since a standard form is employed in which the single-fermion operators always appear on the far right, entailing rearrangement after Hermitian conjugation. As is readily checked, these operators obey the fermion anticommutation rules through the given order in ε.

D. A universal framework for transforming arbitrary fermion operators

While the above methods for obtaining the ε expansion of the unitary transforms of fermion operators are sufficient, a different approach was used for each type of operator. There exists another method whose appeal is that a single framework, indeed, a single algorithm can be implemented for arbitrary fermion operators. This formalism is an extension of one first used by Villars [19] for introducing collective canonically conjugate operators into a Hamiltonian. However, here we apply it to boson operators, which was apparently first done in Ref. [12]. Since the derivations were omitted there, we provide them in the Appendix.

Let $\mathcal{O}$ be an arbitrary fermion operator and let Y, $Y^\dagger$ be fermionic bosons, i.e., fermion operators obeying Eq. (3.8) on some subspace of the fermion space, which will be the ''arena of action'' in the ensuing discussion. From a formal viewpoint, the operators could be defined on any kind of space, not necessarily a many-fermion one, the main requirement being the existence of a boson degree of freedom represented by the pair Y, $Y^\dagger$. Corresponding to the generic operator $\mathcal{O}$, we introduce the fermion operator $\mathring{\mathcal{O}}$ defined by the formal infinite series

$$\mathring{\mathcal{O}}\equiv\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{1}{k!l!}(-Y^\dagger)^k\mathcal{O}(k,l)Y^l,\tag{3.32}$$

where $\mathcal{O}(k,l)$ is the iterated commutator

$$\mathcal{O}(k,l)\equiv\underbrace{[Y,[Y,\cdots[Y,}_{k\ \text{times}}\underbrace{[Y^\dagger,[Y^\dagger,\cdots[Y^\dagger}_{l\ \text{times}},\mathcal{O}]]\cdots],\tag{3.33}$$

and $\mathcal{O}(0,0)\equiv\mathcal{O}$. It is then easily shown (see the Appendix) that $\mathring{\mathcal{O}}$ has the property

$$[Y,\mathring{\mathcal{O}}]=0,\quad [Y^\dagger,\mathring{\mathcal{O}}]=0,\tag{3.34}$$

which means that $\mathring{\mathcal{O}}$ is *invariant* under the unitary transformation $U(\theta)$. While every fermion operator has an associated invariant, in some cases it may be trivially zero. To a given order in ε, only a finite number of terms contribute to the series (3.32). For example, for the single-fermion operators $\alpha_m^\dagger$ and the β_m, the corresponding invariants $\mathring{\alpha}_m^\dagger$ and $\mathring{\beta}_m$, respectively, are given through order ε^4 by

$$\begin{aligned}\mathring{\alpha}_m^\dagger &= \alpha_m^\dagger - \varepsilon X^\dagger \beta_m - \frac{1}{2}\varepsilon^2 X^\dagger X \alpha_m^\dagger \\ &\quad + \varepsilon^3\left(\frac{1}{2}X^\dagger + \frac{3}{2}X^{\dagger 2}X - X^\dagger B\right)\beta_m \\ &\quad + \varepsilon^4\left(\frac{3}{8}X^\dagger X + \frac{11}{8}X^{\dagger 2}X^2 - X^\dagger B X\right)\alpha_m^\dagger \\ &\quad + O(\varepsilon^5), \quad \text{and H.c. eq.,} \\ \mathring{\beta}_m &= \beta_m + \varepsilon X \alpha_m^\dagger - \varepsilon^2\left(\frac{1}{2}X^\dagger X + 1\right)\beta_m \\ &\quad - \varepsilon^3\left(\frac{3}{2}X^\dagger X^2 + X - BX\right)\alpha_m^\dagger \\ &\quad + \varepsilon^4\left(\frac{19}{8}X^\dagger X + \frac{11}{8}X^{\dagger 2}X^2 - 1 - X^\dagger B X - B\right)\beta_m \\ &\quad + O(\varepsilon^5), \quad \text{and H.c. eq.} \end{aligned} \tag{3.35}$$

As proven in the Appendix, the operator $\mathcal{O}$ has the formal decomposition

$$\mathcal{O} = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{1}{k!\,l!}\mathring{\mathcal{O}}(k,l)Y^{\dagger k}(-Y)^l, \tag{3.36}$$

where (k,l) is the invariant associated with the multiple commutator $\mathcal{O}(k,l)$. Although this decomposition is a formal identity, it becomes valuable when both sides are transformed, leading immediately to the result

$$\mathcal{O}(\theta) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{1}{k!\,l!}\mathring{\mathcal{O}}(k,l)[Y^\dagger(\theta)]^k[-Y(\theta)]^l, \tag{3.37}$$

which for $\theta = \pi/2$ becomes

$$\overline{\mathcal{O}} = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{1}{k!\,l!}\mathring{\mathcal{O}}(k,l)(b^\dagger)^k(-b)^l. \tag{3.38}$$

In order to obtain $\mathcal{O}(\theta)$ or $\overline{\mathcal{O}}$ to a given order in ε, only a finite number of the operator coefficients $\mathring{\mathcal{O}}$ (and a finite number of iterated commutators) need to be evaluated, which can be done very efficiently with computer algebra. Equations (3.37) and (3.38) can be applied to *any* fermion operator, be it a pair operator, a single-particle operator, or a Hamiltonian with interactions. Moreover, it is not necessary to calculate $\mathcal{O}(\theta)$ in order to obtain $\overline{\mathcal{O}}$, which can be evaluated directly. This provides an advantage over the method described above for the single-fermion operators. We wrote a REDUCE procedure that evaluates Eqs. (3.37) or (3.38) for any fermion operator. This reproduces the results presented above for the transformation of X and the single-fermion operators but the calculation is much faster. Moreover, this universal algorithm has the advantage of being readily applicable to more realistic nuclear shell models.

IV. THE SUBM: CLOSED FORMS

The previous section described methods for obtaining the ε-expansions of transformed operators. In this section we discuss how to obtain the transformed operators in a closed form within the framework of the SU(2) model. While this approach is not immediately applicable to the general case, it provides insight into the validity of the treatment in Sec. III.

A. Closed forms for Y and $Y^\dagger$

As discussed in Sec. II, eigenvectors of $H_{\rm LMG}$ are linear combinations of the basis vectors $|\gamma;JM\rangle$ belonging to an irreducible representation of the group SU(2). Since the totality of these basis vectors is complete on the Lipkin-model Hilbert space, an arbitrary fermionic operator $\hat{\mathcal{O}}_{\rm F}$ can be expressed in the following dyadic form:

$$\begin{aligned}\hat{\mathcal{O}}_{\rm F} = \sum_{\gamma,\gamma'}\sum_{J,J'}\sum_{M=-J}^{J}\sum_{M'=-J'}^{J'}\langle\gamma';J',M'| \\ \times \hat{\mathcal{O}}_{\rm F}|\gamma;J,M\rangle|\gamma';J',M'\rangle\langle\gamma;J,M|. \end{aligned} \tag{4.1}$$

Thus, the quasispin operators are expressed as

$$\begin{aligned} J_- &= \sum_{\gamma,J}\sum_{\nu=0}^{2J}\sqrt{\nu(2J-\nu+1)}|\gamma;J,-J+\nu-1\rangle \\ &\quad \times\langle\gamma;J,-J+\nu|, \\ J_+ &= (J_-)^\dagger, \end{aligned} \tag{4.2}$$

$$J_0 = \sum_{\gamma,J}\sum_{\nu=0}^{2J}(\nu - J)|\gamma;J,-J+\nu\rangle\langle\gamma;J,-J+\nu|, \tag{4.3}$$

where we set $M = -J + \nu$.

We define the invariant operator $\hat{J}$ as

$$\hat{J} = \sum_{\gamma,J}\sum_{\nu=0}^{2J} J|\gamma;J,-J+\nu\rangle\langle\gamma;J,-J+\nu|, \tag{4.4}$$

which satisfies the following relations:

$$\hat{J}|\gamma,J,-J+\nu\rangle = J|\gamma,J,-J+\nu\rangle, \tag{4.5}$$

$$[\hat{J},J_\pm] = [\hat{J},J_0] = 0, \tag{4.6}$$

$$\hat{J}(\hat{J}+1) = J_0^2 \pm J_0 + J_\mp J_\pm = J^2, \tag{4.7}$$

where J^2 is the Casimir operator.

Next, we introduce the fermionic boson destruction operator Y by

$$Y=\sum_{\gamma,J}\sum_{\nu=0}^{2J}\sqrt{\nu}|\gamma;J,-J+\nu-1\rangle\langle\gamma;J,-J+\nu|, \quad (4.8)$$

in analogy with the dyadic representation of the boson destruction operator

$$b=\sum_{n=0}^{\infty}\sqrt{n}|n-1\rangle\langle n|, \quad (4.9)$$

where $|n\rangle$ denotes the normalized n-boson state. The use of the label Y is justified below. From Eq. (4.8) and its H.c. we immediately obtain

$$Y|\gamma;J,-J+\nu\rangle=\sqrt{\nu}|\gamma;J,-J+\nu-1\rangle, \quad \nu=0,1,\ldots,2J, \quad (4.10)$$

$$Y^\dagger|\gamma;J,-J+\nu\rangle=\sqrt{\nu+1}|\gamma;J,-J+\nu+1\rangle, \quad \nu=0,1,\ldots,2J-1, \quad (4.11)$$

$$Y^\dagger|\gamma;J,J\rangle=0. \quad (4.12)$$

Thus, within each SU(2) irrep, the operator Y corresponds to a boson destruction operator with vacuum $|0\rangle=|\gamma;J,-J\rangle$, while $Y^\dagger$ corresponds to a boson creation operator, *except* for the maximally aligned state $|\gamma;J,J\rangle$. Therefore, the fermionic bosons provide an example of *truncated bosons*, whose mathematical properties are discussed by Hammel in an Internet publication [20]. Of course, Eq. (4.12) represents the inevitable cutoff required by the finite dimensionality of SU(2) irreps.

The fermionic phonon number operator N_Y is defined as

$$N_Y\equiv Y^\dagger Y, \quad (4.13)$$

the dyadic form of which, derived from Eq. (4.8) and its H.c., is

$$N_Y=\sum_{n=0}^{2J}\nu|\gamma,J,-J+\nu\rangle\langle\gamma;J,-J+\nu|. \quad (4.14)$$

It is then easily shown that

$$N_Y|\gamma;J,-J+\nu\rangle=\nu|\gamma;J,-J+\nu\rangle, \quad \nu=0,1,\ldots,2J, \quad (4.15)$$

$$[N_Y,Y^\dagger]=Y^\dagger, \quad [N_Y,Y]=-Y, \quad (4.16)$$

$$N_Y=\hat{J}+J_0. \quad (4.17)$$

The commutation relation between Y and $Y^\dagger$, obtained from Eq. (4.8) and its H.c., is

$$[Y,Y^\dagger]=1-(2\hat{J}+1)P_\Uparrow, \quad (4.18)$$

where $P_\Uparrow$ is the projector to the subspace of maximally spin-aligned vectors, i.e.,

$$P_\Uparrow\equiv\sum_{\gamma J}|\gamma;J,J\rangle\langle\gamma;J,J|. \quad (4.19)$$

Thus, the boson commutation rule $[Y,Y^\dagger]=1$ holds only within the subspace spanned by vectors $|\gamma;J,M\rangle$ with $M\neq J$, but not in the subspace of maximally aligned vectors $|\gamma;J,J\rangle$. It should be noted that Eqs. (4.18) and (4.19) guarantee that Tr $[Y,Y^\dagger]=0$ on the full fermion space, as required by its finite dimensionality.

Noticing that

$$(\hat{J}-J_0)|\gamma;J,-J+\nu\rangle=(2J-\nu)|\gamma;J,-J+\nu\rangle,$$

one obtains the following closed form of Y from Eqs. (4.2) and (4.8):

$$Y=J_-\frac{1}{\sqrt{\hat{J}-J_0+1}}. \quad (4.20)$$

The rhs of this expression is equivalent to $(1/\sqrt{\hat{J}-J_0})J_-$ unless acting on bra vectors in the maximally aligned subspace, where the inverse square root operator breaks down. To verify that Eq. (4.20) gives the ε-expansion for Y obtained in Sec. III, one may proceed as follows. As a root of the quadratic equation in Eq. (4.7), $\hat{J}$ is given in terms of J_0 and $J_\pm$ by

$$\hat{J}=-\frac{1}{2}+\sqrt{\frac{1}{4}+J_0(J_0-1)+J_+J_-}.$$

Therefore, $\sqrt{\hat{J}-J_0+1}$ can be expressed in terms of $X^\dagger X$ and B using Eq. (2.13). In this way we finally obtain

$$Y=X\frac{1}{\sqrt{\frac{1}{2}(1+\varepsilon^2-\varepsilon^2B)+\frac{1}{2}\sqrt{(1+\varepsilon^2-\varepsilon^2B)^2+4\varepsilon^2X^\dagger X}}}. \quad (4.21)$$

This expression represents the summation of the ε-expansion given by Eqs. (3.10) and (3.11). Indeed, it was directly confirmed to very high orders in ε that the Taylor expansion of Eq. (4.21) coincides with the expansion (3.13), and when written in normal order, with Eq. (3.12).

The explicit construction of Y and $Y^\dagger$ presents us with a conundrum—we have consistently assumed the relation $[Y,Y^\dagger]=1$ in all derivations in Sec. III, while the actual commutation rule given by Eq. (4.18) involves the projector $P_\Uparrow$. Some insight is provided by an explicit representation of this projector. From Eq. (4.20) and the relation $J_+J_-=(\hat{J}+J_0)(\hat{J}-J_0+1)$, it follows that

$$[Y,Y^\dagger]=J_-\frac{1}{\hat{J}-J_0+1}J_+-(\hat{J}+J_0). \quad (4.22)$$

Comparison with Eq. (4.18) identifies the projector as

$$P_{\Uparrow}=\frac{1}{2\hat{J}+1}\left(\hat{J}+J_0+1-J_-\frac{1}{\hat{J}-J_0+1}J_+\right)$$
$$=\frac{1}{2\hat{J}+1}(\hat{J}+J_0+1-YY^\dagger). \tag{4.23}$$

This representation is well defined on the entire fermion space. What is remarkable upon first examination is that the ε expansion of $P_{\Uparrow}$ turns out to be precisely zero. This seems to justify *a posteriori* the use of the boson relation $[Y,Y^\dagger]=1$ in deriving ε expansions. The reason for the vanishing of the expansion is easily comprehended. The term $J_-[1/(\hat{J}-J_0+1)]J_+$ has the same ε expansion as the expression $[1/(\hat{J}-J_0)]J_-J_+=[1/(\hat{J}-J_0)](\hat{J}-J_0)(\hat{J}+J_0+1)=\hat{J}+J_0+1$, but the latter, involving $1/(\hat{J}-J_0)$ is valid only in the subspace that *excludes* the maximally aligned vectors $|\gamma;J,J\rangle$. We thus return to the same conclusion reached two paragraphs ago. But now, the inevitability of $[Y,Y^\dagger]=1$ in a Taylor expansion may be better appreciated. Taylor expansions are inherently insensitive to cutoffs. The analogy may be made to the Taylor expansion of an attractive potential well about its minimum—the expansion cannot distinguish between an infinite well and one of finite depth.

A key question remains to be resolved: in which subspace of the *boson-fermion* space are $b(\theta)$ and $Y(\theta)$ harmonic functions, as given by Eq. (3.9)? This is taken up in the next subsection.

B. The principal subspace

Since Y and $Y^\dagger$ are truncated rather than true bosons, Eq. (3.9) cannot hold over the entire boson-fermion space. Indeed, direct evaluation of $b(\theta)$ and $Y(\theta)$ using the multicommutator expansion of the unitary transformation [sometimes referred to as the Baker-Campbell-Hausdorf (BCH) expansion], indicates that in addition to the $\cos\theta$ and $\sin\theta$ terms there is a huge number of terms involving higher harmonics that depend (linearly, of course) on the projector $P_{\Uparrow}$. While, in principle, the additional terms define the action on the whole boson-fermion space, this BCH series is impossible to sum in practice. Nevertheless, we show that the harmonic solution for $b(\theta)$ and $Y(\theta)$ can be realized, not on the whole boson-fermion space, but on a certain subspace that we call the *principal subspace*. This restriction, as it turns out, is perfectly adequate to allow us to achieve our aims.

As a preliminary to constructing the principal subspace, we study the properties of a basis for the entire boson-fermion space $\mathcal{H}$. Now, in accord with the H.c. of Eqs. (4.8) or (4.20), the basis vectors $|\gamma,J-J+\nu\rangle$ for the purely fermionic subspace can be written as

$$|\gamma;J,-J+\nu\rangle=\frac{1}{\sqrt{\nu!}}(Y^\dagger)^\nu|\gamma;J,-J\rangle. \tag{4.24}$$

Therefore, a set of orthonormal basis vectors for the whole boson-fermion space is represented by the product states

$$|\{\gamma,J\};n,\nu\rangle\equiv\frac{1}{\sqrt{\nu!(n-\nu)!}}(b^\dagger)^{n-\nu}(Y^\dagger)^\nu|\gamma;J,-J\rangle;$$
$$\forall\{\gamma,J\},\quad \nu=0,1,\dots,2J;\quad n=\nu,\nu+1,\dots\ . \tag{4.25}$$

Here, the fermion vectors are implicitly extended to the entire boson-fermion space, so that $|\gamma,J,-J+\nu\rangle$ is to be interpreted as $|0\rangle_{\rm B}\otimes|\gamma,J,-J+\nu\rangle_{\rm F}$, etc. Also, in this notation

$$|\{\gamma,J\};\nu,\nu\rangle=|\gamma;J,-J+\nu\rangle, \tag{4.26}$$

which is a pure fermion vector. For later reference, we note here that $b|\{\gamma,J\};\nu,\nu\rangle=0$.

If one defines the total phonon number $N_{\rm t}$ as

$$N_{\rm t}=N_b+N_Y, \tag{4.27}$$

where N_Y is defined by Eq. (4.13) and N_b is the boson number operator

$$N_b\equiv b^\dagger b, \tag{4.28}$$

then the basis vectors $|\{\gamma,J\};n,\nu\rangle$ have the following properties:

$$N_b|\{\gamma,J\};n,\nu\rangle=(n-\nu)|\{\gamma,J\};n,\nu\rangle,$$
$$N_Y|\{\gamma,J\};n,\nu\rangle=\nu|\{\gamma,J\};n,\nu\rangle,$$
$$N_{\rm t}|\{\gamma,J\};n,\nu\rangle=n|\{\gamma,J\};n,\nu\rangle. \tag{4.29}$$

Since $[N_{\rm t},Y^\dagger b]=[N_{\rm t},b^\dagger Y]=0$, $N_{\rm t}$ also commutes with S and therefore with the unitary transformation $U(\theta)$ generated by S. As a consequence, under this unitary transformation, each fermion basis vector remains an eigenvector of $N_{\rm t}$, and, therefore, may be expanded in the set of all such eigenvectors with the same eigenvalue n as follows:

$$U(\theta)|\{\gamma,J\};n,n\rangle$$
$$=\sum_{\nu=0}^{2J}\langle\{\gamma,J\};n,\nu|U(\theta)|\{\gamma,J\};n,n\rangle|\{\gamma,J\};n,\nu\rangle,$$
$$n=0,1,\dots,2J. \tag{4.30}$$

Thus, if $\{|\Psi_{\rm F}(i)\rangle\}$ is any basis for the fermion subspace, then the image basis $\{U(\theta)|\Psi_{\rm F}(i)\rangle\}$ for any θ can be expanded in a subset of the basis (4.25), with the property that the total number of phonons $n\leqslant 2J\leqslant 2\Omega$. The subspace generated by the latter subset is what we call the principal subspace. The formal definition is as follows: the *principal subspace* $\mathcal{H}_0$ of the whole boson-fermion space $\mathcal{H}$ is the set

$$\mathcal{H}_0\equiv\mathrm{span}\{|\{\gamma,J\};n,\nu\rangle:\forall\{\gamma,J\};n=0,1,2,\dots,2J;$$
$$\nu=0,1,2,\dots,n\}. \tag{4.31}$$

Note that $\mathcal{H}_0\supset\mathcal{H}_{\rm F}$, the entire fermion subspace, and, also $\mathcal{H}_0\supset\mathcal{H}_{\rm phys}$, the entire physical subspace, generated by the image basis $\{U(\pi/2)|\Psi_{\rm F}(i)\rangle\}$.

We now consider the restriction of any operator to the principal subspace. To do so, we introduce the projection operator P_0 to the principal subspace $\mathcal{H}_0$, namely,

$$P_0 \equiv \sum_{\gamma,J} \sum_{n=0}^{2J} \sum_{\nu=0}^{n} |\{\gamma,J\};n,\nu\rangle\langle\{\gamma,J\};n,\nu|. \tag{4.32}$$

Then, for any boson-fermion operator $\mathcal{O}$, the restriction to the principal subspace, denoted by $\mathcal{O}_{P_0}$, is defined by

$$\mathcal{O}_{P_0} \equiv P_0 \mathcal{O} P_0. \tag{4.33}$$

Since $U(\theta)$ leaves invariant both the principal subspace and its orthogonal complement, which is the consequence of the property $[N_t, U(\theta)]=0$, it follows that $[U(\theta),P_0] = [U^\dagger(\theta),P_0]=0$. This, in turn, implies that

$$[\mathcal{O}(\theta)]_{P_0} = \mathcal{O}_{P_0}(\theta). \tag{4.34}$$

We are now positioned to prove the harmonic behavior of $b(\theta)$ and $Y(\theta)$ in the principal subspace. As a preliminary, we consider the action of the operator $[Y,Y^\dagger]b$ on the vectors (4.25); thus, from Eqs. (4.9) and (4.18),

$$\begin{aligned}[Y,Y^\dagger]b|\{\gamma,J\};n,\nu\rangle \\ = \sqrt{n-\nu}[1-(2\hat{J}+1)P_\Uparrow]|\{\gamma,J\};n-1,\nu\rangle.\end{aligned}$$

But in $\mathcal{H}_0$, the projector $P_\Uparrow$ has a nonzero action only on vectors with $n=\nu=2J$, in which case $\sqrt{n-\nu}=0$. In other words, acting on vectors in $\mathcal{H}_0$, $[Y,Y^\dagger]b=b$, even if $[Y,Y^\dagger]\neq 1$, or, more formally,

$$\{[Y,Y^\dagger]b\}_{P_0} = b_{P_0}, \tag{4.35}$$

which, because of the commutation of the projection and the unitary transformation, implies that $\{[Y(\theta),Y^\dagger(\theta)]b(\theta)\}_{P_0} = b_{P_0}(\theta)$. Then, the projection of the exact Heisenberg equations (3.5) is given by

$$\frac{db_{P_0}(\theta)}{d\theta} = -Y_{P_0}(\theta),$$

$$\frac{dY_{P_0}(\theta)}{d\theta} = \{[Y(\theta),Y^\dagger(\theta)]b(\theta)\}_{P_0} = b_{P_0}(\theta),$$

$$\text{and H.c. eqs.} \tag{4.36}$$

This proves that the projected operators have the harmonic solution

$$b_{P_0}(\theta) = b_{P_0}\cos\theta - Y_{P_0}\sin\theta,$$

$$Y_{P_0}(\theta) = b_{P_0}\sin\theta + Y_{P_0}\cos\theta, \quad \text{and H.c. eqs.} \tag{4.37}$$

By invoking the following easily proven properties:

$$Y_{P_0} = YP_0, \quad Y^\dagger_{P_0} = P_0 Y^\dagger, \quad b_{P_0} = bP_0, \quad b^\dagger_{P_0} = P_0 b^\dagger, \tag{4.38}$$

one may also write Eqs. (4.37) in the form

$$b(\theta)P_0 = [b\cos\theta - Y\sin\theta]P_0,$$

$$Y(\theta)P_0 = [b\sin\theta + Y\cos\theta]P_0, \quad \text{and H.c. eqs.} \tag{4.39}$$

While Eqs. (4.37) have the same form as Eqs. (3.9), it should be noted that b_{P_0} and Y_{P_0} are not bosons, owing to the presence of projectors. In fact, it is straightforward to show that the commutation rules on $\mathcal{H}$ are given by

$$[b_{P_0}, b^\dagger_{P_0}] = P_0 - (\{N_b\}_{P_0}+1)P_{n_{\max}},$$

$$[Y_{P_0}, Y^\dagger_{P_0}] = P_0 - (\hat{J}+J_0+1)P_{n_{\max}},$$

$$[b^\dagger_{P_0}, Y_{P_0}] = Yb^\dagger P_{n_{\max}}, \quad \text{and H.c. eq.},$$

$$[b_{P_0}, Y_{P_0}] = 0, \quad \text{and H.c. eq.}, \tag{4.40}$$

where $P_{n_{\max}}$, the projector to the subspace having the maximum number of phonons $n_{\max}=2J$ for each value of J, is given by

$$P_{n_{\max}} = \sum_{\gamma J} \sum_{\nu=0}^{2J} |\{\gamma,J\};2J,\nu\rangle\langle\{\gamma,J\};2J,\nu|. \tag{4.41}$$

The operators b_{P_0}, $b^\dagger_{P_0}$ and Y_{P_0}, $Y^\dagger_{P_0}$ are truncated bosons acting within the subspace $\mathcal{H}_0$, the projector $P_{n_{\max}}$ guaranteeing that $b^\dagger_{P_0}$ and $Y^\dagger_{P_0}$ annihilate vectors with $n=2J$ phonons. However, when acting on all other vectors in $\mathcal{H}_0$, the truncated bosons behave just like ordinary bosons. Therefore, the identification of Eqs. (3.9) and (4.37) is complete. No problems arise as a result of treating truncated bosons like true bosons as long as physical matrix elements are constructed within $\mathcal{H}_0$. Thus, the treatment in the preceding subsections is fully justified provided that all operators are regarded as being defined on the principal subspace $\mathcal{H}_0$. In doing so, the boson operators b, $b^\dagger$ are restricted so that they are more on a par with the operators Y, $Y^\dagger$. One could also contemplate the opposite tack: extending the fermion subspace to an infinite-dimensional one so that the operators Y, $Y^\dagger$ could be defined as true bosons on a par with the true bosons b, $b^\dagger$. Indeed, such an approach had been attempted many years ago by Kuriyama *et al.* [13] in their auxiliary-variables treatment of the pairing problem. However, their extension of the fermion space is a purely formal one that has no obvious physical connection with the original many-body problem. It amounts to little more than the assumption of a license to treat all the operators b, $b^\dagger$, Y, and $Y^\dagger$ as true bosons.

C. Transformation of SU(2) generators

The remaining task is the transformation of operators, beginning with the SU(2) generators and then proceeding to single-fermion operators. A convenient starting point is the generator J_0, which according to Eq. (4.17) can be written as

$J_0 = -\hat{J} + N_Y$. Since $[\hat{J}, S] = 0$, $\hat{J}$ is invariant under the unitary transformation $U(\theta)$, i.e.,

$$\hat{J}(\theta) = \hat{J}. \tag{4.42}$$

An operator $\mathcal{O}$ leaves invariant both $\mathcal{H}_0$ and its orthogonal complement $\mathcal{H}_0^\perp$ if and only if

$$P_0 \mathcal{O} P_0 = P_0 \mathcal{O} = \mathcal{O} P_0. \tag{4.43}$$

Since $\hat{J}$, in fact, has this property, it follows that

$$[\hat{J}(\theta)]_{P_0} = \hat{J}_{P_0} = P_0 \hat{J} = \hat{J} P_0. \tag{4.44}$$

Next, making use of the first of Eqs. (4.38) as well as $[U(\theta), P_0] = [U^\dagger(\theta), P_0] = 0$, it is easily shown that

$$[N_Y(\theta)]_{P_0} = (N_Y)_{P_0}(\theta) = Y^\dagger_{P_0}(\theta) Y_{P_0}(\theta). \tag{4.45}$$

Therefore, the projected unitary transformation of J_0 is given by

$$[J_0(\theta t)]_{P_0} = -\hat{J}_{P_0} + Y^\dagger_{P_0}(\theta) Y_{P_0}(\theta). \tag{4.46}$$

However, one can do a little more. Using the first of Eqs. (4.37) for $Y_{P_0}(\theta)$ and its H.c., together with the salient result that N_Y, N_b, $b^\dagger Y$, and $Y^\dagger b$ all have the property (4.43), one calculates

$$Y^\dagger_{P_0}(\theta) Y_{P_0}(\theta) = \mathfrak{N}_Y(\theta) P_0 \equiv \left[N_Y + \frac{1}{2}(N_b - N_Y)(1 - \cos 2\theta) + \frac{1}{2}(b^\dagger Y + Y^\dagger b) \sin 2\theta \right] P_0, \tag{4.47}$$

where the projector P_0 can also be commuted to the far left of the bracket if desired. The complete expression for $[J_0(\theta)]_{P_0}$ is then given by

$$[J_0(\theta)]_{P_0} = \left[-\hat{J} + N_Y + \frac{1}{2}(N_b - N_Y)(1 - \cos 2\theta) + \frac{1}{2}(b^\dagger Y + Y^\dagger b) \sin 2\theta \right] P_0. \tag{4.48}$$

Apart from the presence of the projector, this result exactly agrees with Eq. (3.16), taking into account the relation between B and J_0.

Next consider the transformation of J_-. The starting point is the inverse of Eq. (4.20), which can be written as

$$J_- = \sqrt{\hat{J} - J_0}\, Y. \tag{4.49}$$

Noting that $\sqrt{\hat{J} - J_0} = \sqrt{2\hat{J} - N_Y}$ has the property (4.43), one obtains from Eqs. (4.38) and (4.46) the following projected unitary transform of J_-:

$$\begin{aligned} [J_-(\theta)]_{P_0} &= \sqrt{\hat{J}_{P_0} - [J_0(\theta)]_{P_0}}\, Y_{P_0}(\theta) \\ &= \sqrt{2\hat{J}_{P_0} - Y^\dagger_{P_0}(\theta) Y_{P_0}(\theta)}\, Y_{P_0}(\theta), \end{aligned} \tag{4.50}$$

which, with the aid of Eqs. (4.37), (4.44), and (4.47) can be written as

$$\begin{aligned} [J_-(\theta)]_{P_0} &= \sqrt{2\hat{J} - N_Y - \frac{1}{2}(N_b - N_Y)(1 - \cos 2\theta) - \frac{1}{2}(b^\dagger Y + Y^\dagger b) \sin 2\theta}\,(b \sin\theta + Y \cos\theta) P_0, \\ [J_+(\theta)]_{P_0} &= [J_-(\theta)]^\dagger_{P_0}. \end{aligned} \tag{4.51}$$

We note that the projector P_0 must stand on the far right in the above expression. Apart from the projector, the ε-expansion of $\varepsilon[J_-(\theta)]_{P_0}$ exactly coincides with that of X given by Eq. (3.17), as checked to high orders. Therefore, we conclude that Eq. (4.51) represents the closed summation of Eq. (3.17).

The final transformation, of course, corresponds to the choice $\theta = \pi/2$. At this point, one may just as well replace the projector P_0 to the principal subspace by the projector P_{phys} to the physical subspace since the latter subspace is included in the former and all of the physics has been transported to the physical subspace. It will be recalled that the physical subspace is spanned by all vectors having the property (3.24) or (3.25). We also extend the notation of Eq. (4.33), so that $(\mathcal{O})_{P_{\text{phys}}} \equiv P_{\text{phys}} \mathcal{O} P_{\text{phys}}$, for an arbitrary operator $\mathcal{O}$. In accord with our convention in which $[J_0(\pi/2)]_{P_{\text{phys}}} \equiv (\overline{J_0})_{P_{\text{phys}}}$, and $[J_-(\pi/2)]_{P_{\text{phys}}} \equiv (\overline{J_-})_{P_{\text{phys}}}$, etc., we immediately obtain the following images of the generators:

$$\begin{aligned} (\overline{J_0})_{P_{\text{phys}}} &= (-\hat{J} + N_b) P_{\text{phys}} = P_{\text{phys}}(-\hat{J} + N_b), \\ (\overline{J_-})_{P_{\text{phys}}} &= \sqrt{2\hat{J} - N_b}\, b P_{\text{phys}}, \quad (\overline{J_+})_{P_0} = (\overline{J_-})^\dagger_{P_0} = P_{\text{phys}} b^\dagger \sqrt{2\hat{J} - N_b}. \end{aligned} \tag{4.52}$$

This result is a very familiar one: a slightly extended version (EHP) of the Holstein-Primakoff mapping [17] of the SU(2) generators, the extension corresponding to the operator $\hat{J}$ replacing the *c*-number J. As discussed in the next section, the operator $\hat{J}$ can be written in the form

$$\hat{J}=\frac{1}{2}(\Omega-\mathring{n}), \tag{4.53}$$

where

$$\mathring{n}\equiv\sum_{m}(\mathring{\alpha}_m^{\dagger}\mathring{\alpha}_m+\mathring{\beta}_m^{\dagger}\mathring{\beta}_m) \tag{4.54}$$

is the total number of *ideal particles and holes* (also known as quasifermions) in the terminology of Refs. [12,9]. Indeed, the above EHP representation is identical to the one obtained by other means in these references.

D. Transformation of single-fermion operators

In this subsection we derive closed forms of the transformed single-fermion operators. First, the fermion operators are written in the following dyadic form:

$$\begin{aligned}\alpha_m^{\dagger}=&\sum_{J=1/2}^{\Omega/2}\sum_{\nu=0}^{2J}\sum_{\gamma\gamma'}\langle\gamma';J-1/2,-J+1/2+\nu|\alpha_m^{\dagger}|\gamma;J,-J+\nu\rangle|\gamma';J-1/2,-J+1/2+\nu\rangle\langle\gamma;J,-J+\nu|\\&+\sum_{J=0}^{(\Omega-1)/2}\sum_{\nu=0}^{2J}\sum_{\gamma\gamma'}\langle\gamma';J+1/2,-J+1/2+\nu|\alpha_m^{\dagger}|\gamma;J,-J+\nu\rangle|\gamma';J+1/2,-J+1/2+\nu\rangle\langle\gamma;J,-J+\nu|,\\ \beta_m=&\sum_{J=1/2}^{\Omega/2}\sum_{\nu=0}^{2J}\sum_{\gamma\gamma'}\langle\gamma';J-1/2,-J-1/2+\nu|\beta_m|\gamma;J,-J+\nu\rangle|\gamma';J-1/2,-J-1/2+\nu\rangle\langle\gamma;J,-J+\nu|\\&+\sum_{J=0}^{(\Omega-1)/2}\sum_{\nu=0}^{2J}\sum_{\gamma\gamma'}\langle\gamma';J+1/2,-J-1/2+\nu|\beta_m|\gamma;J,-J+\nu\rangle|\gamma';J+1/2,-J-1/2+\nu\rangle\langle\gamma;J,-J+\nu|,\quad\text{and H.c. eqs.}\end{aligned} \tag{4.55}$$

As shown in Eq. (3.26), these operators are components of a rank-1/2 spherical tensor $t_m^{1/2}$ with

$$t_{m,1/2}^{1/2}=\alpha_m^{\dagger},\quad t_{m,-1/2}^{1/2}=\beta_m. \tag{4.56}$$

Consequently, application of the Wigner-Eckart theorem with explicit evaluation of the Clebsch-Gordan coefficients allows one to write Eqs. (4.55) in terms of the reduced matrix elements as follows:

$$\begin{aligned}\alpha_m^{\dagger}=&-\sum_{J=1/2}^{\Omega/2}\sum_{\nu=0}^{2J}\sum_{\gamma\gamma'}\langle\gamma';J-1/2\|t_m^{1/2}\|\gamma;J\rangle\sqrt{\frac{2J-\nu}{2J+1}}|\gamma';J-1/2,-J+1/2+\nu\rangle\langle\gamma;J,-J+\nu|\\&+\sum_{J=0}^{(\Omega-1)/2}\sum_{\nu=0}^{2J}\sum_{\gamma\gamma'}\langle\gamma';J+1/2\|t_m^{1/2}\|\gamma;J\rangle\sqrt{\frac{\nu+1}{2J+1}}|\gamma';J+1/2,-J+1/2+\nu\rangle\langle\gamma;J,-J+\nu|,\\ \beta_m=&\sum_{J=1/2}^{\Omega/2}\sum_{\nu=0}^{2J}\sum_{\gamma\gamma'}\langle\gamma';J-1/2\|t_m^{1/2}\|\gamma;J\rangle\sqrt{\frac{\nu}{2J+1}}|\gamma';J-1/2,-J-1/2+\nu\rangle\langle\gamma;J,-J+\nu|\\&+\sum_{J=0}^{(\Omega-1)/2}\sum_{\nu=0}^{2J}\sum_{\gamma\gamma'}\langle\gamma';J+1/2\|t_m^{1/2}\|\gamma;J\rangle\sqrt{\frac{2J-\nu+1}{2J+1}}|\gamma';J+1/2,-J-1/2+\nu\rangle\langle\gamma;J,-J+\nu|,\quad\text{and H.c. eqs.}\end{aligned} \tag{4.57}$$

This motivates the introduction of the *quasifermion* operators $\mathring{\alpha}_m^{\dagger}$ and $\mathring{\beta}_m$, defined by

$$\mathring{\alpha}_m^\dagger \equiv -\sum_{J=1/2}^{\Omega/2}\sum_{\nu=0}^{2J-1}\sum_{\gamma\gamma'}\langle\gamma';J-1/2\|t_m^{1/2}\|\gamma;J\rangle\sqrt{\frac{2J}{2J+1}}|\gamma';J-1/2,-J+1/2+\nu\rangle\langle\gamma;J,-J+\nu|$$
$$=\sum_{J=1/2}^{\Omega/2}\sum_{\nu=0}^{2J-1}\sum_{\gamma\gamma'}\langle\gamma';J-1/2,-J+1/2|\alpha_m^\dagger|\gamma;J,-J\rangle|\gamma';J-1/2,-J+1/2+\nu\rangle\langle\gamma;J,-J+\nu|,$$
$$\mathring{\beta}_m \equiv \sum_{J=0}^{(\Omega-1)/2}\sum_{\nu=0}^{2J}\sum_{\gamma\gamma'}\langle\gamma';J+1/2\|t_m^{1/2}\|\gamma;J\rangle|\gamma';J+1/2,-J-1/2+\nu\rangle\langle\gamma;J,-J+\nu|$$
$$=\sum_{J=0}^{(\Omega-1)/2}\sum_{\nu=0}^{2J}\sum_{\gamma\gamma'}\langle\gamma';J+1/2,-J-1/2|\beta_m|\gamma;J,-J\rangle|\gamma';J+1/2,-J-1/2+\nu\rangle\langle\gamma;J,-J+\nu|,\quad \text{and H.c. eqs.} \tag{4.58}$$

The notation for the quasifermions suggests identification with the invariants given by Eq. (3.35), which will be eventually justified. With the help of Eqs. (4.15) and (4.20), one readily obtains from the dyadic form (4.57) the following key linear relation between the fermion and quasifermion operators:

$$\alpha_m^\dagger=\frac{1}{\sqrt{2\hat{J}+1}}\mathring{\alpha}_m^\dagger\sqrt{2\hat{J}-N_Y}+Y^\dagger\mathring{\beta}_m\frac{1}{\sqrt{2\hat{J}+1}},$$

$$\beta_m=-\frac{1}{\sqrt{2\hat{J}+1}}Y\mathring{\alpha}_m^\dagger+\sqrt{2\hat{J}-N_Y}\mathring{\beta}_m\frac{1}{\sqrt{2\hat{J}+1}},$$

$$\text{and H.c. eqs.} \tag{4.59}$$

There are several other important properties of the quasifermion operators. First, the following relations are easily derived from Eqs. (4.3), (4.5), and (4.58):

$$[\hat{J},\mathring{\alpha}_m^\dagger]=-[J_0,\mathring{\alpha}_m^\dagger]=-\frac{1}{2}\mathring{\alpha}_m^\dagger,$$

$$[\hat{J},\mathring{\beta}_m^\dagger]=-[J_0,\mathring{\beta}_m^\dagger]=-\frac{1}{2}\mathring{\beta}_m,\quad \text{and H.c. eqs.} \tag{4.60}$$

Also, from Eqs. (4.13) and (4.58) we obtain

$$[N_Y,\mathring{\alpha}_m^\dagger]=[N_Y,\mathring{\beta}_m^\dagger]=0,\qquad \text{and H.c. eqs.,} \tag{4.61}$$

which is a property expected for invariant operators. On the other hand, as one can straightforwardly verify with the help of Eqs. (4.60) and (4.61)

$$[Y^\dagger,\mathring{\alpha}_m^\dagger]=0,\quad [Y^\dagger,\mathring{\beta}_m^\dagger]=0,$$

$$[Y,\mathring{\alpha}_m^\dagger]=-A_m^\dagger,\quad [Y^\dagger,\mathring{\beta}_m]=B_m,\quad \text{and H.c. eqs.,} \tag{4.62}$$

where $A_m^\dagger$ and B_m can be written as

$$A_m^\dagger\equiv P_\Uparrow\mathring{\alpha}_m^\dagger YP_\Uparrow=P_\Uparrow\mathring{\alpha}_m^\dagger Y=\mathring{\alpha}_m^\dagger YP_\Uparrow,$$

$$B_m\equiv P_\Uparrow Y^\dagger\mathring{\beta}_m P_\Uparrow=P_\Uparrow Y^\dagger\mathring{\beta}_m=Y^\dagger\mathring{\beta}_m P_\Uparrow, \tag{4.63}$$

and $P_\Uparrow$, which is defined by Eq. (4.19), represents the projector to the subspace of aligned quasispin states. The second line of Eq. (4.62) shows that the quasifermions do not exactly commute with both Y and $Y^\dagger$, as one would expect of strictly invariant operators. However, the quasifermions do commute with both Y and $Y^\dagger$ on the fermion subspace that *excludes* the aligned states. Moreover, the ε-*expansions* of the operators $A_m^\dagger$ and B_m *vanish* since they are proportional to $P_\Uparrow$, which has a vanishing expansion, as discussed in Sec. IV A. Therefore, the quasifermions are invariants in the sense of Sec. III D. What is most important, the quasifermions are strictly invariant in the principal subspace as we proceed to show.

First, we note the following relations:

$$\{b^\dagger[Y,\mathring{\alpha}_m^\dagger]\}_{P_0}=0,\quad \{[Y^\dagger,\mathring{\beta}_m]b\}_{P_0}=0,\quad \text{and H.c. eqs.} \tag{4.64}$$

The proof is similar to that of Eq. (4.35): since the commutators are proportional to $P_\Uparrow$ and aligned vectors in $\mathcal{H}_0$ have zero bosons [$n=\nu$ in Eq. (4.31)], the projection P_0 must give a vanishing result. Now, from Eqs. (2.16), (3.2), and (4.62), we obtain the differential equations

$$\frac{d\mathring{\alpha}_m^\dagger(\theta)}{d\theta}=b^\dagger(\theta)[Y(\theta),\mathring{\alpha}_m^\dagger(\theta)],$$

$$\frac{d\mathring{\beta}_m(\theta)}{d\theta}=[\mathring{\beta}_m(\theta),Y^\dagger(\theta)]b(\theta),\quad \text{and H.c. eqs.} \tag{4.65}$$

The projection of these equations into the principal subspace can then be immediately evaluated with the aid of Eqs. (4.64) as follows:

$$\frac{d[\mathring{\alpha}_m^\dagger(\theta)]_{P_0}}{d\theta}=\{b^\dagger(\theta)[Y(\theta),\mathring{\alpha}_m^\dagger(\theta)]\}_{P_0}=0,$$

$$\frac{d[\mathring{\beta}_m(\theta)]_{P_0}}{d\theta}=\{[\mathring{\beta}_m(\theta),Y^\dagger(\theta)]b(\theta)\}_{P_0}=0,\quad \text{and H.c. eqs.} \tag{4.66}$$

Therefore,

$$[\mathring{\alpha}^\dagger_m(\theta)]_{P_0}=\mathring{\alpha}^\dagger_m,\quad [\mathring{\beta}_m(\theta)]_{P_0}=\mathring{\beta}_m,\quad \text{and H.c. eqs.,} \tag{4.67}$$

and $\mathring{\alpha}^\dagger_m$, $\mathring{\beta}_m$ are invariant in the principal subspace.

The stage is now set for the transformation of the single-fermion operators. Thus, transformation of Eqs. (4.59), taking into account the invariance of $\mathring{\alpha}^\dagger_m$, $\mathring{\beta}_m$, Eq. (4.38), and that the operators $1/\sqrt{2\hat{J}+1}$ and $\sqrt{2\hat{J}-N_Y}$ have the property (4.43), yields the result

$$\begin{aligned}[\alpha^\dagger_m(\theta)]_{P_0}&=\frac{1}{\sqrt{2\hat{J}_{P_0}+1}}\mathring{\alpha}^\dagger_m\sqrt{2\hat{J}_{P_0}-Y^\dagger_{P_0}(\theta)Y_{P_0}(\theta)}\\&\quad+Y^\dagger_{P_0}(\theta)\mathring{\beta}_m\frac{1}{\sqrt{2\hat{J}_{P_0}+1}},\\ [\beta_m(\theta)]_{P_0}&=-\frac{1}{\sqrt{2\hat{J}_{P_0}+1}}Y_{P_0}(\theta)\mathring{\alpha}^\dagger_m\\&\quad+\sqrt{2\hat{J}_{P_0}-Y^\dagger_{P_0}(\theta)Y_{P_0}(\theta)}\mathring{\beta}_m\\&\quad\times\frac{1}{\sqrt{2\hat{J}_{P_0}+1}}\ (\text{and H.c. eqs.}).\end{aligned} \tag{4.68}$$

Setting $\theta=\pi/2$, taking into account Eqs. (4.38), (4.44), and (4.47), and, as before, replacing the projector P_0 by P_{phys}, one obtains the final images in the following form:

$$\begin{aligned}(\overline{\alpha^\dagger_m})_{P_{\text{phys}}}&=P_{\text{phys}}\left(\frac{1}{\sqrt{2\hat{J}+1}}\mathring{\alpha}^\dagger_m\sqrt{2\hat{J}-N_b}+b^\dagger\mathring{\beta}_m\frac{1}{\sqrt{2\hat{J}+1}}\right),\\ (\overline{\beta_m})_{P_{\text{phys}}}&=\left(-\frac{1}{\sqrt{2\hat{J}+1}}b\,\mathring{\alpha}^\dagger_m\right.\\&\quad\left.+\sqrt{2\hat{J}-N_b}\mathring{\beta}_m\frac{1}{\sqrt{2\hat{J}+1}}\right)P_{\text{phys}},\quad \text{and H.c. eqs.}\end{aligned} \tag{4.69}$$

In fact, this result, which is the so-called *quantized Bogoliubov transformation* (QBT), formally agrees with that of Refs. [12,9]. Actually, to prove that the expressions in parentheses are identical to those of the earlier papers, it must be shown that the algebraic properties of the quasifermions are the same, which is the next task.

The *anticommutation* relations of the quasifermions can be obtained by a straightforward (though tedious) exercise using the definitions (4.58). One finds the results

$$\begin{aligned}\{\mathring{\alpha}_m,\mathring{\alpha}^\dagger_{m'}\}&=\delta_{mm'}-\frac{1}{2\hat{J}+1}\mathring{\beta}^\dagger_m\mathring{\beta}_{m'}\\&\quad+\left(\mathring{\alpha}^\dagger_{m'}\mathring{\alpha}_m+\frac{1}{2\hat{J}+1}\mathring{\beta}^\dagger_m\mathring{\beta}_{m'}-\delta_{mm'}\right)P_{\Uparrow},\\ \{\mathring{\beta}_m,\mathring{\alpha}^\dagger_{m'}\}&=\delta_{mm'}-\frac{1}{2\hat{J}+1}\mathring{\alpha}^\dagger_m\mathring{\alpha}_{m'}\\&\quad+\left(\mathring{\beta}^\dagger_{m'}\mathring{\beta}_m+\frac{1}{2\hat{J}+1}\mathring{\alpha}^\dagger_m\mathring{\alpha}_{m'}-\delta_{mm'}\right)P_{\Uparrow},\\ \{\mathring{\beta}_{m'},\mathring{\alpha}^\dagger_m\}&=-\frac{1}{2\hat{J}+1}\mathring{\alpha}^\dagger_{m'}\mathring{\beta}_m\\&\quad-\left(\mathring{\beta}_{m'}\mathring{\alpha}^\dagger_m+\frac{1}{2\hat{J}+1}\mathring{\alpha}^\dagger_{m'}\mathring{\beta}_m\right)P_{\Uparrow},\\ &\text{and H.c. eq.,}\\ \{\mathring{\alpha}_m,\mathring{\alpha}_{m'}\}&=0,\quad \{\mathring{\beta}_m,\mathring{\beta}_{m'}\}=0,\\ \{\mathring{\alpha}_m,\mathring{\beta}_{m'}\}&=0,\quad \text{and H.c. eqs.}\end{aligned} \tag{4.70}$$

These anticommutators coincide with those of the quasifermions of Refs. [9,12] only if the terms proportional to the operator $P_{\Uparrow}$ were to vanish. Now, from the definition of the physical subspace given by Eq. (3.24) it is easily seen that

$$P_{\text{phys}}P_{\Uparrow}P_{\text{phys}}=P_{\text{phys}}P_{\Uparrow}=P_{\Uparrow}P_{\text{phys}}=0. \tag{4.71}$$

Therefore, in the physical subspace it is justified to let $P_{\Uparrow}\to 0$ in Eq. (4.70) so that, in fact, the transformation (4.69) entirely agrees with the earlier references. Note that the final anticommutation relations deviate from those of fermions, which is why the operators in question are called quasifermions.

Another property of the quasifermions is expressed by Eqs. (4.53) and (4.54) of the previous subsection. It is a straightforward exercise to prove Eq. (4.53) by inserting the definitions of the quasifermions (4.58) on the right-hand side of Eq. (4.54), making use of the definition of J_0 [Eq. (2.9)] and noting that the result is the same as $\Omega-2\hat{J}$.

Since the aim of this paper is to express all final images in terms of boson and fermion operators, not quasifermions, we must go a little further. Equations (4.59) can be inverted to express the quasifermions in terms of the fermions, and the result substituted in Eqs. (4.68) and rearranged to give the result[3]

$$[\alpha_m^\dagger(\theta)]_{P_0}=P_0\frac{1}{2\hat{J}+1}[(\sqrt{2\hat{J}-\mathfrak{N}_Y(\theta)+1}\sqrt{2\hat{J}-N_Y+1}+Y^\dagger(\theta)Y)\alpha_m^\dagger+(Y^\dagger(\theta)\sqrt{2\hat{J}-N_Y}-\sqrt{2\hat{J}-\mathfrak{N}_Y(\theta)+1}Y^\dagger)\beta_m],$$

$$[\beta_m(\theta)]_{P_0}=\frac{1}{2\hat{J}+1}[(\sqrt{2\hat{J}-\mathfrak{N}_Y(\theta)}Y-Y(\theta)\sqrt{2\hat{J}-N_Y+1})\alpha_m^\dagger+(\sqrt{2\hat{J}-\mathfrak{N}_Y(\theta)}\sqrt{2\hat{J}-N_Y}+Y(\theta)Y^\dagger)\beta_m]P_0,$$

$$[\beta_m^\dagger(\theta)]_{P_0}=P_0\frac{1}{2\hat{J}+1}[(\sqrt{2\hat{J}-\mathfrak{N}_Y(\theta)+1}\sqrt{2\hat{J}-N_Y+1}+Y^\dagger(\theta)Y)\beta_m^\dagger-(Y^\dagger(\theta)\sqrt{2\hat{J}-N_Y}-\sqrt{2\hat{J}-\mathfrak{N}_Y(\theta)+1}Y^\dagger)\alpha_m],$$

$$[\alpha_m(\theta)]_{P_0}=\frac{1}{2\hat{J}+1}[-(\sqrt{2\hat{J}-\mathfrak{N}_Y(\theta)}Y-Y(\theta)\sqrt{2\hat{J}-N_Y+1})\beta_m^\dagger+(\sqrt{2\hat{J}-\mathfrak{N}_Y(\theta)}\sqrt{2\hat{J}-N_Y}+Y(\theta)Y^\dagger)\alpha_m]P_0, \tag{4.72}$$

where $\mathfrak{N}_Y(\theta)$ is defined by Eq. (4.47). It was verified that the ε expansions of these expressions coincide with Eqs. (3.30). Setting $\theta=\pi/2$ and noting that $2\hat{J}-N_Y=\hat{J}-J_0$ provides the final images, equivalent to Eqs. (4.69):

$$(\overline{\alpha_m^\dagger})_{P_{\rm phys}}=P_{\rm phys}\frac{1}{2\hat{J}+1}[(\sqrt{2\hat{J}-N_b+1}\sqrt{\hat{J}-J_0+1}+b^\dagger Y)\times\alpha_m^\dagger+(b^\dagger\sqrt{\hat{J}-J_0}-\sqrt{2\hat{J}-N_b+1}Y^\dagger)\beta_m],$$

$$(\overline{\beta_m})_{P_{\rm phys}}=\frac{1}{2\hat{J}+1}[(\sqrt{2\hat{J}-N_b}Y-b\sqrt{\hat{J}-J_0+1})\alpha_m^\dagger+(\sqrt{2\hat{J}-N_b}\sqrt{\hat{J}-J_0}+bY^\dagger)\beta_m]P_{\rm phys},$$

$$(\overline{\beta_m^\dagger})_{P_{\rm phys}}=P_{\rm phys}\frac{1}{2\hat{J}+1}[(\sqrt{2\hat{J}-N_b+1}\sqrt{\hat{J}-J_0+1}+b^\dagger Y)\times\beta_m^\dagger-(b^\dagger\sqrt{\hat{J}-J_0}-\sqrt{2\hat{J}-N_b+1}Y^\dagger)\alpha_m],$$

$$(\overline{\alpha_m})_{P_{\rm phys}}=\frac{1}{2\hat{J}+1}[-(\sqrt{2\hat{J}-N_b}Y-b\sqrt{\hat{J}-J_0+1})\beta_m^\dagger+(\sqrt{2\hat{J}-N_b}\sqrt{\hat{J}-J_0}+bY^\dagger)\alpha_m]P_{\rm phys}, \tag{4.73}$$

where the operators Y, $Y^\dagger$ are defined by Eq. (4.20). Apart from the projector, this represents the closed summation of the series (3.31).

Having completed the task of mapping all elementary operators into the physical subspace $\mathcal{H}_{\rm phys}$, we conclude this section with a brief discussion of $\mathcal{H}_{\rm phys}$ itself. By definition, $\mathcal{H}_{\rm phys}$ is spanned by the image vectors $\overline{|\gamma;J,M\rangle}\equiv U(\pi/2)|\gamma;J,M\rangle$. Since the base vectors satisfy $Y|\gamma;J,-J\rangle=b|\gamma;J,-J\rangle=0$, it follows that $S|\gamma;J,-J\rangle=0$, implying the invariance $\overline{|\gamma;J,-J\rangle}=|\gamma;J,-J\rangle$. Then from Eq. (4.52), the transformation of a general fermion basis vector $|\gamma;J,M=-J+n\rangle$ is given by

$$\overline{|\gamma;J,-J+n\rangle}=\sqrt{\frac{(2J-n)!}{(2J)!n!}}(\overline{J_+})^n|\gamma;J,-J\rangle=\sqrt{\frac{(2J-n)!}{(2J)!n!}}[b^\dagger(2\hat{J}-N_b)]^n|\gamma;J,-J\rangle=\frac{1}{\sqrt{n!}}(b^\dagger)^n|\gamma;J,-J\rangle,\quad n=0,1,\ldots,2J. \tag{4.74}$$

It is then immediately obvious that the property (3.24) is indeed satisfied on $\mathcal{H}_{\rm phys}$. The vectors (4.74) correspond to the subset of Eq. (4.31) with $\nu=0$. The base states themselves can be generated from the uncorrelated ground state $|0;\Omega/2,-\Omega/2\rangle$ by the action of the quasifermions. For example, a base state with $J=-M=(\Omega-k-l)/2$ is given by

$$\left|m_1\cdots m_k,m_1'^{-1}\cdots m_l'^{-1};\frac{1}{2}(\Omega-k-l),-\frac{1}{2}(\Omega-k-l)\right\rangle=\mathring{\alpha}_{m_1}^\dagger\cdots\mathring{\alpha}_{m_k}^\dagger\mathring{\beta}_{m_1'}^\dagger\cdots\mathring{\beta}_{m_l'}^\dagger\left|0;\frac{1}{2}\Omega,-\frac{1}{2}\Omega\right\rangle, \tag{4.75}$$

which follows from Eqs. (4.60). Such a state can be further expressed in terms of the fermion operators, which can be most conveniently done using the inverse of Eqs. (4.59).

E. Summary and conclusions

The aim of this work is to develop a viable theoretical framework in which prescribed collective excitation modes are selectively treated in terms of bosons while all other degrees of freedom retain their fermionic character. This is in

[3]Note that the expressions have been ordered so that the fermion operators always appear on the far right.

contrast to traditional formalisms in which all degrees of freedom are indiscriminately bosonized. The motivation is based on the belief that the former approach should lead to faster convergence. Our starting point was the improvement of the boson-fermion expansion theory (BFET) of Refs. [3–5], and the final result is a substantially modified approach that we call the selective unitary bosonization method (SUBM). In this paper, we limited the development to SU(2) quasispin models, including the model of Lipkin, Meshkov, and Glick (LMG) and the single-j pairing model. Since these exactly soluble models have already been subjected to almost every applicable many-body approximation, they provide a convenient basis for assessing the validity of new formalisms. A further convenience is the existence of a decoupled collective subspace residing in a single SU(2) irrep, which eliminates ambiguities in the definition of the bosonized collective mode. To be sure, this leaves open for future investigations the question of how well our approach works for more realistic models, in which such a convenience does not exist.

The SUBM belongs in the category of auxiliary-variables techniques. The encompassing Hilbert space is the tensor product of a boson space and the original many-body fermion space. The physics is transferred from the fermion space to a subspace of the boson-fermion space by means of unitary transformations in such a way that the collective mode is selectively preempted by a boson. The key step is the recognition that in the generator of the transformation $S = Y^\dagger b - b^\dagger Y$, where b, $b^\dagger$ are perfect boson operators, the fermionic operators Y, $Y^\dagger$ must also obey boson commutation rules on a suitable *subspace* (called the principal subspace), i.e., they are truncated boson operators. Indeed, the requirement that Y coincide in lowest order with the phonon X, together with the commutation rule $[Y,Y^\dagger]=1$, uniquely determines the expansion of Y in the small parameter $\varepsilon \equiv (1/\Omega)^{1/2}$, and ultimately the expansions of all physical operators. A more detailed analysis of the Hilbert space leads to the identification of Y in closed form and to the demarcation of its domain as a truncated boson annihilation operator, i.e., the principal subspace. Thenceforth, straightforward analysis leads to the derivation of all physical operators in closed form for the SU(2) case. For this case, we have both an expansion theory and also the closed summation of the expansions. In more realistic applications, the latter may not always be achievable, but that remains a problem for future investigation. The expansion theory, which is what one actually needs for numerical computations, can be readily extended to realistic models.

Our final closed-form mappings of the SU(2) generators and also the single-nucleon creation and destruction operators, which are SU(2) tensors, formally agree with the corresponding results of Refs. [9,12], obtained by different means. This testifies to the validity of our approach. In the earlier work, the final mappings were given in terms of bosons and quasifermions (referred to as ''ideal quasiparticles''), obeying complicated anticommutation rules. In the present approach, the quasifermions can be expressed in terms of the original fermion operators, providing a true boson-fermion mapping. In Ref. [12], a secondary derivation was sketched, based on an auxiliary-variables method originally introduced in Ref. [13]. This method, which also involves unitary transformations on a boson-fermion space, is similar to the SUBM, but with one critical difference. In order to justify the commutation rule $[Y,Y^\dagger]=1$ for fermionic operators on a finite-dimensional SU(2) space, the authors attempt to formally extend the space to infinite dimensions. However, as previously remarked, this artificial construction, which obscures the relation to the original many-body problem, seems to amount to little more than a license to use the boson commutation rule for fermionic operators. Moreover, the generalization of this trick to the realistic case is not at all clear. The present work, on the other hand, takes the opposite tack: instead of extending the boson-fermion space, it merely selects a subspace—the principal subspace—on which the relation $[Y,Y^\dagger]=1$ is valid. In this way, one is automatically led to the correct projection operators required in the closed expressions. Thus, a major weakness in the foundation of the auxiliary variables approach has been repaired.

Any method utilizing auxiliary variables requires subsidiary conditions. In our method, these are simply expressed by Eq. (3.24) or Eq. (3.25). Since the final unitary transformation of the fermion basis generates a subspace of the boson-fermion space within which the subsidiary conditions are automatically satisfied, it is only necessary to diagonalize the Hamiltonian (or its expansion to a given order) within this subspace to insure that subsidiary conditions are fulfilled. The indiscriminate use of a larger basis that includes vectors violating the subsidiary conditions may lead to poorer numerical results. Such studies will be presented elsewhere.

APPENDIX: INVARIANT OPERATORS

Let $\mathcal{O}$ be an arbitrary operator defined on the Hilbert space of any system having amongst its degrees of freedom a boson degree of freedom, represented by the operators Y, $Y^\dagger$ with commutator

$$[Y,Y^\dagger]=1. \tag{A1}$$

Let $\mathcal{O}(k,l)$ be the iterated commutator

$$\mathcal{O}(k,l)\equiv\underbrace{[Y,[Y,\cdots[Y,}_{k\ \text{times}}\underbrace{[Y^\dagger,[Y^\dagger,\cdots[Y^\dagger}_{l\ \text{times}},\mathcal{O}]]\cdots], \tag{A2}$$

with $\mathcal{O}(0,0)\equiv\mathcal{O}$. From this definition, it follows immediately that

$$[Y,\mathcal{O}(k,l)]=\mathcal{O}(k+1,l). \tag{A3}$$

The corresponding relation,

$$[Y^\dagger,\mathcal{O}(k,l)]=\mathcal{O}(k,l+1), \tag{A4}$$

is also true but requires a bit more work. From the Jacobi identity together with Eq. (A1), one obtains

$$[Y^\dagger,\mathcal{O}(k,l)]=[Y^\dagger,[Y,\mathcal{O}(k-1,l)]]=[Y,[Y^\dagger,\mathcal{O}(k-1,l)]]. \tag{A5}$$

Application of this rule k times then gives

$$[Y^\dagger,\mathcal{O}(k,l)]=\underbrace{[Y,[Y,\cdots[Y}_{k\ \text{times}},[Y^\dagger,\mathcal{O}(0,l)]]\cdots]=\underbrace{[Y,[Y,\cdots[Y}_{k\ \text{times}},\mathcal{O}(0,l+1)]\cdots=\mathcal{O}(k,l+1)]. \tag{A6}$$

Next, corresponding to the operator $\mathcal{O}$, define the operator $\mathring{\mathcal{O}}$ in terms of the infinite series

$$\mathring{\mathcal{O}}\equiv\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{1}{k!l!}(-Y^\dagger)^k\mathcal{O}(k,l)Y^l. \tag{A7}$$

We show that $\mathring{\mathcal{O}}$ satisfies the conditions

$$[Y,\mathring{\mathcal{O}}]=0,\quad [Y^\dagger,\mathring{\mathcal{O}}]=0. \tag{A8}$$

Such operators will be called *invariants*. Using Eqs. (A1) and (A3), one calculates

$$\begin{aligned}[Y,\mathring{\mathcal{O}}]&=\sum_{k=1}^{\infty}\sum_{l=0}^{\infty}\frac{-1}{(k-1)!l!}(-Y^\dagger)^{k-1}\mathcal{O}(k,l)Y^l+\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{1}{k!l!}(-Y^\dagger)^k\mathcal{O}(k+1,l)Y^l\\&=-\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{1}{k!l!}(-Y^\dagger)^k\mathcal{O}(k+1,l)Y^l+\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{1}{k!l!}(-Y^\dagger)^k\mathcal{O}(k+1,l)Y^l=0,\end{aligned} \tag{A9}$$

after index relabeling in the first sum. The demonstration that $[Y^\dagger,\mathring{\mathcal{O}}]=0$ is entirely parallel, with Eq. (A4) replacing Eq. (A3).

Let us assume that $\mathcal{O}$ can be expanded in terms of the boson degree of freedom, i.e.,

$$\mathcal{O}=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\Omega_{m,n}}{m!n!}(Y^\dagger)^m(-Y)^n, \tag{A10}$$

where the coefficients $\Omega_{m,n}$ are independent of the boson degree of freedom and are therefore invariants, satisfying

$$[Y,\Omega_{m,n}]=0,\quad [Y^\dagger,\Omega_{m,n}]=0. \tag{A11}$$

With the aid of Eqs. (A10) and (A11), the multiple commutator (A2) can be evaluated as follows:

$$\mathcal{O}(k,l)=\sum_{m=k}^{\infty}\sum_{n=l}^{\infty}\frac{(-1)^{n+l}\Omega_{m,n}}{(m-k)!(n-l)!}(Y^\dagger)^{m-k}(Y)^{n-l}=\sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty}\frac{\Omega_{\mu+k,\nu+l}}{\mu!\nu!}(Y^\dagger)^{\mu}(-Y)^{\nu}, \tag{A12}$$

where the change of index $\mu=m-k,\ \nu=n-l$ was used in the second step. This sets the stage to find $\Omega_{m,n}$ in terms of $\mathcal{O}$, Y, and $Y^\dagger$. Since the $\Omega_{m,n}$ are invariants, it is interesting to find the relation to the invariants $\mathring{\mathcal{O}}(k,l)$ based on the operators $\mathcal{O}(k,l)$ as follows:

$$\mathring{\mathcal{O}}(k,l)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{m!n!}(-Y^{\dagger})^{m}\underbrace{[Y,[Y,\cdots[Y,}_{m\ \text{times}}\underbrace{[Y^{\dagger},[Y^{\dagger},\cdots[Y^{\dagger}}_{n\ \text{times}},\mathcal{O}(k,l)]]\cdots]Y^{n}$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{\mu=m}^{\infty}\sum_{\nu=n}^{\infty}\frac{\Omega_{\mu+k,\nu+l}(-1)^{m+n+\nu}}{m!n!(\mu-m)!(\nu-n)!}(Y^{\dagger})^{\mu}(Y)^{\nu}$$

$$=\sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty}\sum_{m=0}^{\mu}\sum_{n=0}^{\nu}\frac{\Omega_{\mu+k,\nu+l}(-1)^{m+n+\nu}}{\mu!\nu!}\binom{\mu}{m}\binom{\nu}{n}(Y^{\dagger})^{\mu}(Y)^{\nu}$$

$$=\sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty}\frac{\Omega_{\mu+k,\nu+l}(-1)^{\nu}}{\mu!\nu!}(Y^{\dagger})^{\mu}(Y)^{\nu}\sum_{m=0}^{\mu}(-1)^{m}\binom{\mu}{m}\sum_{n=0}^{\nu}(-1)^{n}\binom{\nu}{n}$$

$$=\sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty}\frac{\Omega_{\mu+k,\nu+l}(-1)^{\nu}}{\mu!\nu!}(Y^{\dagger})^{\mu}(Y)^{\nu}\delta_{\mu,0}\delta_{\nu,0}=\Omega_{k,l}, \tag{A13}$$

where the following identity for binomial coefficients was used in the last step:

$$\sum_{m=0}^{\mu}(-1)^{m}\binom{\mu}{m}=\delta_{\mu,0}. \tag{A14}$$

This proves that in fact $\Omega_{k,l}=\mathring{\mathcal{O}}(k,l)$, so that the decomposition (A10) of $\mathcal{O}$ is finally given by

$$\mathcal{O}=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{m!n!}\mathring{\mathcal{O}}(m,n)(Y^{\dagger})^{m}(-Y)^{n}. \tag{A15}$$

This relation should be understood as a formal identity with no guarantees of convergence.

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