

Relativistic N -body problem in a separable two-body basis

Cheuk-Yin Wong

*Physics Division, Oak Ridge National Laboratory, Oak Ridge, Tennessee 37831
and Institute of Physics and Applied Physics, Yonsei University, Seoul 120-749, Korea*

Horace W. Crater

Department of Physics, University of Tennessee Space Institute, Tullahoma, Tennessee 37388

(Received 3 October 2000; published 23 March 2001)

We use Dirac's constraint dynamics to obtain a Hamiltonian formulation of the relativistic N -body problem in a separable two-body basis in which the particles interact pairwise through scalar and vector interactions. The resultant N -body Hamiltonian is relativistically covariant. It can be easily separated in terms of the center of mass and the relative motion of any two-body subsystem. It can also be separated into an unperturbed Hamiltonian with a residual interaction. In a system of two-body composite particles, the solutions of the unperturbed Hamiltonian are relativistic two-body internal states, each of which can be obtained by solving a relativistic Schrödinger-like equation. The resultant two-body wave functions can be used as basis states to evaluate reaction matrix elements in the general N -body problem. We prove a relativistic version of the post-prior equivalence which guarantees a unique evaluation of the reaction matrix element, independent of the ways of separating the Hamiltonian into unperturbed and residual interactions. Since an arbitrary reaction matrix element involves composite particles in motion, we show explicitly how such matrix elements can be evaluated in terms of the wave functions of the composite particles and the relevant Lorentz transformations.

DOI: 10.1103/PhysRevC.63.044907

PACS number(s): 24.10.Jv, 24.10.Cn, 21.30.Fe, 25.80.-e

I. INTRODUCTION

The theoretical description of the relativistic N -body problem is an interesting but perplexing problem in nuclear and particle physics. It involves the possibility of relativistic bound states of composite particles on the one hand and the reaction matrix elements between these particles at high energies on the other hand. The relativistic N -body problem is complicated by the fact that the relativistic bound state problem is basically nonperturbative in nature and cannot be solved by the conventional perturbative quantum field theory. One also needs to describe the reaction process relativistically by making use of the results from the bound states.

Much progress has been made in the study of relativistic two-body bound state problems [1–17]. In the 1970s, several authors used Dirac's constraint mechanics [1] to attack the relativistic two-body problem at its classical roots [2], successfully evading the so-called "no interaction theorem" [3]. The quantum version of the constraint approach was extended to pairs of spin one-half particles. The results were two-body quantum bound state equations that correct the defects in the Breit equation, correct the defects in the ladder approximation to the Bethe-Salpeter equation, and control covariantly the relative time and energy variables [4]. Those bound state equations for fermions are the two-body Dirac equations of constraint dynamics, which we shall also call the constraint equations [5]. They possess a number of important desirable features, some of which are unique. For example, they remove the deficiencies of earlier approaches in which the spin dependence of the potentials is a patchwork of semirelativistic corrections determined by field theory [6]. In contrast, the spin dependence in constraint dynamics is determined naturally by the Dirac equation struc-

ture of each of the constraint equations together with the assumed covariant interactions [5,7]. The equations of constraint dynamics are manifestly covariant while yielding simple three-dimensional Schrödinger-type equations, like those of their nonrelativistic counterparts [5,7]. (This particular feature will ultimately be of crucial importance in the discussions given in this paper.) These constraint equations have passed numerous tests showing that they reproduce correct QED perturbative results when solved nonperturbatively [7]. In addition, the Dirac forms of these equations automatically make it unnecessary to introduce *ad hoc* cutoff parameters, which are needed in most other approaches [6] to regularize singular potentials. The relativistic potentials appearing in the constraint equations are related directly to the interactions of perturbative quantum field theory [8]. In nonperturbative QCD as applied to meson spectroscopy, they may be introduced phenomenologically [10] and can be regarded as an anticipation of potentials that may eventually emerge from lattice gauge theory.

In the present paper we will extend the constraint equations beyond the scope of the two-body problem and will, for simplicity, limit ourselves to the constraint description of spinless particles. We first summarize the constraint approach for two spinless particles. For each particle, one ascribes a generalized mass shell constraint which includes the interaction. The constraints must be consistent with each other and this in turn restricts the dependence of the interactions on the relative coordinates, eliminating both the relative time and energy variables in the c.m. system. The resultant equations correspond to a Bethe-Salpeter equation whose kernel and Green's functions are constrained by the requirement of $P \cdot q = 0$, where P and q are the total and relative momenta of the two-body system [8]. In particular, from the Bethe-Salpeter equation with this constraint, one can derive

[10,7] the “quasipotential equation” of Todorov [9], which is a Schrödinger-like inhomogeneous integral equation where the quasipotential Φ is related to the scattering amplitude in perturbative quantum field theory. Other methods of reducing the Bethe-Salpeter equation have also been suggested [12–16]. The application of the harmonic oscillator in a relativistically covariant formalism of the Poincaré group to hadronic bound states was also considered in [17].

At present, the relativistic treatment of the N -body problem and the reaction of composite particles at high energies have not advanced as much as in the relativistic treatment of two-body bound states. An investigation of the relativistic N -body problem was previously carried out by Sazdjian [11]. In contrast, the nonrelativistic N -body problem and the nonrelativistic description of the reaction of composite particles have been well developed [18–21]. One has, for example, the distorted-wave Born approximation method in nuclear reactions [18] and the quark-interchange model in hadron reactions [19] for reactions between composite particles. One first solves for the wave functions and determines the interaction between the constituents using the energy levels of bound states. Then one uses the same set of interactions and wave functions to calculate the reaction matrix elements for the reaction of composite particles. There is also the potentially different values of the post or prior forms of the reaction matrix element, which distinguish whether the interaction occurs before the rearrangement or after the rearrangement of the constituents. The post-prior equality of the reaction matrix elements is attained when the interaction and wave functions that are used in the calculation of the overlap integral are the same as the interaction and the wave functions obtained in the bound state analysis.

The formalism we shall develop can be applied to many processes of interest. For example, in high-energy heavy-ion collisions, the investigation of the dynamics and the properties of the produced hadron matter involve the reaction cross sections and the reaction matrix elements between the produced hadrons at relativistic energies. Most of these cross sections and reaction matrix elements cannot be measured experimentally. A reliable theoretical relativistic model which describes the N -body problem of the constituents is needed for their evaluation. Although the nonrelativistic reaction model of Barnes and Swanson has been used successfully to calculate hadron-hadron reaction cross sections [19,22–28], it is necessary to generalize the nonrelativistic reaction model to study reactions between composite particles at relativistic energies. It is also of interest to study the relativistic N -body problem in order to investigate an assembly of composite particles and their clustering or molecular states.

We shall take advantage of previous advances in the understanding of the relativistic two-body problem in our present study of the relativistic N -body problem. In this work, we specialize in N -body dynamics with just pairwise interactions between particles. We shall write down the Hamiltonian formulation of the relativistic N -body problem which allows an easy separation in terms of the center-of-mass and the relative motion for any two-body system and a simple separation of the unperturbed Hamiltonian and the

residual interaction. We can then use the relativistic two-body bound states as basis states for the investigation of the relativistic N -body problem. We can construct a proof of the “post-prior” equivalence in relativistic dynamics, which guarantees that the reaction matrix element is independent of the different ways of partitioning the unperturbed Hamiltonian and the residual interaction from the N -body Hamiltonian.

This paper is organized as follows. In Sec. II we review important aspects of the constraint approach and introduce the relativistic two-body Hamiltonian and bound state eigenvalue equation. We show that the relativistic two-body solution in the c.m. rest system has the simplicity of its nonrelativistic counterparts. In Sec. III we generalize the two-body Hamiltonian to the case of N particles. We discuss some applications of the present formulation in Sec. IV. We show how to utilize the constituent two-body wave functions for the N -body problem. Because of the analogy to their nonrelativistic counterparts, these states can be used as basis states to evaluate a general reaction matrix element in the general N -body problem. We prove the relativistic version of the post-prior equivalence for the reaction matrix elements. As an illustration and a problem of practical interest, we consider in Sec. V the reaction of four particles in two composite systems where the nonrelativistic treatment has already been formulated by Barnes and Swanson [19]. In Sec. VI, we explicitly obtain the wave function in second-quantized form so as to construct the overlap integral for the reaction matrix element. Such a reaction matrix element involves states of composite particles in motion. Thus we show in Sec. VII explicitly how to evaluate such elements in terms of the wave functions of each composite particle, and develop the relevant Lorentz transformation laws required. Section VIII summarizes and points to future problems.

II. HAMILTONIAN FORMULATION OF THE TWO-BODY PROBLEM FROM CONSTRAINT DYNAMICS

We can formulate the relativistic treatment of the two-body problem for spinless particles [29,30] in a way that has the simplicity of the ordinary nonrelativistic two-body Schrödinger equation and yet maintains relativistic covariance. Including spin and generalizing to different types of interactions can be carried out in a more complete framework [31,4].

For each particle we assume a generalized mass shell constraint of the form

$$\mathcal{H}_i|\psi\rangle=0 \quad \text{for } i=1,2, \quad (2.1)$$

where

$$\mathcal{H}_i=p_i^2-m_i^2-\Phi_i, \quad (2.2)$$

and Φ_1 and Φ_2 are two-body interactions dependent on x_{12} . One constructs the total Hamiltonian \mathcal{H} from these constraints by

$$\mathcal{H}=\lambda_1\mathcal{H}_1+\lambda_2\mathcal{H}_2 \quad (2.3)$$

(with λ_i as Lagrange multipliers). In order that each of these constraints be conserved in time we must have

$$[\mathcal{H}_i, \mathcal{H}]|\psi\rangle = i \frac{d\mathcal{H}_i}{d\tau}|\psi\rangle = 0. \quad (2.4)$$

As a consequence, we have

$$\begin{aligned} & [\mathcal{H}_i, \lambda_1 \mathcal{H}_1 + \lambda_2 \mathcal{H}_2]|\psi\rangle \\ &= \{[\mathcal{H}_i, \lambda_1] \mathcal{H}_1 + \lambda_1 [\mathcal{H}_i, \mathcal{H}_1] \\ &+ [\mathcal{H}_i, \lambda_2] \mathcal{H}_2 + \lambda_2 [\mathcal{H}_i, \mathcal{H}_2]\}|\psi\rangle = 0. \end{aligned} \quad (2.5)$$

Using Eq. (2.1), the above equation leads to the compatibility condition between the two constraints,

$$[\mathcal{H}_1, \mathcal{H}_2]|\psi\rangle = 0. \quad (2.6)$$

Since the mass commutes with the operators, this implies

$$([p_1^2, \Phi_2] + [\Phi_1, p_2^2] + [\Phi_1, \Phi_2])|\psi\rangle = 0. \quad (2.7)$$

The simplest way to satisfy the above equation is to take

$$\Phi_1 = \Phi_2 = \Phi(x_\perp), \quad (2.8)$$

which is a kind of relativistic Newton's third law. Here, the transverse coordinate is defined by

$$x_{\nu\perp} = x_{12}^\mu (\eta_{\mu\nu} - P_\mu P_\nu / P^2), \quad (2.9)$$

where P is the total momentum

$$P = p_1 + p_2. \quad (2.10)$$

The choice of the two-body potential (2.8) leads to

$$[\mathcal{H}_1, \mathcal{H}_2]|\psi\rangle = 2P \cdot \partial_{x_{12}} \Phi(x_\perp)|\psi\rangle = 0, \quad (2.11)$$

and the compatibility condition (2.6) is satisfied.

The two-body Hamiltonian \mathcal{H} determines the dynamics of the two-body system. Its equation of motion is

$$\mathcal{H}|\psi\rangle = 0. \quad (2.12)$$

This equation describes both the center-of-mass motion and the internal relative motion. To characterize the center-of-mass motion, we note that since the potential Φ depends only on the difference of the two coordinates we have

$$[P, \mathcal{H}]|\psi\rangle = 0. \quad (2.13)$$

(This does not require that $[P, \lambda_i] = 0$ since the $\mathcal{H}_i|\psi\rangle = 0$.) Thus, P is a constant of motion and we can take $|\psi\rangle$ to be an eigenstate characterized by a total momentum P .

To separate out the internal relative motion from the center-of-mass motion, we introduce the relative momentum q defined by

$$p_1 = \frac{p_1 \cdot P}{P^2} P + q, \quad (2.14)$$

$$p_2 = \frac{p_2 \cdot P}{P^2} P - q, \quad (2.15)$$

where the first term on the right-hand side of the above two equations is the projection of each momentum onto the total momentum. The above definition of the relative momentum guarantees the orthogonality of the total momentum and the relative momentum,

$$P \cdot q = 0, \quad (2.16)$$

which follows from taking the scalar product of either equation with P . From Eqs. (2.14) and (2.15) this relative momentum can be written in terms of p_1 and p_2 as

$$q = \frac{\varepsilon_2}{\sqrt{P^2}} p_1 - \frac{\varepsilon_1}{\sqrt{P^2}} p_2 \quad (2.17)$$

where

$$\begin{aligned} \varepsilon_1 &= \frac{p_1 \cdot P}{\sqrt{P^2}} = \frac{P^2 + p_1^2 - p_2^2}{2\sqrt{P^2}} \\ \varepsilon_2 &= \frac{p_2 \cdot P}{\sqrt{P^2}} = \frac{P^2 + p_2^2 - p_1^2}{2\sqrt{P^2}} \end{aligned} \quad (2.18)$$

are the projections of the momenta p_1 and p_2 along the direction of the total momentum P . Using Eqs. (2.1) and (2.8) and taking the difference of the two constraints, we obtain

$$(p_1^2 - p_2^2)|\psi\rangle = (m_1^2 - m_2^2)|\psi\rangle. \quad (2.19)$$

Thus on these states $|\psi\rangle$ we have

$$\begin{aligned} \varepsilon_1 &= \frac{P^2 + m_1^2 - m_2^2}{2\sqrt{P^2}} \\ \varepsilon_2 &= \frac{P^2 + m_2^2 - m_1^2}{2\sqrt{P^2}}. \end{aligned} \quad (2.20)$$

Using Eqs. (2.14), (2.15), and Eq. (2.16), we can write \mathcal{H} in terms of P and q :

$$\begin{aligned} \mathcal{H}|\psi\rangle &= \{\lambda_1[\varepsilon_1^2 - m_1^2 + q^2 - \Phi(x_\perp)] \\ &+ \lambda_2[\varepsilon_2^2 - m_2^2 + q^2 - \Phi(x_\perp)]\}|\psi\rangle \\ &= (\lambda_1 + \lambda_2)[b^2(P^2; m_1^2, m_2^2) + q^2 - \Phi(x_\perp)]|\psi\rangle = 0, \end{aligned} \quad (2.21)$$

where

$$\begin{aligned}
b^2(P^2, m_1^2, m_2^2) &= \varepsilon_1^2 - m_1^2 \\
&= \varepsilon_2^2 - m_2^2 \\
&= \frac{1}{4P^2} [P^4 - 2P^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2].
\end{aligned} \tag{2.22}$$

Equation (2.21) contains both the center-of-mass momentum P [through the $b^2(P^2, m_1^2, m_2^2)$] and the relative momentum q . This constraint equation of P and q can then be solved by the method of the separation of variables. That is, we introduce the bound state eigenvalue M to separate Eq. (2.21) into the following two equations for the center-of-mass motion and the internal motion:

$$\{P^2 - M^2\}|\psi\rangle = 0, \tag{2.23}$$

and

$$(\lambda_1 + \lambda_2)\{q^2 - \Phi(x_\perp) + b^2(M^2, m_1^2, m_2^2)\}|\psi\rangle = 0, \tag{2.24}$$

where we have used the first equation on the eigenstate $|\psi\rangle$ so that $b^2(P^2, m_1^2, m_2^2)$ becomes the standard triangle function indicative of the presence of exact relativistic two-body kinematics:

$$b^2(M^2, m_1^2, m_2^2) = \frac{1}{4M^2} \{M^4 - 2M^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2\}. \tag{2.25}$$

The eigenvalue equation, Eq. (2.24), for the relative motion is independent of the Lagrange multipliers. It is nonetheless convenient to choose $\lambda_i = 1/(2m_i)$ so that the resultant Schrödinger equation matches the nonrelativistic two-body Schrödinger equation term by term. Such a choice also helps us obtain useful simplifications in the relativistic N -body problem in later sections. In particular, the N -body Hamiltonian can be easily separated into pairs of two-body Hamiltonians. This separation makes it easy to introduce the unperturbed Hamiltonian and residual interactions.

We note that because of the orthogonality of P and q , we can write Eq. (2.24) in the form

$$\left(\frac{1}{2m_1} + \frac{1}{2m_2}\right)\{q_\perp^2 - \Phi(x_\perp) + b^2(M^2, m_1^2, m_2^2)\}|\psi\rangle = 0, \tag{2.26}$$

where $q_\perp = q - q \cdot P P / P^2 = q$. [Note that if the relative momentum were defined in terms of Eq. (2.20) instead of Eq. (2.18) then we would have

$$q \cdot P |\psi\rangle = 0 \tag{2.27}$$

but not $q \cdot P = 0$ so that $q^2 |\psi\rangle = q_\perp^2 |\psi\rangle$. In either case the coefficients ε_i are invariant and hence Eq. (2.17) the same form regardless of which frame it is evaluated in.]

We show below how the eigenvalue M is related to the eigenvalue obtained in a nonrelativistic Schrödinger equa-

tion. We go to the center-of-momentum system where $q = q_\perp = (0, \mathbf{q})$ and $x_\perp = (0, \mathbf{r})$ (relative energy and time thus being removed from the problem). We then have the equation for the relative motion,

$$\left\{\frac{\mathbf{q}^2}{2\mu} + \frac{\Phi(\mathbf{r})}{2\mu} - \frac{b^2}{2\mu}\right\}|\psi\rangle = 0, \tag{2.28}$$

where μ is the nonrelativistic reduced mass,

$$\mu = \frac{m_1 m_2}{m_1 + m_2}. \tag{2.29}$$

We can cast Eq. (2.28) into the usual form of a nonrelativistic Schrödinger equation. By renaming $\Phi/2\mu$ as V_{12} , and $b^2/2\mu$ as E , Eq. (2.24) becomes

$$\left(\frac{\mathbf{q}^2}{2\mu} + V_{12}\right)|\psi\rangle = E|\psi\rangle. \tag{2.30}$$

The above Schrödinger equation can be solved to give the eigenvalue E . Then, from the equation $b^2(M^2, m_1^2, m_2^2) = 2\mu E$, one can solve for M in terms of E and obtain

$$M = \sqrt{2\mu E + m_1^2} + \sqrt{2\mu E + m_2^2}. \tag{2.31}$$

It is easy to show from this that in the limit of very weak binding, the nonrelativistic limit, we have the familiar result

$$M = m_1 + m_2 + E. \tag{2.32}$$

If one is only interested in the effect of exact two-body relativistic kinematics with V_{12} an energy-independent nonrelativistic potential, the bound state eigenvalue M for the relativistic two-body problem is related to the eigenvalue E of the nonrelativistic problem by Eq. (2.31). It is important to note, however, that the potential V_{12} in relativistic constraint dynamics includes relativistic dynamical corrections as well. These corrections include dependencies of the potential on the c.m. energy M and on the nature of the interaction. For spinless particles interacting by way of a world scalar interaction S , one finds [30,32]

$$V_{12} = \frac{\Phi}{2\mu} = \frac{2m_M S + S^2}{2\mu} \tag{2.33}$$

where

$$m_M = \frac{m_1 m_2}{M}, \tag{2.34}$$

while for (timelike) vector interaction A , one finds [9,30,32]

$$V_{12} = \frac{\Phi}{2\mu} = \frac{2\varepsilon_M A - A^2}{2\mu}, \tag{2.35}$$

where

$$\varepsilon_M = \frac{M^2 - m_1^2 - m_2^2}{2M} \tag{2.36}$$

and for combined spacelike and timelike vector interactions (that reproduce the correct energy spectrum for scalar QED [29])

$$V_{12} = \frac{\Phi}{2\mu} = \frac{2\varepsilon_M A - A^2 + \vec{\nabla}^2 \ln(1 - 2A/M)^{1/2} + [\vec{\nabla} \ln(1 - 2A/M)^{1/2}]^2}{2\mu}. \quad (2.37)$$

The variables m_M and ε_M (which both approach μ in the nonrelativistic limit) were introduced by Todorov [9] in his quasipotential approach and are called the relativistic reduced mass and energy of the fictitious particle of relative motion. In the nonrelativistic limit, Φ approaches $2\mu(S + A)$. In the relativistic case, the dynamical corrections to V_{12} referred to above include both quadratic additions to S and A as well as c.m. energy dependence through m_M and ε_M . This latter point implies that the effective potential V_{12} depends on the eigenvalue E (or M) to be evaluated. One can obtain the mass of the bound state M by an iterative procedure. One starts with an estimated M (or E) value and obtains the potential V_{12} . Equations (2.30) and (2.31) can then be used iteratively to obtain successively improved values of V_{12} and the eigenvalue M (or E).

We note in passing that since

$$b^2 = \varepsilon_M^2 - m_M^2, \quad (2.38)$$

we can write the Schrödinger-like equation for combined scalar and (timelike) vector interactions as [30]

$$\{\mathbf{q}^2 + (m_M + S)^2 - (\varepsilon_M - A)^2\}|\psi\rangle = 0, \quad (2.39)$$

which is suggestive of a Klein-Gordon equation for an effective particle of relative motion. This bound state equation incorporates not only the correct relativistic kinematics but also the correct relativistic dynamical corrections through order $1/c^2$ and higher, depending on the input. It does it without the necessity of introducing complicated momentum-dependent Darwin-like interactions, thereby retaining the simplicity of the nonrelativistic Schrödinger equation. Furthermore, the potentials in these equations are connected to those of Wheeler-Feynman electrodynamics (and its scalar counterpart) [33,34]. They have been obtained systematically from perturbative quantum field theory [7,8,10] and from an eikonal summation of Feynman diagrams [35].

In summary, Eqs. (2.28), (2.30), and (2.31) provide a useful way to obtain the solution of the relativistic two-body problem for spinless particles in scalar and vector interactions. In other works they have been extended to include spin and have been found to give an excellent account of the bound state spectrum of both light and heavy mesons using reasonable input quark potentials [10,34].

III. HAMILTONIAN FORMULATION OF THE N -BODY PROBLEM FROM CONSTRAINT DYNAMICS: SEPARABLE TWO-BODY BASIS

The above treatment of the two-body problem can, to some extent, be generalized to the case of the N -body problem [11]. Our approach differs and extends the work of

Sazdjian in that we choose to formulate the N -body problem in a separable two-body basis.

We consider a system of N particles. For each particle, we specify a generalized mass shell constraint of the form

$$\mathcal{H}_i|\psi\rangle = 0 \quad \text{for } i = 1, \dots, N \quad (3.1)$$

where

$$\mathcal{H}_i = p_i^2 - m_i^2 - \sum_{j, j \neq i}^N \Phi_{ij} - W_i. \quad (3.2)$$

The Φ_{ij} are two-body interactions dependent on x_{ij} , and W_i are possible N -body forces ($N > 2$). We construct the total Hamiltonian for the system as

$$\mathcal{H} = \sum_{i=1}^N \lambda_i \mathcal{H}_i, \quad (3.3)$$

where λ_i 's are the Lagrange multipliers. For each constraint to be conserved in time, we must have

$$[\mathcal{H}_i, \mathcal{H}]|\psi\rangle = i \left(\frac{d}{d\tau} \mathcal{H}_i \right) |\psi\rangle = 0. \quad (3.4)$$

From Eq. (3.3) and Eq. (3.1), we must have the compatibility condition

$$[\mathcal{H}_i, \mathcal{H}_j]|\psi\rangle = 0. \quad (3.5)$$

Now we attempt to expand out the above equation. For a fixed pair of i and j , we have

$$\left(\left[p_i^2, \sum_{k, k \neq j}^N \Phi_{jk} \right] - \left[\sum_{k, k \neq i}^N \Phi_{ik}, p_j^2 \right] + \left[\sum_{l, l \neq i}^N \Phi_{il} + W_i, \sum_{k, k \neq j}^N \Phi_{jk} + W_j \right] - [p_i^2, W_j] - [W_i, p_j^2] \right) |\psi\rangle = 0. \quad (3.6)$$

Motivated by the form of our two-body solution, we assume that

$$\Phi_{ij} = \Phi_{ji} = \Phi_{ij}(x_{ij\perp}) \quad (3.7)$$

in which

$$(x_{ij\perp})_\nu = x_{ij}^\mu [\eta_{\mu\nu} - (P_{ij})_\mu (P_{ij})_\nu / P_{ij}^2], \quad (3.8)$$

and $P_{ij} = p_i + p_j$. This implies that the N -body forces must be present and satisfy

$$\left(\left[\sum_{ll \neq i}^N \Phi_{il} + W_i, \sum_{kk \neq j}^N \Phi_{jk} + W_j \right] - [p_i^2, W_j] - [W_i, p_j^2] \right) |\psi\rangle = 0. \quad (3.9)$$

These are very complicated equations, and, unlike the two-body case, there are no known closed-form solutions [11,36]. Evidently these forces are dependent on the two-body forces themselves [37]. In practice, one often ignores these many-body forces and considers only pair-wise interactions, as we will do in our subsequent computations of reaction matrix elements in Sec. VII. That is, under the approximation in which we set $W_i=0$, we can view the particles as interacting with each other via two-body interactions in a pairwise manner. However, in most of our formal analysis in this paper, we will retain these many-body interactions.

The conservation of the constraints in time depends only upon the compatibility of these constraints and does not depend on the choice of the Lagrange multipliers. This arbitrariness in the choice of the λ_i is similar to a kind of gauge invariance. Choosing a particular set of λ_i is analogous to choosing a gauge and can be done for convenience. In the general N -body formalism we will find it convenient to choose $\lambda_i=1/(2m_i)$. Such a choice has many advantages. First it leads to a simple correspondence with the nonrelativistic two-body and N -body Hamiltonians. Secondly this choice depends only on the particle in question and not what other particle it is linked with. In the general N -body formalism we will find it convenient not to have a preferred pairing of two-body composite subsystems. The choice $\lambda_i=1/(2m_i)$ avoids this. Finally, this allows the N -body Hamiltonian to be conveniently separated into a nonperturbative part and the residual interactions part. Such a correspondence helps one generalize the post-prior equivalence of reaction matrix elements from the nonrelativistic case to the relativistic case.

With this choice, the relativistic N -body Hamiltonian is

$$\begin{aligned} \mathcal{H} &= \sum_{i=1}^N \frac{1}{2m_i} (p_i^2 - m_i^2) - \sum_{i=1}^N \sum_{jj>i}^N \frac{\Phi_{ij}}{2\mu_{ij}} - \sum_i \frac{W_i}{2m_i} \\ &= \sum_{i=1}^N \frac{1}{2m_i} (p_i^2 - m_i^2) - \sum_{i=1}^N \sum_{jj>i}^N V_{ij} - \sum_i \frac{W_i}{2m_i}, \end{aligned} \quad (3.10)$$

where $\mu_{ij}=m_i m_j / (m_i + m_j)$, and we have introduced the simplified notation $V_{ij}=\Phi_{ij}/2\mu_{ij}$. [Note that in light of the above forms of Eqs. (2.33) and (2.36) for Φ_i , the choice $\lambda_i=1/(2m_i)$ gives the correct nonrelativistic limit for V_{ij} .] The dynamics of the relativistic N -body system is determined by the search for the state $|\psi\rangle$ such that

$$\mathcal{H}|\psi\rangle = \left\{ \sum_{i=1}^N \frac{1}{2m_i} (p_i^2 - m_i^2) - \sum_{i=1}^N \sum_{jj>i}^N V_{ij} - \sum_i \frac{W_i}{2m_i} \right\} |\psi\rangle = 0. \quad (3.11)$$

Even without the many-body forces W_i , this equation is very difficult to solve because the potentials V_{ij} depend on the momenta $P_{ij}=p_i+p_j$ (through $x_{ij\perp}$) which are not constants of motion for the N -body system; forces from outside of the ij system produce time-dependent P_{ij} . On the other hand, if one uses the two-body Hamiltonians to generate basis states, then for those states and Hamiltonians, one can regard the P_{ij} as constants of motion. That is, the Hamiltonian can be separated in terms of two-body Hamiltonians plus residual interactions regarded as perturbations. This greatly simplifies the problem.

IV. SOME APPLICATIONS

The quadratic form of the momentum operators p_i in the N -body Hamiltonian (3.11) makes it easy to manipulate the momentum terms to obtain the center-of-mass momentum and other relative momenta. The potential term in the equation appears in a way similar to that in which it appears in the nonrelativistic case.

The relativistic N -body equation can be compared to the nonrelativistic N -body equation. Introducing $\epsilon_i=p_i^0-m_i$, we have

$$\frac{1}{2m_i} (p_i^2 - m_i^2) = \epsilon_i - \frac{\mathbf{p}_i^2 - \epsilon_i^2}{2m_i}. \quad (4.1)$$

Equation (3.11) becomes

$$\mathcal{H}|\psi\rangle = \left\{ E_{\text{NR}} - \sum_{i=1}^N \frac{\mathbf{p}_i^2 - \epsilon_i^2}{2m_i} - \sum_{i=1}^N \sum_{jj>i}^N V_{ij} - \sum_i \frac{W_i}{2m_i} \right\} |\psi\rangle = 0, \quad (4.2)$$

where $E_{\text{NR}}=\sum_i \epsilon_i$. This is identical to the nonrelativistic N -body Hamiltonian with eigenvalue E_{NR} when $|\epsilon_i| \ll m_i$ and the W_i are neglected.

In the next example, we can examine a system of an even number of N particles forming $N/2$ composite particles, as in a system of $N/2$ mesons. For such a system, one can consider an initial state of the form

$$|\psi_a\rangle = |\{(i_1 j_1), (i_2 j_2), \dots, (i_{N/2} j_{N/2})\}\rangle, \quad (4.3)$$

in which particles i_α and j_α form a composite two-body subsystem $(i_\alpha j_\alpha)$. Subsequent dynamics of the system is determined by the evolution operator containing \mathcal{H} and the reaction matrix element $\langle \psi'_a | \mathcal{H} | \psi_a \rangle$ where

$$|\psi'_a\rangle = |\{(i'_1 j'_1), (i'_2 j'_2), \dots, (i'_{N/2} j'_{N/2})\}\rangle. \quad (4.4)$$

For the evaluation of the element $\langle \psi'_a | \mathcal{H} | \psi_a \rangle$ of the Hamiltonian matrix, the N -body Hamiltonian can be separated into an unperturbed Hamiltonian \mathcal{H}_0 and a residual interaction V_I ,

$$\mathcal{H} = H_0 + V_I, \quad (4.5)$$

where the unperturbed Hamiltonian \mathcal{H}_0 is

$$\mathcal{H}_0 = \mathcal{H}_{i_1 j_1} + \mathcal{H}_{i_2 j_2} + \dots + \mathcal{H}_{i_{N/2} j_{N/2}}, \quad (4.6)$$

with

$$\mathcal{H}_{ij} = \frac{1}{2m_i}(p_i^2 - m_i^2) + \frac{1}{2m_j}(p_j^2 - m_j^2) - V_{ij}, \quad (4.7)$$

and V_I , the ‘‘prior’’ form of the residual interaction, is

$$V_I = - \sum_{i=1}^N \sum'_{j>i} V_{ij} - \sum_i \frac{W_i}{2m_i} \quad (4.8)$$

where the summation $\sum_{i=1}^N \sum'_{j>i}$ is carried out with the set $\{ij\}$ different from those of the composite particles $\{i_1j_1\}$, $\{i_2j_2\}$, $\{i_3j_3\}$, . . . , and $\{i_{N/2}j_{N/2}\}$:

$$\sum_{i=1}^N \sum'_{j>i} = \sum_{i=1}^N \sum_{j>i} |\{ij\} \neq \{i_1j_1\}, \{i_2j_2\}, \dots, \{i_{N/2}j_{N/2}\}\rangle. \quad (4.9)$$

Then, since

$$\mathcal{H}_0 | \{(i_1j_1), (i_2j_2), \dots, (i_{N/2}j_{N/2})\} \rangle = 0, \quad (4.10)$$

the transition matrix element $\langle \psi'_a | \mathcal{H} | \psi_a \rangle$ becomes

$$\langle \psi'_a | \mathcal{H} | \psi_a \rangle = \langle \psi'_a | V_I | \psi_a \rangle. \quad (4.11)$$

The separation of \mathcal{H} into \mathcal{H}_0 and V_I is not unique, and it is important to show that the reaction matrix element of the residual interaction is independent of the different ways of separating out the unperturbed Hamiltonian and the residual interaction. We can alternatively choose the unperturbed Hamiltonian to be associated with the state $|\psi'_a\rangle = | \{(i'_1j'_1), (i'_2j'_2), \dots, (i'_{N/2}j'_{N/2})\} \rangle$ such that

$$\mathcal{H}'_0 | \{(i'_1j'_1), (i'_2j'_2), \dots, (i'_{N/2}j'_{N/2})\} \rangle = 0, \quad (4.12)$$

associating with the ‘‘post’’ form of the residual interaction V'_I ,

$$\mathcal{H} = \mathcal{H}'_0 + V'_I. \quad (4.13)$$

The quantities \mathcal{H}'_0 and V'_I are defined in a way similar to that given in Eqs. (4.6)–(4.9). The reaction matrix element between the basis states is

$$\begin{aligned} \langle \psi'_a | V_I | \psi_a \rangle &= \langle \psi'_a | \mathcal{H}_0 + V_I | \psi_a \rangle \\ &= \langle \psi'_a | \mathcal{H} | \psi_a \rangle \\ &= \langle \psi'_a | \mathcal{H}'_0 + V'_I | \psi_a \rangle \\ &= \langle \psi'_a | V'_I | \psi_a \rangle, \end{aligned} \quad (4.14)$$

which indicates that the reaction matrix element is the same for the ‘‘prior’’ form or the ‘‘post’’ form of the residual interactions. It is independent of the way in which we split up the total Hamiltonian. This ‘‘post-prior’’ equivalence guarantees the uniqueness of the reaction matrix element and insures the usefulness of the perturbation expansion.

Using this method of separating the total Hamiltonian, the matrix element of \mathcal{H} between any two basis states can be evaluated. The construction of the matrix of the total Hamil-

tonian will allow one to construct the evolution operator and to follow the dynamics of the system.

V. SCATTERING OF TWO COMPOSITE PARTICLES

As another explicit example, we apply our formalism to a problem of practical interest. Consider four particles with masses m_i and momentum p_i where $i=1,2,3,4$. We have the total Hamiltonian

$$\mathcal{H} = \sum_{i=1}^4 \frac{1}{2m_i}(p_i^2 - m_i^2) - \sum_{i=1}^4 \sum'_{j>i} V_{ij} - \sum_{i=1}^4 \frac{W_i}{2m_i}. \quad (5.1)$$

Using the method of the first section, we can solve for the bound states of mass M_{ij} for the motion of particles i and j interacting with the interaction V_{ij} ,

$$\mathcal{H}_{ij} | \psi_{ij} \rangle = \left[\frac{1}{2m_i}(p_i^2 - m_i^2) + \frac{1}{2m_j}(p_j^2 - m_j^2) - V_{ij} \right] | \psi_{ij} \rangle = 0. \quad (5.2)$$

We can consider the reaction of two composite particles $A(12)$ and $B(34)$ where particles 1 and 3 are particles and particles 2 and 4 are antiparticles as in meson-meson scattering, interacting through a pairwise interaction V_{ij} . We study the relativistic quark-interchange reaction

$$A(12) + B(34) \rightarrow C(14) + D(32), \quad (5.3)$$

as a generalization of the nonrelativistic case investigated by Barnes and Swanson [19]. For this reaction with a momentum transfer t

$$t = (A - C)^2 = m_A^2 + m_C^2 - 2A_0C_0 + 2\mathbf{A} \cdot \mathbf{C}, \quad (5.4)$$

the differential cross section in the first-Born approximation is given by [19]

$$\frac{d\sigma}{dt} = \frac{1}{64\pi s} \frac{\hbar^2}{|\mathbf{p}_A^c|^2} |\mathcal{M}_{fi}|^2, \quad (5.5)$$

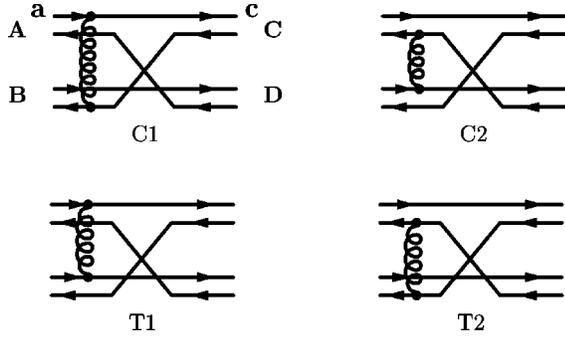
where \mathcal{M}_{fi} is

$$\mathcal{M}_{fi} = (2\pi)^3 \sqrt{2E_A^c 2E_B^c 2E_C^c 2E_D^c} h_{fi}, \quad (5.6)$$

and h_{fi} is the reaction matrix element from the initial state $A(12)B(34)$ to the final state $C(14)D(32)$ initiated by the residual interaction V_I . In the above equation, a kinematic variable with the superscript c refers to that variable evaluated in the center of mass of the colliding mesons (collider system). From the above result, h_{fi} has the dimension of $1/(\text{mass})^2$.

To obtain the reaction matrix element h_{fi} in our relativistic formulation, we need to split the total Hamiltonian into the nonperturbative part \mathcal{H}_0 and the residual interaction V_I . This can be carried out in two different ways. In the ‘‘prior’’ form, it is split as

$$\mathcal{H} = \mathcal{H}_{12} + \mathcal{H}_{34} - V_{13} - V_{14} - V_{23} - V_{24} - \sum_{i=1}^4 \frac{W_i}{2m_i}, \quad (5.7)$$

FIG. 1. ‘‘Prior’’ diagrams for the reaction $A+B \rightarrow C+D$.

where the unperturbed Hamiltonian is

$$\mathcal{H}_0(\text{prior}) = \mathcal{H}_{12} + \mathcal{H}_{34}, \quad (5.8)$$

and the residual interaction is

$$V_I(\text{prior}) = -V_{13} - V_{14} - V_{23} - V_{24} - \sum_{i=1}^4 \frac{W_i}{2m_i}. \quad (5.9)$$

The reaction matrix element is

$$\begin{aligned} & 2\pi \delta^4(P_A + P_B - P_C - P_D) h_{fi}(\text{prior}) \\ &= -\langle \psi_{14} \psi_{23} | V_{13} + V_{14} + V_{23} + V_{24} + \sum_{i=1}^4 \frac{W_i}{2m_i} | \psi_{12} \psi_{34} \rangle. \end{aligned} \quad (5.10)$$

In graphic form, if we represent the interaction V_{ij} by a curly line, the first four terms in the above matrix element are represented by the four diagrams in Fig. 1. The interaction takes place before the rearrangement of the constituents.

On the other hand, if we use the ‘‘post’’ form of splitting the total Hamiltonian, we have

$$\mathcal{H} = \mathcal{H}_{14} + \mathcal{H}_{32} - V_{13} - V_{12} - V_{43} - V_{42} - \sum_{i=1}^4 \frac{W_i}{2m_i}. \quad (5.11)$$

The unperturbed Hamiltonian is

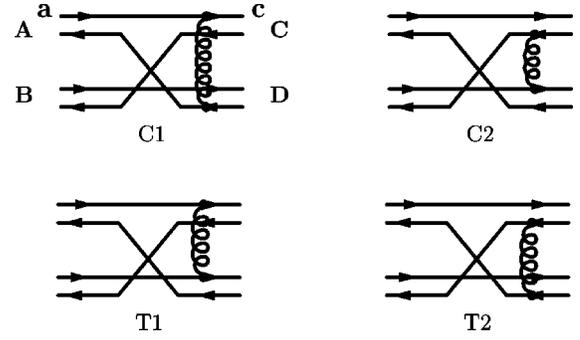
$$\mathcal{H}_0(\text{post}) = \mathcal{H}_{14} + \mathcal{H}_{32}, \quad (5.12)$$

and the residual interaction is

$$V_I(\text{post}) = -V_{13} - V_{12} - V_{43} - V_{42} - \sum_{i=1}^4 \frac{W_i}{2m_i}. \quad (5.13)$$

The reaction matrix element is

$$\begin{aligned} & 2\pi \delta^4(P_A + P_B - P_C - P_D) h_{fi}(\text{post}) \\ &= -\langle \psi_{14} \psi_{23} | V_{12} + V_{13} + V_{42} + V_{43} + \sum_{i=1}^4 \frac{W_i}{2m_i} | \psi_{12} \psi_{34} \rangle. \end{aligned} \quad (5.14)$$

FIG. 2. ‘‘Post’’ diagrams for the reaction $A+B \rightarrow C+D$.

In graphic form, the first four terms in the above matrix element are represented by the four diagrams in Fig. 2. The interaction takes place after the rearrangement of the constituents.

Therefore, if we start with the prior expression for the matrix element, we have $\mathcal{H}_{12}\psi_{12}=0$ and $\mathcal{H}_{34}\psi_{34}=0$, and we have (canceling out the $N>2$ -body potentials)

$$\begin{aligned} & \langle \psi_{14} \psi_{23} | V_{13} + V_{14} + V_{23} + V_{24} | \psi_{12} \psi_{34} \rangle \\ &= \langle \psi_{14} \psi_{23} | -\mathcal{H}_{12} - \mathcal{H}_{34} + V_{13} + V_{14} + V_{23} + V_{24} | \psi_{12} \psi_{34} \rangle \\ &= \langle \psi_{14} \psi_{23} | -\mathcal{H}_{14} - \mathcal{H}_{23} + V_{12} + V_{13} + V_{42} + V_{43} | \psi_{12} \psi_{34} \rangle, \end{aligned} \quad (5.15)$$

where we have used Eq. (5.1) to write out the Hamiltonian for the two-body system. Because $\mathcal{H}_{14}|\psi_{14}\rangle=0$ and $\mathcal{H}_{23}|\psi_{23}\rangle=0$, we have then

$$\begin{aligned} & \langle \psi_{14} \psi_{23} | V_{13} + V_{14} + V_{23} + V_{24} | \psi_{12} \psi_{34} \rangle_1 \\ &= \langle \psi_{14} \psi_{23} | V_{12} + V_{13} + V_{42} + V_{43} | \psi_{12} \psi_{34} \rangle, \end{aligned} \quad (5.16)$$

which leads to the relativistic generalization of the post-prior equivalence of the reaction matrix element h_{fi} ,

$$h_{fi}(\text{prior}) = h_{fi}(\text{post}). \quad (5.17)$$

Just as in nonrelativistic reaction theory [38], the equivalence is possible only when one uses the same internal relative wave function for the composite particles in their scattering process as in the bound state problem for the individual composite particles. The equivalence allows a unique determination of the reaction cross section in the first-Born approximation.

VI. SECOND QUANTIZATION OF PARTICLES IN A BOUND STATE

A practical problem arises when one attempts to evaluate the reaction matrix element $\langle \psi_a | V_I | \psi'_a \rangle$,

$$\langle \psi_a | V_I | \psi'_a \rangle = \langle (i_1 j_1), (i_2 j_2), \dots | - \sum_{i=1}^N \sum_{j \neq i}^N V_{ij} - \sum_i \frac{W_i}{2m_i} | (i'_1 j'_1), (i'_2 j'_2), \dots \rangle, \quad (6.1)$$

where $\{(i_1 j_1), (i_2 j_2), \dots\}$ represents composite two-body subsystems. For a pairwise interaction V_{ij} in the above equation, we have

$$\langle \psi_a | V_{ij} | \psi'_a \rangle = \langle (i \alpha'), (j \beta') | V_{ij} | (i \alpha), (j \beta) \rangle, \quad (6.2)$$

where $\{\alpha' \beta'\}$ is a permutation of $\{\alpha \beta\}$. The composite wave functions $\psi_{i\alpha}$ are usually computed in the c.m. of the $(i\alpha)$ composite particle system (the $q_i \bar{q}_\alpha$ system in our example of a system of mesons). However, this is not the same as the so-called collider frame [the c.m. frame of the $(i\alpha)$ - $(j\beta)$ meson-meson system] either in the bra or ket states. For nonrelativistic reactions, the relative wave function of a composite system in the collider frame is obtained from the wave function for the composite particle at rest by a Galilean boost, and they are related by a simple momentum shift. In the relativistic case, a Lorentz boost is needed in place of a Galilean boost. We need, therefore, to discuss the Lorentz transformation of the state of the composite system.

We represent the state of the composite two-body system with a four-momentum P by

$$\langle P^0 | (12)_P \rangle = 4\varepsilon_1 \varepsilon_2 \int d^4 p_1 d^4 p_2 \delta^4(p_1 + p_2 - P) \theta(p_1^0) \theta(p_2^0) \times \delta(p_1^2 - m_1^2 - \Phi(x_\perp)) \delta(p_2^2 - m_2^2 - \Phi(x_\perp)) \times \psi(p_1, p_2) | p_1 p_2 \rangle, \quad (6.3)$$

where we use the same symbol p_1 and p_2 to denote c -numbers and operators, using the context to distinguish between them. The factor of $4\varepsilon_1 \varepsilon_2$ is included so that we obtain the usual results in the nonrelativistic limit. The delta function containing the composite particle energy P^0 arises from the projection of the energy eigenstate state $|P^0\rangle$ onto the state vector $|(12)_P\rangle$ where $P = \{E, \mathbf{P}\}$ and $E = \sqrt{\mathbf{P}^2 + M^2}$ [see also Eq. (6.22) below]. Hence, we use the notation $\langle P^0 | (12)_{E\mathbf{P}} \rangle$ to denote the projection of the state vector $|(12)_P\rangle$ onto the energy eigenstate state $|P_0\rangle$. The above state is constructed in analogy to the two free-particle states

$$\langle P^0 | (12)_P \rangle = 4\varepsilon_1 \varepsilon_2 \int d^4 p_1 d^4 p_2 \delta^4(p_1 + p_2 - P) \times \theta \times (p_1^0) \theta(p_2^0) \delta(p_1^2 - m_1^2) \times \delta(p_2^2 - m_2^2) \psi(p_1, p_2) | p_1 p_2 \rangle. \quad (6.4)$$

However, unlike the free-particle state, the state (6.3) satisfies the simultaneous constraint conditions of Eq. (2.1)

$$\mathcal{H}_i | (12)_P \rangle = [p_i^2 - m_i^2 - \Phi(x_\perp)] | (12)_P \rangle = 0 \quad \text{for } i=1,2, \quad (6.5)$$

so that neither particle is on mass shell. The definition (6.3) also introduces a momentum space wave function $\psi(p_1, p_2)$ defined so that it has positive constituent energies. We emphasize that the momentum eigenstates $|p_1 p_2\rangle$ in this expansion are off shell. That is

$$(p_i^2 - m_i^2) | p_1 p_2 \rangle \neq 0. \quad (6.6)$$

Our first step is to show that the above bound state composite is a sharp state being zero unless $P^2 = M^2$ where M is the meson bound state mass. By using the total and relative momentum operators and Eq. (2.20) [so that Eq. (2.27) is satisfied and not $P \cdot q = 0$], the product of the two delta functions can be written as

$$\begin{aligned} & \delta(p_1^2 - m_1^2 - \Phi(x_\perp)) \delta(p_2^2 - m_2^2 - \Phi(x_\perp)) \\ &= \delta\left(\frac{\varepsilon_1^2 P^2}{M^2} + q^2 - m_1^2 - \Phi(x_\perp) + \frac{2\varepsilon_1 P \cdot q}{M}\right) \\ & \quad \times \delta\left((m_1^2 - m_2^2) \frac{(P^2 - M^2)}{M^2} + 2P \cdot q\right) \\ &= \delta\left(\frac{\varepsilon_1 \varepsilon_2 (P^2 - M^2)}{M^2} + \mathcal{H}_q\right) \\ & \quad \times \delta\left((m_1^2 - m_2^2) \frac{(P^2 - M^2)}{M^2} + 2P \cdot q\right), \quad (6.7) \end{aligned}$$

where $\mathcal{H}_q = q^2 - \Phi(x_\perp) + b^2$. We assume that the momentum space wave function $\psi(p_1, p_2)$ is an eigenfunction of \mathcal{H}_q , so that $\mathcal{H}_q \psi(p_1, p_2) = 0$. Thus, on such states the above equation becomes

$$\begin{aligned} & \delta(p_1^2 - m_1^2 - \Phi(x_\perp)) \delta(p_2^2 - m_2^2 - \Phi(x_\perp)) \\ &= \frac{M^2}{2\varepsilon_1 \varepsilon_2} \delta(P^2 - M^2) \delta(P \cdot q). \quad (6.8) \end{aligned}$$

This shows, as anticipated, that the state defined in Eq. (6.3) is sharp and would satisfy Eq. (2.23) in addition to Eq. (2.24).

We allow the delta function arguments to operate on the momentum states, and use the positive energy condition. Then from Eq. (2.17) we have

$$\begin{aligned} & \theta(p_1^0) \theta(p_2^0) \delta(p_1^2 - m_1^2 - \Phi(x_\perp)) \delta(p_2^2 - m_2^2 - \Phi(x_\perp)) \\ &= \theta(p_1^0) \theta(p_2^0) \frac{M^2}{4\varepsilon_1 \varepsilon_2 E^2} \delta\left(p_1^0 - \frac{E}{M} \varepsilon_1 - \mathbf{P} \cdot \mathbf{q}\right) \\ & \quad \times \delta\left(p_2^0 - \frac{E}{M} \varepsilon_2 + \mathbf{P} \cdot \mathbf{q}\right), \quad (6.9) \end{aligned}$$

where $E = \sqrt{\mathbf{P}^2 + M^2}$. We would like to express this in terms of a Lorentz transformation from the c.m. system (in which the meson has a mass M).

When the composite particle is boosted by Λ to a momentum \mathbf{P} , the meson's velocity is

$$\mathbf{V} = \frac{\mathbf{P}}{E}. \quad (6.10)$$

Thus with $\hat{u} = \hat{\mathbf{V}} = \mathbf{P}/|\mathbf{P}|$, the components of Λ are

$$\Lambda_k^i = \delta_{ik} + (\gamma - 1) \hat{u}_i \hat{u}_k = \delta_{ik} + \left(\frac{E}{M} - 1 \right) \frac{\mathbf{P}_i \mathbf{P}_k}{\mathbf{P}^2},$$

$$\Lambda_0^i = \Lambda_i^0 = \hat{u}_i \sqrt{(\gamma^2 - 1)} = \frac{\mathbf{P}_i}{M},$$

$$\Lambda_0^0 = \Lambda_i^i = \gamma = \frac{E}{M}. \quad (6.11)$$

We use the notation $p_1^* = (\varepsilon_1, \mathbf{p}_1^*)$ to represent the four-momenta of the i th constituent in the composite particle rest frame, and $p_i = p_{i\Lambda} = (\varepsilon_{i\Lambda}, \mathbf{p}_{i\Lambda})$ to represent the four-momentum of the i th constituent in the frame boosted from \mathbf{p}_i^* by Λ . Then we have

$$p_i^0 \equiv p_{i\Lambda}^0 = (\Lambda p_1^*)^0 = \gamma p_i^{*0} + \hat{u} \cdot \mathbf{p}_i^* = \frac{E \varepsilon_i}{M} + \frac{\mathbf{P} \cdot \mathbf{p}_i^*}{M} \equiv \varepsilon_{i\Lambda},$$

$$(i=1,2) \quad (6.12)$$

and

$$\mathbf{p}_i \equiv \mathbf{p}_{i\Lambda} = \frac{\mathbf{P}}{M} \varepsilon_i + \mathbf{p}_i^* + \left(\frac{E}{M} - 1 \right) \frac{\mathbf{P} \mathbf{P} \cdot \mathbf{p}_i^*}{\mathbf{P}^2}, \quad (i=1,2). \quad (6.13)$$

Because $\mathbf{p}_1^* + \mathbf{p}_2^* = 0$ we have therefore

$$p_1^0 + p_2^0 = \sqrt{\mathbf{P}^2 + M^2} = E, \quad (6.14)$$

$$\mathbf{p}_1 + \mathbf{p}_2 = \frac{\mathbf{P}}{M} (\varepsilon_1 + \varepsilon_2) = \mathbf{P}. \quad (6.15)$$

Furthermore, using Eq. (2.17) we have

$$\mathbf{P} \cdot \mathbf{q} = \frac{E}{M} \mathbf{P} \cdot \mathbf{q}^* = \frac{E}{M} \mathbf{P} \cdot \mathbf{p}_1^* = - \frac{E}{M} \mathbf{P} \cdot \mathbf{p}_2^*. \quad (6.16)$$

Hence, Eq. (6.9) can be rewritten as

$$\theta(p_1^0) \theta(p_2^0) \delta(p_1^2 - m_1^2 - \Phi(x_\perp)) \delta(p_2^2 - m_2^2 - \Phi(x_\perp))$$

$$= \theta(p_1^0) \theta(p_2^0) \frac{M^2}{4\varepsilon_1 \varepsilon_2 E^2} \delta(p_1^0 - (\Lambda p_1^*)^0) \delta(p_2^0 - (\Lambda p_2^*)^0). \quad (6.17)$$

We have therefore

$$\langle P^0 | (12)_{EP} \rangle = \frac{M^2}{E^2} \int d^4 p_1 d^4 p_2 \delta^4(P - p_1 - p_2)$$

$$\times \delta(p_1^0 - (\Lambda p_1^*)^0) \delta(p_2^0 - (\Lambda p_2^*)^0)$$

$$\times \psi(p_1, p_2) | p_1, p_2 \rangle. \quad (6.18)$$

Now the p_1^0 and p_2^0 part of the $d^4 p_1$ and $d^4 p_2$ can be integrated out and the result is

$$\langle P^0 | (12)_{EP} \rangle = \frac{M^2}{E^2} \delta(P^0 - p_1^0 - p_2^0) \int d\mathbf{p}_1 d\mathbf{p}_2$$

$$\times \delta^3(\mathbf{P} - \mathbf{p}_1 - \mathbf{p}_2) \psi(p_1^0 \mathbf{p}_1, p_2^0 \mathbf{p}_2) | p_1^0 \mathbf{p}_1, p_2^0 \mathbf{p}_2 \rangle, \quad (6.19)$$

where $p_i^0 = (\Lambda p_i^*)^0$ and $p_1^0 + p_2^0 = E$. In the c.m. system, $p_1^0 + p_2^0 = M$ and the state vector is

$$\langle P^0 | (12)_{M0} \rangle = \delta(P_0 - M) \int d\mathbf{p}_1^* d\mathbf{p}_2^* \delta^3(\mathbf{p}_1^* + \mathbf{p}_2^*)$$

$$\times \psi_M(\varepsilon_1 \mathbf{p}_1^*, \varepsilon_2 \mathbf{p}_2^*) | \varepsilon_1 \mathbf{p}_1^*, \varepsilon_2 \mathbf{p}_2^* \rangle. \quad (6.20)$$

We introduce the notation $|M(\mathbf{P})\rangle$ defined as

$$|M(\mathbf{P})\rangle = \int d\mathbf{p}_1 d\mathbf{p}_2 \delta^3(\mathbf{P} - \mathbf{p}_1 - \mathbf{p}_2)$$

$$\times \psi(p_1^0 \mathbf{p}_1, p_2^0 \mathbf{p}_2) | p_1^0 \mathbf{p}_1, p_2^0 \mathbf{p}_2 \rangle, \quad (6.21)$$

so that, since in a general frame $p_1^0 + p_2^0 = \sqrt{\mathbf{P}^2 + M^2} = E$,

$$\langle P^0 | (12)_{EP} \rangle = \frac{M^2}{E^2} \delta(P^0 - \sqrt{\mathbf{P}^2 + M^2}) |M(\mathbf{P})\rangle. \quad (6.22)$$

The projection of the state vector $|(12)_{EP}\rangle$ onto the time component of the center of mass [39] is then

$$\langle t | (12)_{EP} \rangle = \int \langle t | P^0 \rangle dP^0 \langle P^0 | (12)_{EP} \rangle = e^{-iEt} \frac{M^2}{E^2} |M(\mathbf{P})\rangle. \quad (6.23)$$

From the above results, the energies of the constituents in a composite particle take on fixed values ($p_i^{*0} = \varepsilon_i$) in the center-of-mass system, while their off-shell component \mathbf{p}_i^* takes on continuous variations with a distribution. On the other hand, when boosted by the Lorentz transformation Λ , the energy of the i th constituent is given by $p_i^0 = (\Lambda p_i^*)^0$ in a moving composite particle, but their sum, $p_1^0 + p_2^0$, remains a constant. Even though the timelike components of the constituents have these well-defined values which depend on the frame of reference, they are often not written out explicitly, for brevity of notation.

For a proper constraint treatment of spin, we should parallel the treatment in the above sections except using Dirac operators instead of Klein-Gordon operators. Alternatively, we can adapt the above spinless results to the case of spin by using the fact that we can reduce, for two particles, the two-

body Dirac equations to Schrödinger-like forms above but with Φ depending on spin degrees of freedom [10,34]. In a future paper we shall include the spin dependent features in more detail.

In order to deal with multiparticle configurations, we introduce creation and annihilation operators of the constituents in a composite particle in its c.m. frame

$$|\mathbf{p}_1^*, \mathbf{p}_2^*\rangle = b^\dagger(\mathbf{p}_1^*) d^\dagger(\mathbf{p}_2^*) |0\rangle. \quad (6.24)$$

(We suppress spin, flavor, and color indices here and in the wave functions below.) We assume the general expression

$$\begin{aligned} \{b^\dagger(\mathbf{p}_1^*), b(\mathbf{p}_1^*)\} &= N(\mathbf{p}_1^*) \delta(\mathbf{p}_1^* - \mathbf{p}_1^*), \\ \{d^\dagger(\mathbf{p}_2^*), d(\mathbf{p}_2^*)\} &= N(\mathbf{p}_2^*) \delta(\mathbf{p}_2^* - \mathbf{p}_2^*), \end{aligned} \quad (6.25)$$

and thus

$$\langle \mathbf{p}_1'^*, \mathbf{p}_2'^* | \mathbf{p}_1^*, \mathbf{p}_2^* \rangle = N(\mathbf{p}_1^*) \delta(\mathbf{p}_1'^* - \mathbf{p}_1^*) N(\mathbf{p}_2^*) \delta(\mathbf{p}_2'^* - \mathbf{p}_2^*). \quad (6.26)$$

In the case in which free isolated particles are created and annihilated, one traditionally takes either $N(\mathbf{p}^*)=1$ or $N(\mathbf{p}^*)=2\sqrt{\mathbf{p}^{*2}+m^2}$. We emphasize, however, that the above momentum in the creation and annihilation operators are not on mass shell but on energy shell. In the context of the constraint approach, the individual creation and annihilation operators do not produce free-particle states, but rather constituent states within a composite associated with a definite total mass and total momentum. Since the aim of this paper is a description of the relativistic N -body problem in a separable two-body basis, this is plausible. To achieve this, we must determine how the creation and annihilation operators will transform under a Lorentz transformation. Let $U(\Lambda)$ be our unitary boost operator defined so that

$$U(\Lambda) b^\dagger(\mathbf{p}_1^*) U^{-1}(\Lambda) = C(\mathbf{p}_{1\Lambda}) b^\dagger(\mathbf{p}_{1\Lambda}) \quad (6.27)$$

and

$$U(\Lambda) d^\dagger(\mathbf{p}_2^*) U^{-1}(\Lambda) = C(\mathbf{p}_{2\Lambda}) d^\dagger(\mathbf{p}_{2\Lambda}), \quad (6.28)$$

where $\mathbf{p}_{i\Lambda}$ is the three-vector part of Λp_i^* . In the case in which free isolated particles are produced, the above two conventions lead respectively to either $C(\mathbf{p}) = \sqrt{(\Lambda p)^0/p^0}$ or $C(\mathbf{p}) = 1$. In order to see what these factors become now in the constraint approach, we consider

$$\begin{aligned} U(\Lambda) \{b^\dagger(\mathbf{p}_1^*), b(\mathbf{p}_1^*)\} U^{-1}(\Lambda) \\ = N(\mathbf{p}_1^*) \delta(\mathbf{p}_1'^* - \mathbf{p}_1^*) \\ = C(\mathbf{p}_{1\Lambda}) C^*(\mathbf{p}'_{1\Lambda}) N(\mathbf{p}_{1\Lambda}) \delta(\mathbf{p}'_{1\Lambda} - \mathbf{p}_{1\Lambda}). \end{aligned} \quad (6.29)$$

To make use of this equation, we need to express the boosted momenta in terms of the unboosted momenta. The Lorentz transformation matrix Λ , defined in Eq. (6.11), is independent of the momentum of the system being boosted. For the

same two-body system, the spacelike part (6.13) of the Lorentz transformation on two different momenta (for the same quark, that is, $\varepsilon_1 = \varepsilon'_1$) yields

$$\mathbf{p}_{1\Lambda} - \mathbf{p}'_{1\Lambda} = \left[\mathbf{1} + \left(\frac{E}{M} - 1 \right) \frac{\mathbf{P}\mathbf{P}}{\mathbf{P}^2} \right] (\mathbf{p}_1^* - \mathbf{p}_1'^*). \quad (6.30)$$

Using the fact that $\delta^3(\mathbf{A}\mathbf{r}) = \delta^3(\mathbf{r})/\det|\mathbf{A}|$, we find that

$$\delta(\mathbf{p}_{1\Lambda} - \mathbf{p}'_{1\Lambda}) = \frac{M}{E} \delta(\mathbf{p}_1'^* - \mathbf{p}_1^*) \quad (6.31)$$

so that Eq. (6.29) implies

$$N(\mathbf{p}_1^*) = C(\mathbf{p}_{1\Lambda}) C^*(\mathbf{p}'_{1\Lambda}) N(\mathbf{p}_{1\Lambda}) \frac{M}{E}. \quad (6.32)$$

We choose the normalization $N=1$. Using phase convention with real C , the simplest choice is

$$C = \sqrt{\frac{E}{M}}. \quad (6.33)$$

This C is associated with the motion of the composite particle. Thus we have

$$U(\Lambda) b^\dagger(\mathbf{p}_1^*) U^{-1}(\Lambda) = \sqrt{\frac{E}{M}} b^\dagger(\mathbf{p}_{1\Lambda}) \quad (6.34)$$

and

$$U(\Lambda) d^\dagger(\mathbf{p}_2^*) U^{-1}(\Lambda) = \sqrt{\frac{E}{M}} d^\dagger(\mathbf{p}_{2\Lambda}). \quad (6.35)$$

(As anticipated above, this contrasts with the on mass shell factor given in standard texts [see [40], Eq. (4.2.12)] which refer only to the constituent momenta.) Thus we have

$$\begin{aligned} |M(\mathbf{P})\rangle &= U(\Lambda) |M(\mathbf{0})\rangle \\ &= \int d^3 p_1^* d^3 p_2^* \psi_M(\mathbf{p}_1^*, \mathbf{p}_2^*) \\ &\quad \times \delta(\mathbf{p}_1^* + \mathbf{p}_2^*) \frac{E}{M} b^\dagger(\mathbf{p}_{1\Lambda}) d^\dagger(\mathbf{p}_{2\Lambda}) |0\rangle. \end{aligned} \quad (6.36)$$

We emphasize that the transformation equations [Eq. (6.34) and Eq. (6.35)] are valid for arbitrary \mathbf{p}_1 or \mathbf{p}_2 , not just ones that satisfy the rest condition of $\mathbf{p}_1 + \mathbf{p}_2 = 0$. This implies that the creation and annihilation operators that go into making up the individual two-body interactions will transform in a similar way. [See Eq. (7.10) below.]

Next we change variables so that the delta function reflects the new total momentum \mathbf{P} . The inverse of the above Lorentz transformation gives for $i=1,2$

$$\mathbf{p}_i^* = -\frac{\mathbf{P}\varepsilon_i}{E} + \mathbf{p}_{i\Lambda} + \left(\frac{M}{E} - 1 \right) \frac{\mathbf{P}\mathbf{P} \cdot \mathbf{p}_{i\Lambda}}{\mathbf{P}^2} \quad (6.37)$$

which in turn gives

$$(\Lambda p_i^*)^0 = \frac{\varepsilon_i M}{E} + \frac{\mathbf{P} \cdot \mathbf{p}_{i\Lambda}}{E}. \quad (6.38)$$

Note that with $\mathbf{p}_{1\Lambda} + \mathbf{p}_{2\Lambda} = \mathbf{P}$ we have

$$\mathbf{p}_1^* + \mathbf{p}_2^* = \mathbf{p}_{1\Lambda} + \mathbf{p}_{2\Lambda} - \mathbf{P} = 0. \quad (6.39)$$

Computing the Jacobian of the above transformation (6.37) gives

$$d^3 p_i^* = \frac{M}{E} d^3 p_{i\Lambda}. \quad (6.40)$$

[Note again how this contrasts with the case of free on-shell particles where $d^3 p_i = d^3 p_{i\Lambda} p_i^0 / (\Lambda p_i)^0$.] Hence,

$$\begin{aligned} |M(\mathbf{P})\rangle &= \frac{M}{E} \int d^3 p_{1\Lambda} d^3 p_{2\Lambda} \psi_M(\mathbf{p}_1^*, \mathbf{p}_2^*) \\ &\times \delta(\mathbf{P} - \mathbf{p}_{1\Lambda} - \mathbf{p}_{2\Lambda}) b^\dagger(\mathbf{p}_{1\Lambda}) d^\dagger(\mathbf{p}_{2\Lambda}) |0\rangle. \end{aligned} \quad (6.41)$$

The wave function $\psi_M(\mathbf{p}_1^*, \mathbf{p}_2^*)$ is actually $\psi_M(\mathbf{p}_1^*(\mathbf{p}_{1\Lambda}), \mathbf{p}_2^*(\mathbf{p}_{2\Lambda}))$ where the vectors $\{\mathbf{p}_1^*(\mathbf{p}_{1\Lambda}), \mathbf{p}_2^*(\mathbf{p}_{2\Lambda})\}$ are $\{(\mathbf{p}_{1\Lambda})_{\Lambda^{-1}}, (\mathbf{p}_{2\Lambda})_{\Lambda^{-1}}\}$. A simple relabeling (not a transformation) gives the representation of a general state for a composite particle with momentum \mathbf{P} ,

$$\begin{aligned} |M(\mathbf{P})\rangle &= \frac{M}{E} \int d^3 p_1 d^3 p_2 \psi_M(\mathbf{p}_{1\Lambda^{-1}}, \mathbf{p}_{2\Lambda^{-1}}) \\ &\times \delta(\mathbf{P} - \mathbf{p}_1 - \mathbf{p}_2) b^\dagger(\mathbf{p}_1) d^\dagger(\mathbf{p}_2) |0\rangle. \end{aligned} \quad (6.42)$$

In the nonrelativistic limit, this becomes

$$\begin{aligned} |M(\mathbf{P})\rangle &= \int d^3 p_1 d^3 p_2 \psi_M \left(\mathbf{p}_1 - \frac{m_1}{m_1 + m_2} \mathbf{P}, \right. \\ &\left. \times \mathbf{p}_2 - \frac{m_2}{m_1 + m_2} \mathbf{P} \right) \delta(\mathbf{P} - \mathbf{p}_1 - \mathbf{p}_2) b^\dagger(\mathbf{p}_1) d^\dagger(\mathbf{p}_2) |0\rangle. \end{aligned} \quad (6.43)$$

So, each composite state, Eq. (6.22) with Eq. (6.42), differs from that of the nonrelativistic limit by not only replacing Galilean boosts with inverse Lorentz boosts but also by a factor of the ratio of the total c.m. energy to the lab energy.

In Appendix A we show that the scalar product for two systems in the same internal state is given by the three-dimensional momentum delta function times a covariant form

$$\langle M(\mathbf{P}') | M(\mathbf{P}) \rangle = \frac{\delta^3(\mathbf{P}' - \mathbf{P}) M^3}{E^2} \int d^4 p \delta(p \cdot P) |\psi_M(p)|^2, \quad (6.44)$$

with the wave function having the same dimensions as in the nonrelativistic case.

Given these preliminaries, we consider how to use this formulation in the calculation of meson-meson scattering amplitudes. In this problem one starts with a state $| (12)(34) \rangle$ consisting of two quark-antiquark states. We model the interaction by the exchange of an (effective) gluon corresponding to V . At lowest order, the exchange could not produce a final state $| (12)(34) \rangle$ but only $| (14)(23) \rangle$ since the emission of a virtual gluon would leave the resultant initial state as two color octet mesons rather than singlet mesons. One would thus need to evaluate a typical matrix element of the form

$$\langle (14)(23) | V(\hat{x}_{14\perp}) | (12)(34) \rangle. \quad (6.45)$$

Inserting $\int dt |t\rangle \langle t| = 1$ and using Eq. (6.23) into the above expression, we can carry out the integration in t to obtain a delta function which describes the condition of total energy conservation. We have

$$\begin{aligned} &\langle (14)(23) | V(\hat{x}_{14\perp}) | (12)(34) \rangle \\ &= 2\pi \delta(E_{12} + E_{34} - E_{13} - E_{24}) \left(\frac{M_{12} M_{14} M_{13} M_{24}}{E_{12} E_{14} E_{13} E_{24}} \right)^2 \\ &\times \langle M(\mathbf{P}_{14}), M(\mathbf{P}_{23}) | V(\hat{x}_{14\perp}) | M(\mathbf{P}_{12}), M(\mathbf{P}_{34}) \rangle, \end{aligned} \quad (6.46)$$

where

$$\begin{aligned} |M(\mathbf{P}_{12})\rangle &= \frac{M_{12}}{E_{12}} \int d^3 p_1 d^3 p_2 \psi(\mathbf{p}_{1\Lambda_{12}^{-1}}, \mathbf{p}_{2\Lambda_{12}^{-1}}) \\ &\times \delta(\mathbf{P}_{12} - \mathbf{p}_1 - \mathbf{p}_2) b^\dagger(\mathbf{p}_1) d^\dagger(\mathbf{p}_2) |0\rangle, \end{aligned} \quad (6.47)$$

in which \mathbf{P}_{12} is the momentum of the composite with c.m. energy M_{12} so that $E_{12} = \sqrt{\mathbf{P}_{12}^2 + M_{12}^2}$ and Λ_{12}^{-1} is the inverse boost to the rest system of the composite. Similar expressions appear for $|M(\mathbf{P}_{34})\rangle$, $|M(\mathbf{P}_{14})\rangle$, and $|M(\mathbf{P}_{23})\rangle$.

VII. REACTION MATRIX ELEMENT

In order to compute matrix elements of the potential, we need its second quantized version. We evaluate it in the rest frame of the two interacting constituents and we assume that the second quantized form of the potential has the same relation to its first quantized form as in the nonrelativistic case. For the reaction $A(12) + B(34) \rightarrow C(14) + D(32)$, the matrix elements for the interaction between a particle and an anti-particle consists of the C1 and the C2 diagram in Fig. 1. We consider the C1 diagram as a representative case. The interaction corresponding to this C1 diagram takes place between particles 1 and 4 and is given by

$$\begin{aligned}
V(\hat{x}_{14\perp})|_{\mathbf{P}_{14}=0} &= V(\hat{\mathbf{x}}_{14}) = \int d^3x'_1 d^3x'_4 d^3x''_1 d^3x''_4 V(\mathbf{x}'_{14}) \delta(\mathbf{x}'_1 - \mathbf{x}''_1) \delta(\mathbf{x}'_4 - \mathbf{x}''_4) |\mathbf{x}'_1 \mathbf{x}'_4\rangle \langle \mathbf{x}''_1 \mathbf{x}''_4| \\
&= \int d^3p_1 d^3p_4 d^3p'_1 d^3p'_4 \delta(\mathbf{p}_1 + \mathbf{p}_4 - \mathbf{p}'_1 - \mathbf{p}'_4) \tilde{V}(\mathbf{p}_1 - \mathbf{p}'_4) b^\dagger(\mathbf{p}_1) d^\dagger(\mathbf{p}_4) d(\mathbf{p}'_4) b(\mathbf{p}'_1), \quad (7.1)
\end{aligned}$$

in which the integrals include sums over spin, flavor, and color. In this form the indices and primes serve to label the quark color and flavor as well as the momentum. The annihilation and creation operators with momentum p_i and p'_i apply only to particle i . To represent the interaction $V(\hat{\mathbf{x}}_{14})$, we choose to represent the annihilation and creation operators in the rest frame of the (particle 1)-(particle 4) pair, i.e., the ($\mathbf{P}_{14}=0$) frame. In this frame, let the total momenta of the two initial meson composite systems be $\mathbf{P}=\mathbf{P}_{12}+\mathbf{P}_{34}$, with \mathbf{P}_{12} and \mathbf{P}_{34} to be the individual incoming momenta of the two meson composite systems. The total energy of the two meson system is given in terms of the respective c.m. energies of the composite particles

$$\sqrt{s} = E_{12} + E_{34} = \sqrt{\mathbf{P}_{12}^2 + M_{12}^2} + \sqrt{\mathbf{P}_{34}^2 + M_{34}^2}. \quad (7.2)$$

The total momentum \mathbf{P} of the two (composite) particle system is conserved in the scattering process so that we can label the matrix element as

$$\langle M(\mathbf{P}_{14}), M(\mathbf{P}_{23}); \mathbf{P} | V(\hat{\mathbf{x}}_{14}) | M(\mathbf{P}_{12}), M(\mathbf{P}_{34}); \mathbf{P} \rangle. \quad (7.3)$$

Let us evaluate the matrix element in the ‘‘collider frame’’ defined by

$$\mathbf{P}_{12}^c + \mathbf{P}_{34}^c = \mathbf{0}. \quad (7.4)$$

The Lorentz boost to that frame is given by

$$\begin{aligned}
\Lambda_k^i &= \delta_{ik} + \left(\frac{\sqrt{\mathbf{P}^2 + s}}{\sqrt{s}} - 1 \right) \frac{\mathbf{P}_i \mathbf{P}_k}{\mathbf{P}^2}, \\
\Lambda_0^i &= \Lambda_i^0 = -\frac{\mathbf{P}_i}{\sqrt{s}}, \\
\Lambda_0^0 &= \gamma = \frac{\sqrt{\mathbf{P}^2 + s}}{\sqrt{s}}, \quad (7.5)
\end{aligned}$$

and takes us to

$$\langle M(\mathbf{P}_{14}), M(\mathbf{P}_{23}); \mathbf{P} | V(\hat{\mathbf{x}}_{14}) | M(\mathbf{P}_{12}), M(\mathbf{P}_{34}); \mathbf{P} \rangle = \langle M(\mathbf{P}_{14}^c), M(\mathbf{P}_{23}^c); \mathbf{0} | U(\Lambda) V(\hat{\mathbf{x}}_{14}) U^{-1}(\Lambda) | M(\mathbf{P}_{12}^c), M(\mathbf{P}_{34}^c); \mathbf{0} \rangle, \quad (7.6)$$

where

$$\begin{aligned}
|M(\mathbf{P}_{12}^c), M(\mathbf{P}_{34}^c); \mathbf{0} \rangle &= \frac{M_{12}}{E_{12}} \int d^3p_1 d^3p_2 \psi(\mathbf{p}_1 \Lambda_{12}^{-1}, \mathbf{p}_2 \Lambda_{12}^{-1}) \delta(\mathbf{P}_{12}^c - \mathbf{p}_1 - \mathbf{p}_2) b^\dagger(\mathbf{p}_1) d^\dagger(\mathbf{p}_2) \\
&\times \frac{M_{34}}{E_{34}} \int d^3p_3 d^3p_4 \psi(\mathbf{p}_3 \Lambda_{34}^{-1}, \mathbf{p}_4 \Lambda_{34}^{-1}) \delta(\mathbf{P}_{34}^c - \mathbf{p}_3 - \mathbf{p}_4) b^\dagger(\mathbf{p}_3) d^\dagger(\mathbf{p}_4) |0\rangle, \quad (7.7)
\end{aligned}$$

$$\begin{aligned}
\langle M(\mathbf{P}_{14}^c), M(\mathbf{P}_{23}^c); \mathbf{0} | &= \frac{M_{14}}{E_{14}} \langle 0 | \int d^3p''_1 d^3p''_4 \psi^*(\mathbf{p}''_1 \Lambda_{14}^{-1}, \mathbf{p}''_4 \Lambda_{14}^{-1}) \delta(\mathbf{P}_{14}^c - \mathbf{p}''_1 - \mathbf{p}''_4) d(\mathbf{p}''_4) b(\mathbf{p}''_1) \\
&\times \frac{M_{23}}{E_{23}} \int d^3p''_2 d^3p''_3 \psi^*(\mathbf{p}''_2 \Lambda_{23}^{-1}, \mathbf{p}''_3 \Lambda_{23}^{-1}) \delta(\mathbf{P}_{23}^c - \mathbf{p}''_2 - \mathbf{p}''_3) d(\mathbf{p}''_3) b(\mathbf{p}''_2), \quad (7.8)
\end{aligned}$$

and $E_{ij} \equiv \sqrt{\mathbf{P}_{ij}^c{}^2 + M_{ij}^2}$.

Note that the integrals in the above state vectors include color summation but not flavor. The above matrix element then becomes

$$\begin{aligned}
& \langle M(\mathbf{P}_{14}^c), M(\mathbf{P}_{23}^c); \mathbf{0} | U(\Lambda) V(\hat{\mathbf{x}}_{14}) U^{-1}(\Lambda) | M(\mathbf{P}_{12}^c), M(\mathbf{P}_{34}^c); \mathbf{0} \rangle \\
&= \langle M(\mathbf{P}_{14}^c), M(\mathbf{P}_{23}^c); \mathbf{0} | \int d^3 p_1 d^3 p_4 d^3 p'_1 d^3 p'_4 \delta(\mathbf{p}_1 + \mathbf{p}_4 - \mathbf{p}'_1 - \mathbf{p}'_4) \tilde{V}(\mathbf{p}_1 - \mathbf{p}'_4) \\
&\quad \times U(\Lambda) b^\dagger(\mathbf{p}_1) d^\dagger(\mathbf{p}_4) d(\mathbf{p}'_4) b(\mathbf{p}'_1) U(\Lambda)^{-1} | M(\mathbf{P}_{12}^c), M(\mathbf{P}_{34}^c); \mathbf{0} \rangle. \tag{7.9}
\end{aligned}$$

From our earlier arguments, the transformations defined in Eqs. (6.34) and (6.35) are independent of the total momentum of the two-body system. Thus, using this, we obtain

$$\begin{aligned}
& \langle M(\mathbf{P}_{14}^c), M(\mathbf{P}_{23}^c); \mathbf{0} | U(\Lambda) V(\hat{\mathbf{x}}_{14}) U^{-1}(\Lambda) | M(\mathbf{P}_{12}^c), M(\mathbf{P}_{34}^c); \mathbf{0} \rangle \\
&= \langle M(\mathbf{P}_{14}^c), M(\mathbf{P}_{23}^c); \mathbf{0} | \int d^3 p_1 d^3 p_4 d^3 p'_1 d^3 p'_4 \delta(\mathbf{p}_1 + \mathbf{p}_4 - \mathbf{p}'_1 - \mathbf{p}'_4) \tilde{V}(\mathbf{p}_1 - \mathbf{p}'_4) \\
&\quad \times b^\dagger(\mathbf{p}_{1\Lambda}) d^\dagger(\mathbf{p}_{4\Lambda}) d(\mathbf{p}'_{4\Lambda}) b(\mathbf{p}'_{1\Lambda}) | M(\mathbf{P}_{12}^c), M(\mathbf{P}_{34}^c); \mathbf{0} \rangle \frac{E_{14}^2}{M_{14}^2}. \tag{7.10}
\end{aligned}$$

We can point out that this last factor, due to interaction transformations, is not present in the nonrelativistic limit. This matrix element is evaluated in Appendix B, and we find

$$\begin{aligned}
& \langle M(\mathbf{P}_{14}^c), M(\mathbf{P}_{23}^c); \mathbf{0} | U(\Lambda) V(\hat{\mathbf{x}}_{14}) U^{-1}(\Lambda) | M(\mathbf{P}_{12}^c), M(\mathbf{P}_{34}^c); \mathbf{0} \rangle \\
&= -\delta^3(\mathbf{P}_{34} - \mathbf{P}_{14} + \mathbf{P}_{12} - \mathbf{P}_{23}) \frac{M_{23}}{E_{23}} \frac{E_{14}}{M_{14}} \frac{M_{12}}{E_{12}} \frac{M_{34}}{E_{34}} \times \int d^3 p_1 d^3 p_4 \psi_D^*(\mathbf{p}_{1\Lambda\Lambda_{14}^{-1}}, -\mathbf{p}_{1\Lambda\Lambda_{14}^{-1}}) \psi_C^* [(-\mathbf{P}_{34}^c + \mathbf{p}_{4\Lambda})_{\Lambda_{23}^{-1}}, \\
&\quad -(-\mathbf{P}_{34}^c + \mathbf{p}_{4\Lambda})_{\Lambda_{23}^{-1}}] \times \psi_A [(-\mathbf{P}_{23}^c + \mathbf{P}_{34}^c - \mathbf{p}_{4\Lambda})_{\Lambda_{12}^{-1}}, -(-\mathbf{P}_{23}^c + \mathbf{P}_{34}^c - \mathbf{p}_{4\Lambda})_{\Lambda_{12}^{-1}}] \psi_B(-\mathbf{p}_{4\Lambda_{\Lambda_{34}^{-1}}}, \mathbf{p}_{4\Lambda_{\Lambda_{34}^{-1}}}) \tilde{V}(\mathbf{p}_1 - \mathbf{p}_4). \tag{7.11}
\end{aligned}$$

From Eqs. (5.10), (6.46), and (7.11), the reaction matrix element for the interaction V_{14} is

$$h_{fi[14]} = \left(\frac{M_{23}}{E_{23}} \frac{M_{12}}{E_{12}} \frac{M_{34}}{E_{34}} \right)^3 \frac{M_{14}}{E_{14}} \int d^3 p_1 d^3 p_4 \psi_A(\mathbf{p}_A, -\mathbf{p}_A) \psi_B(\mathbf{p}_B, -\mathbf{p}_B) \times \psi_C^*(\mathbf{p}_C, -\mathbf{p}_C) \psi_D^*(\mathbf{p}_D, -\mathbf{p}_D) \tilde{V}(\mathbf{p}_1 - \mathbf{p}_4), \tag{7.12}$$

where

$$\mathbf{p}_A = (-\mathbf{P}_{23}^c + \mathbf{P}_{34}^c - \mathbf{p}_{4\Lambda})_{\Lambda_{12}^{-1}}, \tag{7.13}$$

$$\mathbf{p}_B = -\mathbf{p}_{4\Lambda_{\Lambda_{34}^{-1}}}, \tag{7.14}$$

$$\mathbf{p}_C = (-\mathbf{P}_{34}^c + \mathbf{p}_{4\Lambda})_{\Lambda_{23}^{-1}}, \tag{7.15}$$

$$\mathbf{p}_D = \mathbf{p}_{1\Lambda\Lambda_{14}^{-1}}. \tag{7.16}$$

Note that the dimension of the wave function leads to an h_{fi} which has the dimension of $1/\text{mass}^2$ as required earlier. In the above expression, $\mathbf{p}_{1\Lambda\Lambda_{23}^{-1}}$, for example, is the space part of $(\Lambda_{23}^{-1}\Lambda p_1)$ and the four energy ratios correspond to the transformations associated with the (23),(14),(12),(34) composite particles. The energy ratio for the (14) composite differs from the others due to the transformation of the (14) interaction term. The term here corresponds to the C1 diagram in Fig. 1.

It is interesting to note that compared to the nonrelativistic case, the overlap matrix element now involves two major differences. First, the momentum arguments in the wave function need to be inversely boosted back to the frame in which the composite particles are at rest, as they should be. Second, there are factors of the type M_{ij}/E_{ij} appropriate for the composite particle in the collider frame. Both effects can lead to substantial modification of the magnitude of the reaction cross sections.

VIII. CONCLUSION AND SUMMARY

We seek a relativistic formulation of the many-body problem involving both bound states and reaction between constituents of composite particles. As a first example, we have focused our attention on a system of spinless particles interacting with a scalar and/or vector interaction.

We began by examining the relativistic two-body bound state problem and introduced Lagrangian multipliers to write down the most general two-body Hamiltonian. The formulation using the constraint dynamics allowed a simple separation of the center-of-mass and the relative motion. A two-

body equation was obtained in the form of a nonrelativistic Schrödinger equation which connects naturally to the corresponding nonrelativistic problem in the nonrelativistic limit. The two-body equation is independent of the Lagrange multipliers. The bound state mass is related to the eigenvalue of the nonrelativistic problem by a simple algebraic equation in the case in which one considers only relativistic kinematics. Further relativistic effects show up in this algebraic relationship when the energy dependence accompanying the scalar and vector interaction is taken into account.

For a many-particle system, we considered pairwise interaction between particles. In constructing the total Hamiltonian, a good choice of the Lagrange multiplier provides a simple way to separate the N -body Hamiltonian into the unperturbed Hamiltonian and residual interactions. It also presents a systematic way to use the two-body solution as basis states for multiparticle dynamics. The study of the dynamics involves the evaluation of the reaction matrix elements of a general two-body interaction in terms of the wave functions of the composite particles.

In rearrangement reactions, because there can be many ways to divide the total Hamiltonian, the evaluation of the reaction matrix elements should not depend on the choice of the unperturbed Hamiltonian and basis states. With our formulation, this “post-prior” equivalence can be shown explicitly, allowing for a meaningful definition of the perturbation expansion and treatment of the reaction dynamics.

Finally, we give an explicit formula for the reaction matrix elements in terms of the composite wave functions. In the relativistic treatment, the important effects include the

inverse boost of the relative momentum to the frame in which the composite particles are at rest, so as obtain the correct wave function. Furthermore, there are factors of M_{ij}/E_{ij} in the collider frame for the composite particles. These relations will be useful when we apply the present formulation to many problems in nuclear and particle physics such as meson-meson scattering. An aim would be to see how this approach modifies the results of the nonrelativistic formalism as present in [19].

The results we have obtained are very encouraging. We should in future work carry out a calculation for the relativistic $I=2\pi\pi$ scattering, to compare with the nonrelativistic results of Barnes and Swanson [19] and with experimental data. We should also extend our considerations to include the spin degree of freedom and more complicated interactions in the constraint description.

ACKNOWLEDGMENTS

The authors would like to thank Professor T. Barnes and Professor E. S. Swanson for stimulating discussions. One of us (C.Y.W.) would like to thank Professor Su Hounq Lee for helpful discussions and for his kind hospitality at Yonsei University where this work was completed. This work was supported by the U.S. Department of Energy under Contract No. DE-AC05-00OR22725 with UT-Battelle, LLC.

APPENDIX A: SCALAR PRODUCTS

Let us consider the general scalar product $\langle M(\mathbf{P}')|M(\mathbf{P})\rangle$

$$\begin{aligned} \langle M(\mathbf{P}')|M(\mathbf{P})\rangle &= \frac{M}{E'} \frac{M}{E} \int d^3p'_{1\Lambda'} d^3p'_{2\Lambda'} \langle 0|d(\mathbf{p}'_{2\Lambda'}) b(\mathbf{p}'_{1\Lambda'}) \psi_M^*(\mathbf{p}'_1, \mathbf{p}'_2) \delta^3(\mathbf{P}' - \mathbf{p}'_{1\Lambda'} - \mathbf{p}'_{2\Lambda'}) \\ &\quad \times \int d^3p_{1\Lambda} d^3p_{2\Lambda} \psi_M(\mathbf{p}_1, \mathbf{p}_2) \delta(\mathbf{P} - \mathbf{p}_{1\Lambda} - \mathbf{p}_{2\Lambda}) b^\dagger(\mathbf{p}_{1\Lambda}) d^\dagger(\mathbf{p}_{2\Lambda}) 0 \rangle \\ &= \frac{M}{E'} \frac{M}{E} \int d^3p'_{1\Lambda'} d^3p'_{2\Lambda'} \int d^3p_{1\Lambda} d^3p_{2\Lambda} \psi_M^*(\mathbf{p}'_1, \mathbf{p}'_2) \psi_M(\mathbf{p}_1, \mathbf{p}_2) \delta^3(\mathbf{P}' - \mathbf{P}) \delta^3(\mathbf{P} - \mathbf{p}_{1\Lambda} - \mathbf{p}_{2\Lambda}) \\ &\quad \times \delta^3(\mathbf{p}_{1\Lambda} - \mathbf{p}'_{1\Lambda'}) \delta^3(\mathbf{p}_{2\Lambda} - \mathbf{p}'_{2\Lambda'}), \end{aligned} \quad (\text{A1})$$

where $E = \sqrt{\mathbf{P}^2 + M^2}$, $E' = \sqrt{\mathbf{P}'^2 + M^2}$.

Now the total momentum delta function makes $\Lambda = \Lambda'$, which in turn implies that the two delta functions that come from the creation and annihilation operator force the arguments of the two wave functions to be the same. Thus

$$\begin{aligned} \langle M(\mathbf{P}')|M(\mathbf{P})\rangle &= \delta^3(\mathbf{P}' - \mathbf{P}) \frac{M^2}{E^2} \int d^3p_{1\Lambda} d^3p_{2\Lambda} \psi_M^*(\mathbf{p}_1, \mathbf{p}_2) \psi_M(\mathbf{p}_1, \mathbf{p}_2) \delta^3(\mathbf{P} - \mathbf{p}_{1\Lambda} - \mathbf{p}_{2\Lambda}) \\ &= \delta^3(\mathbf{P}' - \mathbf{P}) \frac{M^2}{E^2} \int d^3p_1 d^3p_2 \psi_M^*(\mathbf{p}_{1\Lambda^{-1}}, \mathbf{p}_{2\Lambda^{-1}}) \psi_M(\mathbf{p}_{1\Lambda^{-1}}, \mathbf{p}_{2\Lambda^{-1}}) \delta^3(\mathbf{P} - \mathbf{p}_1 - \mathbf{p}_2). \end{aligned} \quad (\text{A2})$$

On the right-hand side let

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2,$$

$$\mathbf{p} = \frac{\varepsilon_2}{M} \mathbf{p}_1 - \frac{\varepsilon_1}{M} \mathbf{p}_2, \quad (\text{A3})$$

so that the integral becomes

$$\begin{aligned} & \int d^3 P d^3 p \psi_M^* \left(\left(\frac{\varepsilon_1}{M} \mathbf{P} + \mathbf{p} \right)_{\Lambda^{-1}}, \left[\left(\frac{\varepsilon_2}{M} \mathbf{P} - \mathbf{p} \right) \right]_{\Lambda^{-1}} \right) \psi_M \left(\left(\frac{\varepsilon_1}{M} \mathbf{P} + \mathbf{p} \right)_{\Lambda^{-1}}, \left[\left(\frac{\varepsilon_2}{M} \mathbf{P} - \mathbf{p} \right) \right]_{\Lambda^{-1}} \right) \delta(\mathbf{P} - \mathbf{P}) \\ &= \int d^3 p \psi_M^* \left(\left(\frac{\varepsilon_1}{M} \mathbf{P} + \mathbf{p} \right)_{\Lambda^{-1}}, \left(\frac{\varepsilon_2}{M} \mathbf{P} - \mathbf{p} \right)_{\Lambda^{-1}} \right) \psi_M \left(\left(\frac{\varepsilon_1}{M} \mathbf{P} + \mathbf{p} \right)_{\Lambda^{-1}}, \left[\left(\frac{\varepsilon_2}{M} \mathbf{P} - \mathbf{p}' \right) \right]_{\Lambda^{-1}} \right) \\ &= \int d^3 p \psi_M^* ((\mathbf{0} + \mathbf{p}')_{\Lambda^{-1}}, ((\mathbf{0} - \mathbf{p}))_{\Lambda^{-1}}) \psi_M [(\mathbf{0} + \mathbf{p})_{\Lambda^{-1}}, ((\mathbf{0} - \mathbf{p}))_{\Lambda^{-1}}] \\ &\equiv \int d^3 p |\psi_M(\mathbf{p}_{\Lambda^{-1}})|^2 \\ &= \int d^4 p \delta(p^0) |\psi_M(\Lambda^{-1} p)|^2. \end{aligned} \quad (\text{A4})$$

But $p^0 = p \cdot \Lambda^{-1} P / M$. So taking $p = \Lambda \bar{p}$ and using $d^4 p' = d^4 \bar{p}$, we have the following manifestly covariant scalar product:

$$\int d^4 p \delta(p^0) |\psi_M(\Lambda^{-1} p)|^2 = M \int d^4 \bar{p} \delta(\bar{p} \cdot P') |\psi_M(\bar{p})|^2. \quad (\text{A5})$$

APPENDIX B: EVALUATION OF THE REACTION MATRIX ELEMENT

In this appendix we evaluate the matrix element of Eq. (7.10),

$$\begin{aligned} & \langle M(\mathbf{P}_{14}^c), M(\mathbf{P}_{23}^c); \mathbf{0} | U(\Lambda) V(\hat{\mathbf{x}}_{14}) U^{-1}(\Lambda) | M(\mathbf{P}_{12}^c), M(\mathbf{P}_{34}^c); \mathbf{0} \rangle \\ &= \langle M(\mathbf{P}_{14}^c), M(\mathbf{P}_{23}^c); \mathbf{0} | \int d^3 p_1''' d^3 p_4''' d^3 p_1' d^3 p_4' \delta(\mathbf{p}_1''' + \mathbf{p}_4''' - \mathbf{p}_1' - \mathbf{p}_4') \tilde{V}(\mathbf{p}_1''' - \mathbf{p}_4') \\ &\quad \times U(\Lambda) b^\dagger(\mathbf{p}_1''') d^\dagger(\mathbf{p}_4''') d(\mathbf{p}_4') b(\mathbf{p}_1') U(\Lambda)^{-1} | M(\mathbf{P}_{12}^c), M(\mathbf{P}_{34}^c); \mathbf{0} \rangle \\ &= \langle M(\mathbf{P}_{14}^c), M(\mathbf{P}_{23}^c); \mathbf{0} | \int d^3 p_1''' d^3 p_4''' d^3 p_1' d^3 p_4' \delta(\mathbf{p}_1''' + \mathbf{p}_4''' - \mathbf{p}_1' - \mathbf{p}_4') \\ &\quad \times \tilde{V}(\mathbf{p}_1''' - \mathbf{p}_4') b^\dagger(\mathbf{p}_{1\Lambda}''') d^\dagger(\mathbf{p}_{4\Lambda}''') d(\mathbf{p}_{4\Lambda}') b(\mathbf{p}_{1\Lambda}') | M(\mathbf{P}_{12}^c), M(\mathbf{P}_{34}^c); \mathbf{0} \rangle \frac{\mathbf{P}_{14}^{c'2} + M_{14}^2}{M_{14}^2}, \end{aligned} \quad (\text{B1})$$

in which the integrals include sums over flavor and color. This requires us to compute

$$\begin{aligned} & \langle 0 | d(\mathbf{p}_4'') b(\mathbf{p}_1'') d(\mathbf{p}_3'') b(\mathbf{p}_2'') b^\dagger(\mathbf{p}_{1\Lambda}''') d^\dagger(\mathbf{p}_{4\Lambda}''') d(\mathbf{p}_{4\Lambda}') b(\mathbf{p}_{1\Lambda}') b^\dagger(\mathbf{p}_1) d^\dagger(\mathbf{p}_2) b^\dagger(\mathbf{p}_3) d^\dagger(\mathbf{p}_4) | 0 \rangle \\ &= \langle 0 | [\delta(\mathbf{p}_{4\Lambda}''' - \mathbf{p}_3'') \delta(\mathbf{p}_{1\Lambda}''' - \mathbf{p}_2'') d(\mathbf{p}_4'') b(\mathbf{p}_1'') - \delta(\mathbf{p}_{4\Lambda}''' - \mathbf{p}_4'') \delta(\mathbf{p}_{1\Lambda}''' - \mathbf{p}_2'') d(\mathbf{p}_3'') b(\mathbf{p}_1'') - \delta(\mathbf{p}_{4\Lambda}''' - \mathbf{p}_3'') \delta(\mathbf{p}_{1\Lambda}''' - \mathbf{p}_1'') d(\mathbf{p}_4'') b(\mathbf{p}_2'') \\ &\quad + \delta(\mathbf{p}_{4\Lambda}''' - \mathbf{p}_4'') \delta(\mathbf{p}_{1\Lambda}''' - \mathbf{p}_1'') d(\mathbf{p}_3'') b(\mathbf{p}_2'')] \times [\delta(\mathbf{p}_{4\Lambda}' - \mathbf{p}_2) \delta(\mathbf{p}_{1\Lambda}' - \mathbf{p}_1) b^\dagger(\mathbf{p}_3) d^\dagger(\mathbf{p}_4) - \delta(\mathbf{p}_{4\Lambda}' - \mathbf{p}_4) \\ &\quad \times \delta(\mathbf{p}_{1\Lambda}' - \mathbf{p}_1) b^\dagger(\mathbf{p}_3) d^\dagger(\mathbf{p}_2) - \delta(\mathbf{p}_{4\Lambda}' - \mathbf{p}_2) \delta(\mathbf{p}_{1\Lambda}' - \mathbf{p}_3) b^\dagger(\mathbf{p}_1) d^\dagger(\mathbf{p}_4) + \delta(\mathbf{p}_{4\Lambda}' - \mathbf{p}_4) \delta(\mathbf{p}_{1\Lambda}' - \mathbf{p}_3) b^\dagger(\mathbf{p}_1) d^\dagger(\mathbf{p}_2)] | 0 \rangle, \end{aligned} \quad (\text{B2})$$

in which the delta functions include flavor and color indices. If we assume that flavors for 1 and 2 are distinct from those of 3 and 4, then of the 16 terms above the only one that survives is

$$\langle 0 | \delta(\mathbf{p}_{4\Lambda}''' - \mathbf{p}_4'') \delta(\mathbf{p}_{1\Lambda}''' - \mathbf{p}_1'') \delta(\mathbf{p}_{4\Lambda}' - \mathbf{p}_4) \delta(\mathbf{p}_{1\Lambda}' - \mathbf{p}_1) \delta(\mathbf{p}_3' - \mathbf{p}_3) \delta(\mathbf{p}_2' - \mathbf{p}_2) | 0 \rangle. \quad (\text{B3})$$

Thus (using the notation $E_{ij} = \sqrt{\mathbf{P}_{ij}^{c2} + M_{ij}^2}$), we obtain

$$\begin{aligned}
& \langle M(\mathbf{P}_{14}^c), M(\mathbf{P}_{23}^c); \mathbf{0} | U(\Lambda) V(\hat{\mathbf{x}}_{14}) U^{-1}(\Lambda) | M(\mathbf{P}_{12}^c), M(\mathbf{P}_{34}^c); \mathbf{0} \rangle \\
&= - \int d^3 p_4''' d^3 p_1''' d^3 p_4' d^3 p_1' d^3 p_3'' d^3 p_2'' \psi_D^*(\mathbf{p}_{1\Lambda\Lambda_{14}}''', \mathbf{p}_{4\Lambda\Lambda_{14}}''') \delta(\mathbf{P}_{14}^c - \mathbf{p}_{1\Lambda}''' - \mathbf{p}_{4\Lambda}''') \psi_C^*(\mathbf{p}_{2\Lambda_{23}}'', \mathbf{p}_{3\Lambda_{23}}'') \\
&\quad \times \delta(\mathbf{P}_{23}^c - \mathbf{p}_2'' - \mathbf{p}_3'') \psi_A(\mathbf{p}_{1\Lambda\Lambda_{12}}', \mathbf{p}_{2\Lambda_{12}}'') \delta(\mathbf{P}_{12}^c - \mathbf{p}_1' - \mathbf{p}_{1\Lambda}') \psi_B(\mathbf{p}_{3\Lambda_{34}}'', \mathbf{p}_{4\Lambda_{34}}'') \delta(\mathbf{P}_{34}^c - \mathbf{p}_{4\Lambda}'' - \mathbf{p}_3'') \\
&\quad \times \delta(\mathbf{p}_1''' + \mathbf{p}_4''' - \mathbf{p}_1' - \mathbf{p}_4') \tilde{V}(\mathbf{p}_1''' - \mathbf{p}_4') \frac{M_{23}}{E_{23}} \frac{E_{14}}{M_{14}} \frac{M_{12}}{E_{12}} \frac{M_{34}}{E_{34}}. \tag{B4}
\end{aligned}$$

We perform four of the remaining six volume integrals of the first set of integrals. In particular, we perform integrations over $d^3 p_4''' d^3 p_1' d^3 p_3'' d^3 p_2''$. We use the first delta function $\mathbf{p}_{4\Lambda}''' = \mathbf{P}_{14}^c - \mathbf{p}_{1\Lambda}'''$, the second gives $\mathbf{p}_2'' = \mathbf{P}_{23}^c - \mathbf{p}_3''$, the third gives $\mathbf{p}_{1\Lambda}' = \mathbf{P}_{12}^c - \mathbf{p}_1''$, and the fourth gives $\mathbf{p}_3'' = \mathbf{P}_{34}^c - \mathbf{p}_{4\Lambda}''$. The argument of the remaining delta function is then

$$\mathbf{p}_1''' + \mathbf{p}_4''' - \mathbf{p}_1' - \mathbf{p}_4' = \mathbf{P}_{14\Lambda}^c - \mathbf{P}_{12\Lambda}^c + \mathbf{P}_{34\Lambda}^c - \mathbf{P}_{23\Lambda}^c = \mathbf{P}_{34} - \mathbf{P}_{14} + \mathbf{P}_{12} - \mathbf{P}_{23}, \tag{B5}$$

which corresponds to overall momentum conservation in the frame in which we evaluate the matrix element (the $\mathbf{P}_{14} = 0$ frame). Then, the reaction matrix element becomes

$$\begin{aligned}
& \langle M(\mathbf{P}_{14}^c), M(\mathbf{P}_{23}^c); \mathbf{0} | U(\Lambda) V(\hat{\mathbf{x}}_{14}) U^{-1}(\Lambda) | M(\mathbf{P}_{12}^c), M(\mathbf{P}_{34}^c); \mathbf{0} \rangle \\
&= - \delta^3(\mathbf{P}_{34} - \mathbf{P}_{14} + \mathbf{P}_{12} - \mathbf{P}_{23}) \int d^3 p_1''' d^3 p_4' \psi_D^*[\mathbf{p}_{1\Lambda\Lambda_{14}}''', (\mathbf{P}_{14}^c - \mathbf{p}_{1\Lambda}')_{\Lambda_{14}}^{-1}] \times \psi_C^*(\mathbf{P}_{23}^c - \mathbf{P}_{34}^c + \mathbf{p}_{4\Lambda}')_{\Lambda_{23}}^{-1}, \\
&\quad (\mathbf{P}_{34}^c - \mathbf{p}_{4\Lambda}')_{\Lambda_{23}}^{-1}) \psi_A[(\mathbf{P}_{12}^c - \mathbf{P}_{23}^c + \mathbf{P}_{34}^c - \mathbf{p}_{4\Lambda}')_{\Lambda_{12}}^{-1}, (\mathbf{P}_{23}^c - \mathbf{P}_{34}^c + \mathbf{p}_{4\Lambda}')_{\Lambda_{12}}^{-1}] \\
&\quad \times \psi_B[(\mathbf{P}_{34}^c - \mathbf{p}_{4\Lambda}')_{\Lambda_{34}}^{-1}, \mathbf{p}_{4\Lambda\Lambda_{34}}'] \tilde{V}(\mathbf{p}_1''' - \mathbf{p}_4') \frac{M_{23}}{E_{23}} \frac{E_{14}}{M_{14}} \frac{M_{12}}{E_{12}} \frac{M_{34}}{E_{34}}. \tag{B6}
\end{aligned}$$

But $\mathbf{P}_{14\Lambda_{14}}^c = 0 = \mathbf{P}_{12\Lambda_{12}}^c = \mathbf{P}_{34\Lambda_{34}}^c = \mathbf{P}_{23\Lambda_{23}}^c$ so that

$$\begin{aligned}
& \langle M(\mathbf{P}_{14}^c), M(\mathbf{P}_{23}^c); \mathbf{0} | U(\Lambda) V(\hat{\mathbf{x}}_{14}) U^{-1}(\Lambda) | M(\mathbf{P}_{12}^c), M(\mathbf{P}_{34}^c); \mathbf{0} \rangle \\
&= - \delta^3(\mathbf{P}_{34} - \mathbf{P}_{14} + \mathbf{P}_{12} - \mathbf{P}_{23}) \int d^3 p_1''' d^3 p_4' \psi_D^*(\mathbf{p}_{1\Lambda\Lambda_{14}}''', -\mathbf{p}_{1\Lambda\Lambda_{14}}''') \psi_C^*[-(\mathbf{P}_{34}^c + \mathbf{p}_{4\Lambda}')_{\Lambda_{23}}^{-1}, (\mathbf{P}_{34}^c - \mathbf{p}_{4\Lambda}')_{\Lambda_{23}}^{-1}] \\
&\quad \times \tilde{V}(\mathbf{p}_1''' - \mathbf{p}_4') \psi_A[-(\mathbf{P}_{23}^c + \mathbf{P}_{34}^c - \mathbf{p}_{4\Lambda}')_{\Lambda_{12}}^{-1}, (\mathbf{P}_{23}^c - \mathbf{P}_{34}^c + \mathbf{p}_{4\Lambda}')_{\Lambda_{12}}^{-1}] \psi_B(-\mathbf{p}_{4\Lambda\Lambda_{34}}', \mathbf{p}_{4\Lambda\Lambda_{34}}') \frac{M_{23}}{E_{23}} \frac{E_{14}}{M_{14}} \frac{M_{12}}{E_{12}} \frac{M_{34}}{E_{34}}, \tag{B7}
\end{aligned}$$

which corresponds to the indicated amplitude in which a gluon is exchanged between particle 1 and particle 4. This produces the form in the text (where we dropped the primes on the two integration variables).

-
- [1] P. A. M. Dirac, *Can. J. Math.* **2**, 129 (1950); *Proc. R. Soc. London, Ser. A* **246**, 326 (1958); *Lectures on Quantum Mechanics* (Yeshiva University, New York, 1964).
[2] The following references detail the first applications of Dirac's constraint formalism to the two-body problem. M. Kalb and P. Van Alstine, *Yale Reports*, C00-3075-146 (1976), C00-3075-156 (1976); P. Van Alstine, Ph.D. dissertation, Yale University, 1976; I. T. Todorov, *Dubna Joint Institute for Nuclear Research No. E2-10175*, 1976; *Ann. Inst. Henri Poincaré, Sect. A* **28**, 207 (1978); L. P. Horwitz and F. Rohrlich, *Phys. Rev. D* **24**, 1528 (1981); F. Rohrlich, *ibid.* **23**, 1305 (1981); *Nucl. Phys.* **B161**, 469 (1979); A. Komar, *Phys. Rev. D* **18**, 1881 (1978); **18**, 1887 (1978); P. Droz-Vincent, *Rep. Math. Phys.* **8**, 79 (1975).

- [3] D. G. Currie, T. F. Jordan, and E. C. G. Sudarshan, *Rev. Mod. Phys.* **35**, 350 (1963).
[4] H. W. Crater, C. W. Wong, and C. Y. Wong, *Int. J. Mod. Phys. E* **5**, 589 (1996); J. Mourad and H. Sazdjian, *J. Phys. G* **21**, 267 (1995).
[5] P. Van Alstine and H. W. Crater, *J. Math. Phys.* **23**, 1697 (1982); H. W. Crater and P. Van Alstine, *Ann. Phys. (N.Y.)* **148**, 57 (1983); supersymmetry discussions in the above references were based in part on C. A. P. Galvao and C. Teitelboim, *J. Math. Phys.* **21**, 1863 (1980); see also A. Barducci, R. Casalbuoni, and L. Lusanna, *Nuovo Cimento A* **32**, 377 (1976); T. Takabayasi, *Prog. Theor. Phys.* **117**, 292 (1979); H. Sazdjian, *Phys. Rev. D* **33**, 3401 (1986).
[6] See, for example, D. P. Stanley and D. Robson, *Phys. Rev. D*

- 21**, 3180 (1980); S. Godfrey and N. Isgur, *ibid.* **32**, 189 (1985).
- [7] H. W. Crater, R. Becker, Cheuk-Yin Wong, and P. Van Alstine, *Phys. Rev. D* **46**, 5117 (1992).
- [8] H. Sazdjian, *Phys. Lett.* **156B**, 381 (1985); *Extended Objects and Bound Systems*, Proceedings of the Karuizawa International Symposium, 1992, edited by O. Hara, S. Ishida, and S. Nake (World Scientific, Singapore, 1992), p. 117; H. Jollouli and H. Sazdjian, *Ann. Phys. (N.Y.)* **253**, 376 (1997).
- [9] I. T. Todorov, in *Properties of the Fundamental Interactions*, edited by A. Zichichi (Editrice Compositori, Bologna, 1973), Vol. 9, part C, pp. 953–979; *Phys. Rev. D* **3**, 2351 (1971).
- [10] H. W. Crater and P. Van Alstine, *Phys. Rev. D* **37**, 1982 (1988).
- [11] H. Sazdjian, *Ann. Phys. (N.Y.)* **191**, 52 (1989); *Phys. Rev. D* **33**, 3401 (1986).
- [12] R. Blankenbecler and R. Sugar, *Phys. Rev.* **142**, 1051 (1966).
- [13] G. P. Lepage, *Phys. Rev. A* **16**, 863 (1977).
- [14] V. B. Mandelzweig and S. J. Wallace, *Phys. Lett. B* **197**, 469 (1987); S. J. Wallace and V. B. Mandelzweig, *Nucl. Phys. A* **503**, 673 (1989); D. R. Phillips and S. J. Wallace, *Phys. Rev. C* **54**, 507 (1996).
- [15] J. Bijtebier and J. Broekaert, *Nucl. Phys. A* **612**, 279 (1997).
- [16] F. Gross, *Phys. Rev.* **186**, 1448 (1969); *Phys. Rev. C* **26**, 2203 (1982); F. Gross, J. W. Van Orden, and K. Holinde, *ibid.* **45**, 2094 (1992).
- [17] Y. S. Kim and M. E. Noz, *Theory and Applications of the Poincaré Group* (Reidel, Dordrecht, 1986).
- [18] G. R. Satchler, *Direct Nuclear Reactions* (Oxford University Press, New York, 1980).
- [19] T. Barnes and E. S. Swanson, *Phys. Rev. D* **46**, 131 (1992).
- [20] G. E. Brown, *Many-Body Problem* (North-Holland, Amsterdam, 1972).
- [21] P. Ring and P. Schuck, *The Nuclear Many-Body Problem* (Springer, New York, 1980).
- [22] T. Barnes, E. S. Swanson, and J. Weinstein, *Phys. Rev. D* **46**, 4868 (1992).
- [23] T. Barnes and E. S. Swanson, *Phys. Rev. C* **49**, 1166 (1994).
- [24] T. Barnes, S. Capstick, M. D. Kovarik, and E. S. Swanson, *Phys. Rev. C* **48**, 539 (1993).
- [25] T. Barnes, N. Black, D. J. Dean, and E. S. S. Swanson, *Phys. Rev. C* **60**, 045202 (1999).
- [26] C. Y. Wong, E. S. Swanson, and T. Barnes, *Phys. Rev. C* **62**, 045201 (2000).
- [27] C. Y. Wong, E. S. Swanson, and T. Barnes, Proceedings of the Hirscheegg 2000 Workshop, Hadrons in Dense Matter, Hirscheegg, Austria, 2000, nucl-th/0002034.
- [28] T. Barnes, E. S. Swanson, and C. Y. Wong, Proceedings of the Dubna Meeting on Heavy Quark Physics Dubna, 2000, nucl-th/0006012.
- [29] H. W. Crater and P. Van Alstine, *Phys. Rev. D* **30**, 2585 (1984).
- [30] H. Crater and P. Van Alstine, *Phys. Lett.* **100B**, 166 (1981).
- [31] H. W. Crater and P. Van Alstine, *J. Math. Phys.* **31**, 1998 (1990); P. Long and H. W. Crater, *ibid.* **39**, 124 (1998).
- [32] H. W. Crater and D. Yang, *J. Math. Phys.* **32**, 2374 (1991); H. W. Crater and P. Van Alstine, *Phys. Rev. D* **46**, 766 (1992).
- [33] H. W. Crater and P. Van Alstine, *Phys. Rev. D* **36**, 3007 (1987).
- [34] H. W. Crater and P. Van Alstine, *Found. Phys.* **24**, 297 (1994).
- [35] H. Jallouli and H. Sazdjian, *Phys. Lett. B* **366**, 409 (1996).
- [36] H. Sazdjian, *Lett. Math. Phys.* **5**, 319 (1981); *Ann. Phys. (N.Y.)* **136**, 136 (1981).
- [37] F. Coester and W. N. Polyzou, *Phys. Rev. D* **26**, 1348 (1982), and references cited therein; F. Coester, in *Constraint's Theory and Relativistic Dynamics*, edited by G. Longhi and L. Lusanna (World Scientific, Singapore, 1987), p. 159; in *The Three-Body Force in the Three Nucleon System*, edited by B. L. Berman and B. F. Gibson (Springer, New York, 1986), p. 472; see also recent work by J. Bijtebier, *Nucl. Phys. A* **623**, 498 (1997).
- [38] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill, New York, 1968), pp. 384–387.
- [39] This is the time component of the relativistic center-of-mass position operator X^μ satisfying $[X^\mu, P^\nu] = i\eta^{\mu\nu}$ and whose classical form is $X^\mu = (\varepsilon_1 x_1^\mu + \varepsilon_2 x_2^\mu) / M + (m_1^2 - m_2^2) x P P^\mu / (P^2)^2$ (see [29], p. 2590). This commutator leads to $\langle t | P^0 \rangle = \exp(-iP^0 t)$.
- [40] Steven Weinberg, *The Quantum Theory of Fields* (Cambridge University Press, Cambridge, 1995), Vol. 1.