Quark-antiquark bound states in the relativistic spectator formalism

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The original model of $q\bar{q}$-bound states, developed by Gross and Milana, which uses the relativistic spectator (Gross) equations to give a manifestly covariant description of confinement in Minkowski space that is consistent with chiral symmetry, is improved and extended. These improvements include (i) derivation of the normalization condition for the relativistic wave functions, (ii) proof that confinement automatically prohibits decays by implying the vanishing of the vertex function when both quarks are on shell, (iii) extension of the model to the strange quark sector and to sectors with unequal quark masses, (iv) removal of unphysical singularities associated with the confining interaction, and (v) inclusion of a realistic one-gluon-exchange interaction. We use phenomenological quark mass functions to build chiral symmetry into the theory and to explain the connection between the current quark and constituent quark masses. We obtain reasonable results for pions and kaons, establishing that the formalism, designed to work well in the heavy quark sectors, can also be extended to the light quark sector.

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I. INTRODUCTION

Description of simple hadrons in terms of quark-gluon degrees of freedom has long been an active area in physics. With the advent of Jefferson Laboratory, which operates at intermediate energies and therefore probes the structure of hadrons, there are new opportunities to test simple theoretical descriptions of quark interactions. The first natural step in this direction is to obtain a thorough understanding of how to treat the relativistic quark-antiquark bound state problem. In this context, Nambu–Jona-Lasinio (NJL) inspired models have gained popularity in recent years [1–3]. The common goal of these works is to bridge the gap between nonrelativistic quark models and more rigorous approaches, such as lattice gauge theory or Feynman-Schwinger calculations. While the Euclidean metric based calculations avoid the complicated singularities present in Minkowski space, the required extrapolations limit their applicability to light bound states such as the pion and the kaon. Therefore, it is important to develop Minkowski metric based models that can be used over a wider scale of energies. One such work using the spectator formalism was developed in Refs. [1,2]. In those works a relativistic generalization of the linear potential was developed and the pion was shown to be massless in the chiral limit. However, the calculations involved some approximations and related conceptual problems. In this work we improve and simplify the model presented in those works and address in detail some of the conceptual issues related to confinement.

If a quark-antiquark pair (referred to collectively as ''quarks'') is confined to a meson bound state with mass μ, then the bound state cannot decay into two free quarks, even if the sum of the quark masses is less than the bound state mass. This trivial statement can be realized by two possible mechanisms: either (a) the quark propagators are free of timelike mass poles, [3,4] or (b) the vertex function of the bound state *vanishes* when both quarks are on-shell. In this work we *prove* that the Gross equation supports the second mechanism of confinement. The first mechanism, which is commonly used in Euclidean metric based calculations, is a stronger constraint since it forbids any free quark states. On the other hand, it has been recently shown [5,6] that *the lack of physical mass poles may be a consequence of a poor approximation to the physics, and not a consequence of confinement.* We emphasize that the second mechanism using the Gross equation allows one of the two quarks in a meson to be on-shell, but insures that, in the presence of a confining interaction, the matrix element that couples the bound state to *two* free quarks vanishes automatically. The spectator formalism facilitates the use of the Minkowski metric, and the confinement mechanism of this approach has a closer resemblance to nonrelativistic models.

The organization of the paper is as follows. In Sec. II we review the formalism for nonrelativistic confinement in momentum space. This discussion is carried out in momentum space in order to prepare for the relativistic generalization, which can only be done in momentum space. Two different methods of defining the momentum space confining interaction are introduced, discussed, and compared. Both are needed for Sec. III, where we outline the general philosophy of the spectator treatment of confined systems, examine the implications of confinement for the scattering amplitude, and prove that the relativistic linear potential used in earlier works automatically ensures that $\mu\to q+\bar{q}$ vanishes at the momentum where decay of the state into two physical quarks would otherwise be kinematically possible. The treatment is first presented for scalar particles, and then generalized to fermions. The development extends the initial work of Gross and Milana [1,2] and permits us to find the covariant normalization condition for the first time. We then introduce a new form for the linear confining kernel in momentum space, Eq. (3.21), which allows us to remove the unphysical singularities present in the work of Ref. [1] and extend the calculations to the strange quark sector. Use of the kernel (3.21) requires a different treatment of quark self energies than originally published [1,2], and in Sec. IV we construct quark mass functions with the correct chiral limit. These mass functions are consistent with asymptotic freedom, and allow us to choose parameters that give good numerical results for

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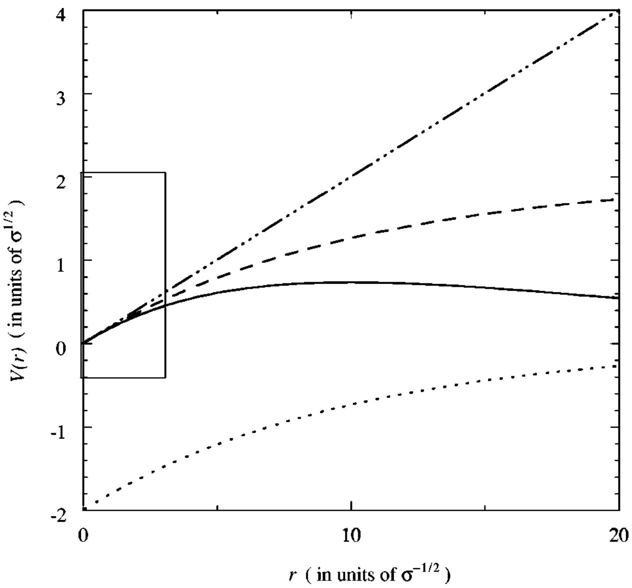

FIG. 1. The linear potential in coordinate space for $\epsilon = 0.1\sqrt{\sigma}$ and $\sigma = 0.2$. The solid line is $\tilde{V}_S(r)$, the dashed line is $\tilde{V}_L(r)$, the dotted line is $\tilde{V}_A(r)$, and the dot-dashed line is $\tilde{V}(r)$. For ''small'' $r < 0.3/\epsilon$ (the region inside the small box) $\tilde{V}_L(r)$ and $\tilde{V}_S(r)$ are both approximately equal to σr.

pseudoscalar bound states. The results are presented in Sec. V, and some conclusions are given in Sec. VI.

II. NONRELATIVISTIC CONFINEMENT IN MOMENTUM SPACE

A. Alternative approximations for the nonrelativistic linear potential

We start by reviewing the discussion of confinement within the context of the nonrelativistic Schrödinger equation given in Ref. [1]. We will denote potentials in coordinate space by $\tilde{V}$ and in momentum space by V. The nonrelativistic linear potential is

$$\tilde{V}(r) = \sigma r. \tag{2.1}$$

This potential can be constructed from familiar Yukawa-like potentials in two different ways:

$$\tilde{V}(r) = \lim_{\epsilon\to 0} \times \begin{cases} \tilde{V}_S(r) \equiv \sigma r e^{-\epsilon r} & \text{(2.2a)} \\ \tilde{V}_L(r) \equiv -\dfrac{\sigma}{\epsilon}(e^{-\epsilon r} - 1) = \tilde{V}_A(r) + \dfrac{\sigma}{\epsilon}. & \text{(2.2b)} \end{cases}$$

These various potentials are shown in Fig. 1 for the illustrative case of $\epsilon = 0.1\sqrt{\sigma}$ and $\sigma = 0.2$. Note that the two potentials $\tilde{V}_S(r)$ and $\tilde{V}_L(r)$ both approximate the linear potential $\tilde{V}(r)$ when $r \ll 1/\epsilon$, but these two approximate potentials behave very differently at large r.

The potential $\tilde{V}_S(r) \to 0$ at large r, so that, strictly speaking, it does not confine particles at all. This potential always permits scattering, and we will therefore refer to this as the *scattering form* of the linear potential. If ϵ is small the scattering is strongly resonant, and the wave function is significant at small r only for energies near one of the allowed resonances. The width of these resonance states becomes narrower, and their wave function approaches that of a bound state, as $\epsilon \to 0$.

In contrast, the potential $\tilde{V}_L(r) \to 1/\epsilon$ as $r \to \infty$ and therefore binds particles with energies $E < 1/\epsilon$. We will refer to this as the *bound state form* potential. As $\epsilon \to 0$ this potential does not permit scattering at any energy; it has a spectrum of bound states only.

These two approximate forms of the linear confining potential are very different, yet for sufficiently small ϵ it should be possible to move freely from one of these potentials to the other, and the results obtained with either form should be equivalent. *The ability to move freely from one form to the other is very helpful to the discussion of normalization and scattering, and will be assumed without formal proof.* We will return to this later in this section.

B. Two equivalent forms of the Schrödinger equation with confinement

Next we write a Schrödinger equation appropriate for each of the potentials (2.2). Following Ref. [1], the momentum space form of $\tilde{V}_L$ given in Eq. (2.2b) can be written as

$$V_L(\mathbf{q}) = \lim_{\epsilon\to 0}\left[V_A(\mathbf{q}) - \delta^3(q)\int d^3q'\, V_A(\mathbf{q}')\right], \tag{2.3}$$

where

$$V_A(\mathbf{q}) = -\frac{8\pi\sigma}{(\mathbf{q}^2 + \epsilon^2)^2}. \tag{2.4}$$

Note that the second term (the ''subtraction term'') ensures that

$$\int d^3q\, V_L(\mathbf{q}) = 0, \tag{2.5}$$

which is the momentum space form of the statement that $\tilde{V}(r=0) = 0$. This subtraction has been previously used in Refs. [1,2], and also by Adler and Davis [7] to regularize their treatment of confinement in the coulomb gauge. The Fourier transform of V_A, for finite ϵ, is

$$\tilde{V}_A(\mathbf{r}) = \int \frac{d^3q}{(2\pi)^3} e^{-iq\cdot r} V_A(\mathbf{q}) \tag{2.6}$$

$$= -\sigma\frac{e^{-\epsilon r}}{\epsilon} \simeq \lim_{\epsilon\to 0}\sigma\left(r - \frac{1}{\epsilon}\right), \tag{2.7}$$

and the subtraction term cancels the singular $1/\epsilon$ term insuring that the linear part of the potential has the correct behavior in the limit as $\epsilon \to 0$ and that it vanishes at the origin ($r = 0$).

Adding a potential V_C, constant in coordinate space

$$V_C(r) = -C,$$

$$V_C(\mathbf{q}) = -(2\pi)^3 \delta^3(q) C, \tag{2.8}$$

to the potential (2.2b), and inserting the total potential into the momentum space Schrödinger equation gives

$$\left[\frac{\mathbf{p}^2}{2m_R} - E\right]\Psi_A(\mathbf{p},p_0) = -\int \frac{d^3k}{(2\pi)^3} V_A(\mathbf{p}-\mathbf{k})\,[\Psi_B(\mathbf{k},p_0) - \Psi_A(\mathbf{p},p_0)] + C\,\Psi_A(\mathbf{p},p_0), \tag{2.9}$$

or, alternatively,

$$\left[\frac{\mathbf{p}^2}{2m_R} - E - \tilde{V}_A(0)\right]\Psi_A(\mathbf{p},p_0) = -\int \frac{d^3k}{(2\pi)^3} V_A(\mathbf{p}-\mathbf{k})\,\Psi_A(\mathbf{k},p_0). \tag{2.10}$$

Here m_R is the reduced mass, E is the energy,

$$p_0^2 = 2m_R E, \tag{2.11}$$

and, for the linear potential introduced above,

$$\tilde{V}_A(0) = -\frac{\sigma}{\epsilon}. \tag{2.12}$$

The constant potential is used to adjust the energy scale. These equations will be referred to as the *bound state* form of the equation. The name comes from the fact that the spectrum of Eq. (2.9) consists of bound states and scattering states, where

$$E < \frac{\sigma}{\epsilon} + C = E_{\rm crit} \quad \text{bound states,}$$

$$E > \frac{\sigma}{\epsilon} + C = E_{\rm crit} \quad \text{scattering states.}$$

This shows clearly how, in the limit $\epsilon \to 0$, the equations confine quarks of any energy.

While Eq. (2.10) has no scattering states for $E < E_{\rm crit}$, it is clear that the equation obtained by replacing $\tilde{V}_L(r)$ by its counterpart, $\tilde{V}_S(r)$ defined in Eq. (2.2a), has scattering states for all $E > 0$. This potential has no subtraction, so its momentum space Schrödinger equation is simply

$$\left[\frac{\mathbf{p}^2}{2m_R} - E\right]\Psi_S(\mathbf{p},p_0) = -\int \frac{d^3k}{(2\pi)^3} V_S(\mathbf{p}-\mathbf{k})\,\Psi_S(\mathbf{k},p_0). \tag{2.13}$$

This will be referred to as the *scattering* form of the equation. As stated above, we will assume that the two equations (2.10) and (2.13) give equivalent results when ϵ is very small.

It is important to our relativistic discussion in the next section to be very clear in what sense the two equations (2.10) and (2.13) are equivalent. To develop this idea more precisely, introduce wave functions suggested by the bound (2.10) and scattering (2.13) forms of the Schrödinger equation. These wave functions are

$$\Psi_A(\mathbf{p},p_0) = -\frac{2m_R\,\gamma(\mathbf{p},p_0)}{\mathbf{p}^2 - p_0^2}, \tag{2.14}$$

$$\Psi_S(\mathbf{p},p_0) = \mathcal{P}\left\{(2\pi)^3\,\eta\,\delta^3(p-p') - \frac{2m_R\,M_S(\mathbf{p},\mathbf{p}')}{\mathbf{p}^2 - p_0^2}\right\}, \tag{2.15}$$

where γ is the nonrelativistic *vertex* function (defined in analogy with the relativistic vertex function) and M_S is the half off-shell scattering amplitude, $\mathbf{p}'^2 = p_0^2$, and $\mathcal{P}$ is the operator that projects out the quantum numbers of the initial state (the same spin, angular momentum, and internal symmetries as the bound described by Ψ_A). The wave function (2.15) has the form of the usual scattering wave function, with the δ function describing the asymptotic plane-wave part. We have chosen to multiply this plane-wave part by a (small) parameter η. This parameter can be removed by dividing the wave function and the half off-shell scattering amplitude by η, so it is, strictly speaking, an arbitrary scale factor.

However, the size of η is fixed by physical considerations. When ϵ is very small and the energy $E \ll E_{\rm crit}$, the wave function Ψ_S is, in general, very small at short distances r. The exception occurs at special *resonant* energies near the bound state energies of Eq. (2.10). At these energies the boundary conditions can be satisfied by a scattering solution large at small r, with a small oscillatory tail escaping to infinity (as shown in Fig. 2). If we require that the scattering solution (2.15) and the bound state solution (2.14) be of *comparable size at short distances*, it is necessary to choose η small, as illustrated in Fig. 2. This complicated limiting process will be summarized by the equation

$$\Psi_S(\mathbf{p},p_0) \leftrightarrow \Psi_A(\mathbf{p},p_0), \tag{2.16}$$

where the $\leftrightarrow$ symbol means that the spectrum of resonance scattering states obtained from Eq. (2.13) converges to the spectrum of bound states obtained from Eq. (2.10), and that, in the region of confinement (i.e., where $r \leqslant E/\sigma$), the resonant wave functions (2.15) are comparable to the bound state wave functions (2.14), and the nonresonant solutions to Eq. (2.13) are very small.

The equivalence (2.16) can also be stated in terms of the scattering amplitude and the vertex function

$$\mathcal{P}M_S(\mathbf{p},\mathbf{p}') \leftrightarrow \gamma(\mathbf{p},p_0). \tag{2.17}$$

These amplitudes satisfy the following equations:

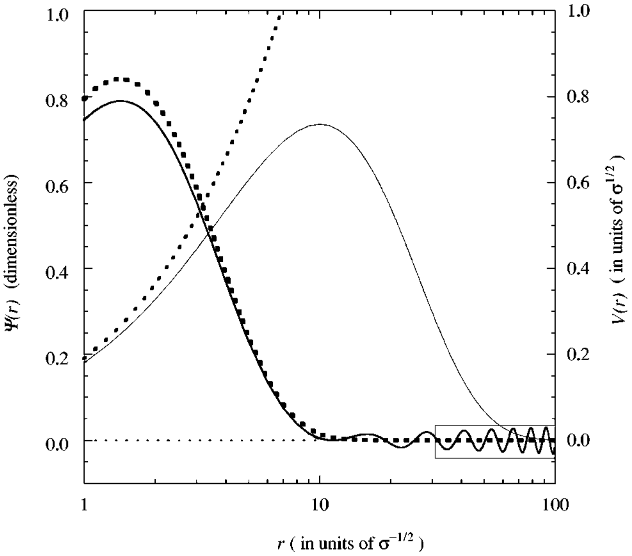

FIG. 2. Comparison of possible wave functions $\Psi_A(r)$ (falling dotted line) and $\Psi_S(r)$ (heavy solid line). [For reference, the potentials $\tilde{V}_S(r)$ (thin solid line) and $\tilde{V}_L(r)$ (rising dotted line) are also shown.] The normalization is chosen so that $\Psi_A \leftrightarrow \Psi_S$, making the plane-wave tail of Ψ_S (shown in the box) small. In this example $\eta \simeq 0.05$.

$$M_S(\mathbf{p},\mathbf{p}') = \eta\, V_S(\mathbf{p}-\mathbf{p}') - 2m_R \int \frac{d^3k}{(2\pi)^3} V_S(\mathbf{p}-\mathbf{k}) \times \frac{M_S(\mathbf{k},\mathbf{p}')}{\mathbf{k}^2 - p_0^2} + \frac{2m_R C\, M_S(\mathbf{p},\mathbf{p}')}{\mathbf{p}^2 - p_0^2}, \tag{2.18}$$

$$\gamma(\mathbf{p},p_0) = -2m_R \int \frac{d^3k}{(2\pi)^3} V_A(\mathbf{p}-\mathbf{k}) \times \left[\frac{\gamma(\mathbf{k},p_0)}{\mathbf{k}^2 - p_0^2} - \frac{\gamma(\mathbf{p},p_0)}{\mathbf{p}^2 - p_0^2}\right] + \frac{2m_R C\, \gamma(\mathbf{p},p_0)}{\mathbf{p}^2 - p_0^2}, \tag{2.19}$$

where, in the limit $\epsilon \to 0$ (and $\eta \to 0$), the two amplitudes M_S and γ are equivalent [in the sense of Eq. (2.17)].

In the next section we will use the generalization of M_S when ϵ is very small but nonzero, and γ when we want to discuss exact confinement ($\epsilon = 0$). Only Eq. (2.19) has a well defined mathematical limit when $\epsilon \to 0$. It will always be assumed that either M_S or γ may be used with equivalent results.

C. Implications of confinement in nonrelativistic theory

We now derive an interesting restriction on the vertex function γ that will have an important physical interpretation in the next section. To obtain this result, look at Eq. (2.19) when $\mathbf{p}^2 \to p_0^2$. Expand γ around p_0^2 using the fact that γ depends only on $\mathbf{p}^2$,

$$\gamma(\mathbf{p},p_0) = \gamma(\mathbf{p}_0,p_0) + (\mathbf{p}^2 - p_0^2)\, \mathcal{R}(\mathbf{p},p_0), \tag{2.20}$$

and then substitute this into Eq. (2.19) [with $C = 0$ for the moment], giving

$$\gamma(\mathbf{p},p_0) = -2m_R\, \gamma(\mathbf{p}_0,p_0) \int \frac{d^3k}{(2\pi)^3} V_A(\mathbf{p}-\mathbf{k}) \times \left[\frac{1}{\mathbf{k}^2 - p_0^2} - \frac{1}{\mathbf{p}^2 - p_0^2}\right] - 2m_R \int \frac{d^3k}{(2\pi)^3} \times V_A(\mathbf{p}-\mathbf{k})\, [\mathcal{R}(\mathbf{k},p_0) - \mathcal{R}(\mathbf{p},p_0)]. \tag{2.21}$$

All terms on the (right-hand side) of this equation should be regular as $\mathbf{p}^2 \to p_0^2$. Because of the subtraction, the term involving $\mathcal{R}$ is finite, and, because of our choice of p_0, only *one* of the two remaining terms is zero if ϵ is *finite*,

$$\lim_{\mathbf{p}^2 \to p_0^2} \int \frac{d^3k}{(2\pi)^3} \frac{V_A(\mathbf{p}-\mathbf{k})}{\mathbf{k}^2 - p_0^2} = -\frac{\sigma}{\epsilon^2}\left(\frac{\epsilon \pm 2ip_0}{4p_0^2 + \epsilon^2}\right) \to \text{finite},$$

$$\lim_{\mathbf{p}^2 \to p_0^2} \int \frac{d^3k}{(2\pi)^3} \frac{V_A(\mathbf{p}-\mathbf{k})}{\mathbf{p}^2 - p_0^2} = -\frac{\sigma}{\epsilon} \lim_{\mathbf{p}^2 \to p_0^2}\left(\frac{1}{\mathbf{p}^2 - p_0^2}\right) \to \infty. \tag{2.22}$$

Hence the subtraction term will be singular unless

$$\gamma(\mathbf{p}_0,p_0) = 0. \tag{2.23}$$

This condition also ensures that the constant term is not singular. We will discuss the physical interpretation of this result in the next section.

III. CONFINEMENT IN THE SPECTATOR FORMALISM

A. Introduction

At this point it is very tempting to generalize the nonrelativistic linear potential (2.3) by simply replacing the three vector $\mathbf{q}$ by a four vector q

$$\mathcal{V}(\mathbf{q}) \overset{?}{\to} \lim_{\epsilon \to 0} \left[V_A(q) - \delta^4(q) \int d^4q'\, V_A(q') \right] + (2\pi)^3 \delta^4(q) C. \tag{3.1}$$

This, seemingly obvious, generalization will not reduce to the correct nonrelativistic limit because of the unconstrained behavior of the $\int dq_0'\, V_A(q')$ integral. Lacking a four dimensional expression for the linear interaction that reduces to the correct nonrelativistic limit, we rephrase our question: Can one find a covariant equation that reduces to the Schrödinger equation (2.19) with a linear interaction? The confining relativistic bound state equation should be a relativistic generalization of Eq. (2.19).

A covariant equation with the correct nonrelativistic limit is the Gross equation [8,9]. If two quarks with masses $m_1 \geqslant m_2$ are not identical, the one-channel equation may be used. It has the feature that the four-dimensional loop integrals are constrained so that the heavier constituent (with mass m_1 in this example) is restricted to its positive-energy

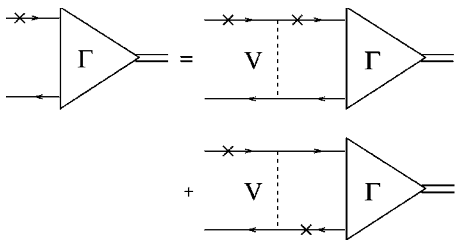

FIG. 3. One of the two-channel Gross equations for the bound state vertex function Γ. In this figure the $\times$ means that the particle is on the mass shell.

mass shell (provided $M_B > 0$; see Ref. [10]). It has been shown by Zeng, Van Orden, and Roberts [11] that the one-channel spectator equations can be used to give a good account of heavy measons.

If the particles are identical and the mass M_B of the bound state is comparable to m, a symmetrized two-channel equation should be used. This is illustrated in Fig. 3. In this case an average of the contributions in which either particle 1 (channel 1) or particle 2 (channel 2) are on their positive-energy mass shell are included, and this leads to a set of equations in which the two channels are coupled. The symmetrized two-channel equation has been used previously to describe low-energy NN scattering [12].

However, as is well known [13], both the one-channel equation and the symmetrized two-channel equations develop pathological behavior when the mass of the bound state approaches zero. The reasons for this, and the way to fix it were first discussed in Ref. [1]. Briefly, as the mass of the bound state $\mu \to 0$, distant singularities that may be normally neglected move into close range and give very large contributions. If these singularities are included from the start, the equation has a smooth behavior as $\mu \to 0$ and is consistent with chiral symmetry [1]. To include these extra contributions more channels are needed, and if the particles are identical, one needs to use a four-channel equation. This four channel equation is a symmetrized version of the unsymmetrized two channel equation used in Refs. [1,2]. One of the purposes of this paper is to present calculations using the four-channel equation, and to improve on the work of Ref. [1].

We emphasize that once we know how to handle the ''hard'' problem of treating light quarks in nearly massless bound states, we will be in a position to combine it with the successes already achieved in the heavy meson-meson sector [11] and build a comprehensive model of mesons. In view of the complexity of the light quark sector, our purpose here is to see if this promising approach can be used to describe light mesons.

In this section we will first review the one-channel equations, and using a relativistic generalization of the two forms of confinement given in Eq. (2.2) we will show that the relativistic normalization condition for a confined wave function is the same as for a conventional bound state wave function. We will then obtain a relativistic generalization of the condition (2.23), and discuss its interpretation. Finally, we will show how to discuss scattering in the presence of confinement. This last discussion indicates how the recent work of Isgur, Jeschonnek, Melnitchouk, and Van Orden on quark hadron duality might be extended to this formalism [14].

B. One channel scattering equations for scalar quarks

We will begin with the one-channel equation. The momentum and mass of the quark are p_1 and m_1, the momentum and mass of the antiquark are p_2 and m_2, the total momentum is P, and the relative momentum is p, where

$$P = p_1 + p_2,$$

$$p = \frac{1}{2}(p_1 - p_2). \tag{3.2}$$

Since the equations are *manifestly* covariant [1,2], they may be solved in any frame, and it is convenient to solve them in the rest frame where

$$P = p_1 + p_2 = \{M_B, \mathbf{0}\}. \tag{3.3}$$

The quark will be on mass-shell, and the symbol p_1^+ will be used to denote the particle on its *positive* energy mass-shell [i.e., $p_1^{+\,2} = m_1^2$ and in the rest frame $(p_1^+)_0 = E_1(p) = \sqrt{m_1^2 + \mathbf{p}^2}$]. The scattering amplitude $\mathcal{M}(p_1^+, p_2, p'_1{}^+, p'_2)$ is denoted $\mathcal{M}_{11}(\mathbf{p},\mathbf{p}',P)$, or in the one-channel case where there can be no confusion, simply by $\mathcal{M}(\mathbf{p},\mathbf{p}',P)$. Then, introducing a relativistic generalization of the potential V_S and writing the equation in the rest frame (this will be the convention from now on), the one-channel equation for the scattering of scalar ''quarks'' $(m_1 \geqslant m_2)$ can be written

$$\mathcal{M}_S(\mathbf{p},\mathbf{p}',P) = \eta V_S(\mathbf{p},\mathbf{p}',P) - \frac{2m_1 m_2}{(2\pi)^3} \times \int \frac{d^3k}{E_1(k)} \frac{V_S(\mathbf{p},\mathbf{k},P)\, \mathcal{M}_S(\mathbf{k},\mathbf{p}',P)}{m_2^2 - (P-k_1^+)^2} + \frac{2m_2 C\, \mathcal{M}_S(\mathbf{p},\mathbf{p}',p)}{m_2^2 - (P-p_1^+)^2}. \tag{3.4}$$

This equation is the relativistic generalization of Eq. (2.18).

Alternatively, the bound state form of the scattering equation is

$$\mathcal{M}_A(\mathbf{p},\mathbf{p}',P) = -\frac{2m_1 m_2}{(2\pi)^3} \int \frac{d^3k}{E_1(k)} V_A(\mathbf{p},\mathbf{k},P) \times \left[\frac{\mathcal{M}_A(\mathbf{k},\mathbf{p}',P)}{m_2^2 - (P-k_1^+)^2} - \frac{\mathcal{M}_A(\mathbf{p},\mathbf{p}',P)}{m_2^2 - (P-p_1^+)^2} \right] + \frac{2m_2 C\, \mathcal{M}_A(\mathbf{p},\mathbf{p}',P)}{m_2^2 - (P-p_1^+)^2}. \tag{3.5}$$

This is the analog of Eq. (2.19) and has a smooth limit as $\epsilon \to 0$. The kernels V_S and V_A will be specified later [see Eqs. (3.20) and (3.21) below]. Equations (3.4) and (3.5) will be our starting points for this section.

C. One channel bound state equation for scalar quarks

In the vicinity of a bound state of mass M_B, or a very narrow resonance with mass and width $M_B = M_R + iM_I$, the scattering amplitude has the form

$$\mathcal{M}_X(\mathbf{p},\mathbf{p}',P) = -\frac{\Gamma_X(\mathbf{p},M_B)\,\Gamma_X(\mathbf{p}',M_B)}{M_B^2 - P^2} + \mathcal{R}_X(\mathbf{p},M_B), \tag{3.6}$$

where $X = A$ or S, depending on which of the two forms (3.4) or (3.5) we are using. If ϵ is finite and we are using Eq. (3.4), the width $M_I \neq 0$. If we use Eq. (3.4) the width is zero for all states with mass below some critical mass $M_\epsilon \to \infty$ as $\epsilon \to 0$.

Substituting the form (3.6) into either Eq. (3.4) or Eq. (3.5), and equating residues at the pole (real or complex) gives the bound state equations for the vertex functions Γ_X

$$\Gamma_S(\mathbf{p},M_B) = -2m_1m_2\int \frac{d^3k V_S(p,k,M_B)}{(2\pi)^3\,E_1(k)}\,\frac{\Gamma_S(\mathbf{k},M_B)}{m_2^2-(M_B-k_1^+)^2} + \frac{2m_2\,C\,\Gamma_S(\mathbf{p},M_B)}{m_2^2-(M_B-p_1^+)^2}, \tag{3.7}$$

$$\Gamma_A(\mathbf{p},M_B) = -2m_1m_2\int \frac{d^3k V_A(p,k,M_B)}{(2\pi)^3\,E_1(k)} \times\left[\frac{\Gamma_A(\mathbf{k},M_B)}{m_2^2-(M_B-k_1^+)^2} - \frac{\Gamma_A(\mathbf{p},M_B)}{m_2^2-(M_B-p_1^+)^2}\right] + \frac{2m_2\,C\,\Gamma_A(\mathbf{p},M_B)}{m_2^2-(M_B-p_1^+)^2}, \tag{3.8}$$

where we use a mixed notation with M_B denoting both the mass and the four vector $\{M_B,0\}$, the difference being clear from the context.

As with the scattering amplitudes, the two vertex functions are equivalent in the limit $\epsilon\to 0$

$$\Gamma_S(\mathbf{p},M_B) \leftrightarrow \Gamma_A(\mathbf{p},M_B), \tag{3.9}$$

but the vertex function Γ_A is more convenient to calculate in the limit $\epsilon \to 0$.

D. Normalization condition and charge conservation

The bound state equation and the normalization condition for the bound state wave function can be derived from a nonlinear form of Eq. (3.4) [15]. In this paper the derivative $\partial V_S/\partial P_\mu = 0$ in the rest frame, so the result is (for equal mass particles)

$$2P^\mu = \frac{\partial}{\partial P_\mu}\int \frac{d^3k}{(2\pi)^3\,2E_1(k)}\left\{\frac{\Gamma_S(\mathbf{k},M_B)\,\Gamma_S(\mathbf{k},M_B)}{m_1^2-(P-k_1^+)^2}\right\}. \tag{3.10}$$

In view of the relation (3.9) this relation can also be written

$$2P^\mu = \frac{\partial}{\partial P_\mu}\int \frac{d^3k}{(2\pi)^3\,2E_1(k)}\left\{\frac{\Gamma_A(\mathbf{k},M_B)\,\Gamma_A(\mathbf{k},M_B)}{m_1^2-(P-k_1^+)^2}\right\} = \int \frac{d^3k}{(2\pi)^3\,E_1(k)}\,\frac{\Gamma_A(\mathbf{k},M_B)\,(P-k_1^+)^\mu\,\Gamma_A(\mathbf{k},M_B)}{(m_1^2-(P-k_1^+)^2)^2}. \tag{3.11}$$

This is a familiar result, which will be generalized to the spin-1/2 case later.[1]

This normalization condition also follows from the conservation of charge (or, alternatively, can be used to prove that charge is conserved). For the scalar case being discussed in this section we assume that the two particles 1 and 2 have equal masses but (possibly) unequal charges e_1 and e_2. If there are also no interaction currents the full current operator will be given by the relativistic impulse approximation (RIA) discussed in Refs. [16,17]. The electromagnetic current of the bound state in the RIA consists of the two terms resulting from the coupling of the photon to particle 1 and 2. The current is related to the form factor by

$$\begin{aligned} e_B F(q^2)(P+P')^\mu &= e_1\int \frac{d^3k}{(2\pi)^3\,2E_2(k)} \times\left\{\frac{\Gamma_{2A}(\mathbf{k},M_B)\,(P+P'-2k^+)^\mu\,\Gamma_{2A}(\mathbf{k},M_B)}{(m_1^2-(P'-k^+)^2)(m_1^2-(P-k^+)^2)}\right\} \\ &\quad + e_2\int \frac{d^3k}{(2\pi)^3\,2E_2(k)} \times\left\{\frac{\Gamma_{1A}(\mathbf{k},M_B)\,(P+P'-2k^+)^\mu\,\Gamma_{1A}(\mathbf{k},M_B)}{(m_2^2-(P'-k^+)^2)(m_2^2-(P-k^+)^2)}\right\} \\ &= (e_1+e_2)\int \frac{d^3k}{(2\pi)^3\,2E_1(k)} \times\left\{\frac{\Gamma_A(\mathbf{k},M_B)(P+P'-2k^+)^\mu\,\Gamma_A(\mathbf{k},M_B)}{(m_1^2-(P'-k^+)^2)(m_1^2-(P-k^+)^2)}\right\}, \end{aligned} \tag{3.12}$$

where the second line follows if we use $m_1 = m_2$ and the fact that, for equal mass particles, the vertex function Γ_1 (particle 1 on-shell) = Γ_2 (particle 2 on-shell) = Γ. At $q=0$, this expression becomes

[1] In the nonrelativistic limit the normalization condition reduces to the normalization condition of the Schrödinger wave function. This indicates that although the relativistic wave function does not have a probability interpretation it is a relativistic generalization of the nonrelativistic Schrödinger wave function.

$$e_B F(0)\, 2P^\mu = (e_1+e_2)\int \frac{d^3k}{(2\pi)^2 E_1(k)} \times \frac{\Gamma_A(\mathbf{k},M_B)\,(P-k^+)^\mu\,\Gamma_A(\mathbf{k},M_B)}{[m_1^2-(P-k^+)^2]^2} = (e_1+e_2)\,2P^\mu. \tag{3.13}$$

Hence, the normalization condition guarantees charge conservation

$$e_B F(0) = e_B = e_1+e_2. \tag{3.14}$$

E. Symmetrized two channel equation for equal mass scalar quarks

If the quarks have equal mass $(m_1=m_2=m)$, and the bound state mass is positive and not too small, a symmetrized two channel equation is needed. The two channels will be labeled 1 and 2 depending on whether the quark or antiquark is on mass-shell, and the symbol p_1^+ denotes that the particle is on its *positive* energy mass-shell [i.e., $p_1^{+\,2}=m^2$ and $p_0^+=E(p)=\sqrt{m^2+\mathbf{p}^2}$]. Starting from Eq. (3.8), and suppressing the subscript A, the vertex functions for the two channels are denoted

$$\Gamma_1(\mathbf{p},M_B)=\Gamma(p_1^+,p_2), \quad \Gamma_2(\mathbf{p},M_B)\equiv\Gamma(p_1,p_2^+). \tag{3.15}$$

With this notation, the symmetrized two channel equation for equal mass scalar ''quarks'' with a confining interaction can be written

$$\Gamma_i(\mathbf{p},M_B) = -m^2\sum_j \int \frac{d^3k V_{ij}(p,k)}{(2\pi)^3 E_j(k)} \times\left[\frac{\Gamma_j(\mathbf{k},M_B)}{m^2-(P-k_j^+)^2} - \frac{\Gamma_i(\mathbf{p},M_B)}{m^2-(P-p_i^+)^2}\right] + \frac{2mC\,\Gamma_i(\mathbf{p},M_B)}{m^2-(P-p_i^+)^2}, \tag{3.16}$$

where i and j label which of the two quarks is on-shell, and

$$k_j^+ = \{E(k),(-)^{j+1}\,\mathbf{k}\} \tag{3.17}$$

is the momentum of the on-shell quark. Note that the strength of the V_{ij} term has been multiplied by 1/2, reflecting the fact that the interaction is an *average* of the strengths in two channels that are equal in the nonrelativistic limit. This equation uses the same subtraction for both the $i=j$ and the $i\neq j$ terms. This prescription differs from that previously used in Ref. [1]. In this work the kernel below will not, in general, be singular when $i\neq j$, and the subtraction used above is sufficient to preserve the nonrelativistic limit (see below).

In order to complete the description we need to specify the form of covariant interaction V_{ij}. A natural choice that reduces to the correct nonrelativistic limit is [1]

$$V_{ij}(p,k)\equiv V_A(q_{ij}) = -\frac{8\pi\sigma}{(q_{ij}^2-\epsilon^2)^2}, \tag{3.18}$$

where the four-momentum transfer depends on whether or not $i=j$:

$$q_{11}^2=q_{22}^2=[E(k)-E(p)]^2-(\mathbf{k}-\mathbf{p})^2,$$
$$q_{12}^2=q_{21}^2=[M_B-E(k)-E(p)]^2-(\mathbf{k}+\mathbf{p})^2. \tag{3.19}$$

A similar form could be used for the kernel V_S (which we will not need)

$$V_{ij}(p,k)\equiv V_S(q_{ij}) = -8\pi\sigma\left\{\frac{1}{(q_{ij}^2-\epsilon^2)^2}+\frac{4\epsilon^2}{(q_{ij}^2-\epsilon^2)^3}\right\}. \tag{3.20}$$

However, the form (3.18) has two drawbacks. First, at large $\mathbf{p}\simeq\mathbf{k}$ the kernel converges slowly, and the equation is ultraviolet divergent. In Refs. [1,2] a form factor was introduced to regularize this divergence. Second, using this form it is difficult to regularize the infrared $(q^2=0)$ singularities that appear in the $\epsilon=0$ limit. In the nonrelativistic case the infrared singularity occurs only at $\mathbf{q}=0$ and can be regulated by the δ function subtraction in Eq. (2.3). However, in the relativistic case infrared singularities occur not only when $q^\mu=0$, but also (for the $i\neq j$ kernels) when the momentum transfer is lightlike, so that $q^2=0$ but $q^\mu\neq 0$. These ''*off-diagonal*'' singularities are not regulated by the subtraction term, and their removal spoils the simplicity of this approach [1].

Since the role of V_A is to model the linear interaction, and the principle requirement is that it reduces to the correct nonrelativistic limit, both of these problems are eliminated very simply if V_A is defined as follows:

$$V_A(q_{ij}) = -\frac{8\pi\sigma}{q_{ij}^4+(P\cdot q_{ij})^4/P^4}, \tag{3.21}$$

where P is the total four-momentum of the bound state. This form has the following advantages: (i) the denominator is not singular unless both q^2 and $P\cdot q$ are zero, so the singularities are restricted to $q^\mu=0$; (ii) no ultraviolet regularization is needed; (iii) the interaction does *not* depend on the bound state momentum P in the bound state rest frame; and (iv) it has the correct nonrelativistic dependence on $\mathbf{q}^2$. One disadvantage of the form (3.21) is its dependence on the total momentum P of the particle pair. However, since this kernel confines particles in pairs that cannot be separated, they are naturally associated as a pair and we do not view this as a serious limitation. Another feature of the form (3.21) is that its off-diagonal couplings are singular only when $W=2E(p)$ (because $\mathbf{k}+\mathbf{p}=0$ also). This is only possible for excited states and, as we will prove below, confinement requires the vertex function to be zero at this point, controlling this singularity automatically.

The introduction of the definition (3.21) considerably simplifies the solution of the relativistic equation (3.16), but will introduce electromagnetic interaction currents if the photon four-momentum is not zero. These will be discussed in a subsequent paper.

Both Eqs. (3.8) and (3.16) have the correct nonrelativistic limit with confinement. Consider the one-channel equation (3.8) first, and let m_1 and $m_2 \to \infty$. Then the energy transferred by the on-shell quark, $E_1(k)-E_1(p)\to 0$ and $V_A(q_{11})\to V_A(\mathbf{q})$. Furthermore, if $M_B=m_2+m_1+E$, then to first order in the small quantities $\mathbf{k}^2$ and $m_R E$, the relativistic propagator reduces to

$$\frac{1}{m_2^2-k_2^2}\to\frac{m_R}{m_2(\mathbf{k}^2-2m_R E)}, \tag{3.22}$$

and substituting this into Eq. (3.8) gives Eq. (2.19). In the two-channel case $q_{11}\to q_{12}$ as $m\to\infty$ and the kernels $V_{11}\to V_{12}$. Since the subtraction in the two channels is also identical, the contributions from the two channels are equal and the coupled equations reduce to the single equation (2.19).

F. Proof of confinement

While one can visualize the potential in the nonrelativistic case and get a picture of the physics, it is less possible to visualize the covariant interaction. What are the criteria with which one can judge whether a given interaction really confines? If the particles are bound in a state of total mass larger than the sum of the masses of the constituents ($M_B > m_1 + m_2$), the bound state could, in principle, decay into free constituents. Confinement prevents this from happening in one of two possible ways: (i) the quark propagators will not have any physical mass poles [18], or, as we will now prove for this model, (ii) the vertex function will vanish when the quarks are simultaneously on-shell.

The proof is identical to the nonrelativistic proof given above and we will summarize it only for the one-channel equation. Setting $C=0$, the one channel bound state equation (3.8) can be written

$$\begin{aligned}\Gamma_A(\mathbf{p},M_B)=&-2m_1m_2\int\frac{d^3k V_A(p,k)}{(2\pi)^3E_1(k)}\\&\times\left[\frac{\Gamma_A(\mathbf{k},M_B)-\Gamma_A(\mathbf{p},M_B)}{m_2^2-k_2^2}\right]\\&+2m_1m_2\Gamma_A(\mathbf{p},M_B)\int\frac{d^3kV_A(p,k)}{(2\pi)^3E_1(k)}\\&\times\left\{\frac{p_2^2-k_2^2}{(m_2^2-p_2^2)(m_2^2-k_2^2)}\right\}.\end{aligned} \tag{3.23}$$

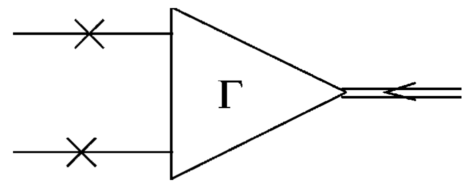

FIG. 4. The confinement condition for the Gross vertex function.

Since the first quark is on-shell, the second quark is on its *positive* energy mass shell when the magnitude of the relative three-momentum $|\mathbf{p}|=p_0$ is

$$\sqrt{m_1^2+p_0^2}+\sqrt{m_2^2+p_0^2}=M_B. \tag{3.24}$$

This occurs when p_0^2 is given by

$$4M_B^2p_0^2=[M_B^2-(m_1+m_2)^2][M_B^2-(m_1-m_2)^2]. \tag{3.25}$$

As in the nonrelativistic case, the singularity at $\mathbf{p}=\mathbf{k}$ is integrable, and hence the second term on the rhs of Eq. (3.23) will be singular for any vector $\mathbf{p}_0$ (with $|\mathbf{p}_0|=p_0$) unless

$$\Gamma_A(\mathbf{p}_0,M_B)=0. \tag{3.26}$$

Therefore, the vertex function vanishes when both particles are on their mass shell. This condition is illustrated diagrammatically in Fig. 4.

Note that the subtraction term in Eqs. (3.16) and (3.23) plays two central roles: (i) it regularizes the singular interaction at $\mathbf{p}=\mathbf{k}$ and makes it zero at $r=0$, and (ii) it is singular when $p_2^2\to m_2^2$, forcing condition (3.26). The subtraction term is essential to the self consistent description of confinement. As in the nonrelativistic case the proof did not depend on the specific form of the interaction.

We now discuss how confinement affects the stability of bound states under external disturbances.

G. Excitation of bound states

A consistent description of confinement implies that two free quarks cannot be liberated from a bound state, even under the influence of an energetic external photon or other probe. This requirment implies that the usual Born term (shown in Fig. 5) is either canceled by the rescattering term, or is a diagram that does not exist in the formalism. If the Born term does not exist, the rescattering term, illustrated in

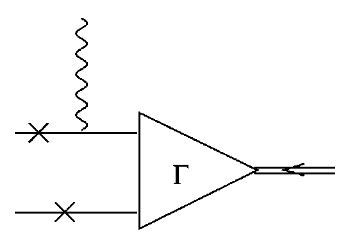

FIG. 5. The Born term, which cannot exist if the quarks are confined.

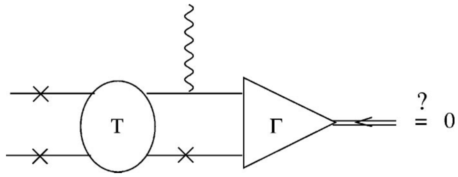

FIG. 6. Can an external photon probe disintegrate the bound state?

Fig. 6, must be zero if the final state quarks are all on-shell. How are these restrictions built into the formalism?

When particles are confined there are no free two-particle states and the two-body propagator must always include an infinite number of interactions. Since there are no free particle states, a perturbation theory for confined particles built around the free propagator cannot be constructed. This feature is built-in automatically if the two body propagators satisfy *homogeneous* integral equations with *no free particle contribution.*

To illustrate these ideas, we review the formalism for the scattering amplitude and its relation to the two-body propagator. It is convenient to work with the scattering form of the equation. In operator notation, Eq. (2.13) is

$$\begin{aligned} M(\mathbf{p},\mathbf{p}',P) &= \eta\, V(p,p') - V(p,k) G_0(\mathbf{k},\mathbf{k}',P) M(\mathbf{k}',\mathbf{p}',P) \\ &= \eta\, V(p,p') - M(\mathbf{p},\mathbf{k},P) G_0(\mathbf{k},\mathbf{k}',P) V(k',p'), \end{aligned} \tag{3.27}$$

where $G_0(\mathbf{k},\mathbf{k}',P)$ is the free two body propagator [containing a factor of $\delta^3(k-k')$], integration over d^3k and d^3k' is implied, and we have dropped the subscript S for simplicity. The parameter η was introduced in the discussion following Eq. (2.15) and is very small, approaching zero as $\epsilon\to 0$.

Now the dressed propagator G is related to the scattering amplitude M by

$$\begin{aligned} G(\mathbf{p},\mathbf{p}',P) &= \zeta\, G_0(\mathbf{p},\mathbf{p}',P) - G_0(\mathbf{p},\mathbf{k},P) M(\mathbf{k},\mathbf{k}',P) \\ &\quad \times G_0(\mathbf{k}',\mathbf{p}',P), \end{aligned} \tag{3.28}$$

where ζ, to be determined, is a parameter proportional to the strength of the free particle scattering. If the potential confines there should be *no inhomogeneous term* and $\zeta=0$. To determnine ζ and the equation for G, substitute Eq. (3.27) into Eq. (3.28) giving

$$\begin{aligned} G(\mathbf{p},\mathbf{p}',P) &= \zeta\, G_0(\mathbf{p},\mathbf{p}',P) \\ &\quad - \eta G_0(\mathbf{p},\mathbf{k},P) V(k,k') G_0(\mathbf{k}',\mathbf{p}',P) \\ &\quad + G_0(\mathbf{p},\mathbf{k},P) V(k,k') G_0(\mathbf{k}',\mathbf{k}'',P) \\ &\quad \times M(\mathbf{k}'',\mathbf{k}''',P) G_0(\mathbf{k}''',\mathbf{p}',P) \\ &= \zeta\, G_0(\mathbf{p},\mathbf{p}',P) + (\zeta-\eta)\, G_0(\mathbf{p},\mathbf{k},P) V(k,k') \\ &\quad \times G_0(\mathbf{k}',\mathbf{p}',P) - G_0(\mathbf{p},\mathbf{k},P) V(k,k') \\ &\quad \times G(\mathbf{k}',\mathbf{p}',P). \end{aligned} \tag{3.29}$$

The second term is eliminmated by choosing $\zeta=\eta$, and gives familiar equations for the dressed propagator

$$\begin{aligned} G(\mathbf{p},\mathbf{p}',P) & \\ &= \eta\, G_0(\mathbf{p},\mathbf{p}',P) - G_0(\mathbf{p},\mathbf{k},P) V(k,k') G(\mathbf{k}',\mathbf{p}',P) \\ &= \eta\, G_0(\mathbf{p},\mathbf{p}',P) - G(\mathbf{p},\mathbf{k},P) V(k,k') G_0(\mathbf{k}',\mathbf{p}',P), \end{aligned} \tag{3.30}$$

where the second form parallels the second form of Eq. (3.27).

The interpretation of Eqs. (3.28) and (3.30) for the dressed propagator follows from the interpretation of Eq. (3.27) for the scattering amplitude. As $\epsilon\to 0$, the parmeter $\eta\to 0$ and the inhomogeneous term vanishes. In this limit both the scattering amplitude and the propagator satisfy homogeneous equations.

The amplitude $\mathcal{J}$ for inelastic scattering induced by a probe γ can be obtained from the dressed propagator by striping off the final free propagators. If J is the current describing the coupling of the probe to the quark, then the inelastic scattering amplitude is [9]

$$\begin{aligned} \mathcal{J}(\mathbf{p},P,q) &= G_0^{-1}(\mathbf{p},\mathbf{k},P+q) G(\mathbf{k},\mathbf{p}',P+q) \\ &\quad \times J(P+q,P)\Psi(P) \\ &= \{\eta + M(\mathbf{p},\mathbf{k},P+q) G_0(\mathbf{k},\mathbf{p}',P+q)\} \\ &\quad \times J(P+q,P)\Psi(P). \end{aligned} \tag{3.31}$$

Here the first term proportional to η is the Born term shown in Fig. 5, and we see that there is no Born term in the limit of exact confinment (i.e., $\eta=0$). Furthermore, in the presence of confinement the scattering matrix satisfies the same homogeneous equation satisfed by the bound states [Eq. (3.27) with $\eta=0$], and an extension of the proof given in Sec. III F above shows that the scattering matrix in Fig. 6 must be zero if both final state quarks are on shell.

We have constructed a self-consistent description of confinement within the context of relativistic spectator equations.

H. Generalization to fermions

If the quarks have spin, the kernel in the spectator equation will be an operator in the Dirac space of the two quarks. This operator can be written

$$\mathcal{V}(p,k) = \sum_{i=1}^{3} \alpha_i\, O_{i1}\, O_{i2} V_i(p,k), \tag{3.32}$$

where the Dirac matrices O, which operate on the Dirac indices of particles 1 and 2, describe the spin-dependent structure of quark-antiquark interaction. The α_i are parameters determined either empirically (by fitting the spectrum), from lattice calculations, or from the theory. In this paper we consider only three possible spin structures: scalar $O_{1j}=\mathbb{1}_j$, pseudoscalar $O_{2j}=\gamma_{5j}$, and vector $O_{3j}=\gamma_{\mu j}/2$. With this notation the one channel spectator equation for spin 1/2 particles with constant masses $m_1 \gg m_2$ is given by

$$\Gamma(p,P) = -\int \frac{d^3k}{(2\pi)^3 E_1(k)} \sum_i V_i(p,k) O_{i1}(m_1 + \not{k}_1^+) \times \left\{ \frac{\Gamma(k,P)}{m_2^2 - k_2^2} \right\} (m_2 - \not{k}_2) O_{i2}, \tag{3.33}$$

where the quark has mass m_1 and is on shell, so that $k_1^{+\,2} = m_1^2 = p_1^{+\,2}$, and the antiquark has mass m_2. Therefore, the momentum transfered by the interaction is

$$(p^+ - k^+)^2 = [E_1(p) - E_1(k)]^2 - (\mathbf{p} - \mathbf{k})^2 \equiv q. \tag{3.34}$$

As in the nonrelativistic case, we consider a kernel composed of linear, constant, and one-gluon exchange pieces. The interaction kernel for the linear part of the potential $\mathcal{V}_L$ is

$$\mathcal{V}_L(p,k) \equiv \sum_{i=1}^{3} \alpha_{Li} O_{i1} O_{i2} V_L(p,k), = \left(\alpha_s \mathbb{1}_1 \mathbb{1}_2 + \alpha_{ps} \gamma_{51}\gamma_{52} + \frac{1}{4} \alpha_v \gamma_{\mu 1} \gamma_2^{\mu} \right) V_L(p,k), \tag{3.35}$$

where $V_L(p,k)$ is

$$V_L(p,k) = V_A(q_{11}(p,k)) - E_1(k)\, \delta^3(p-k) \int d^3k' \frac{V_A[q_{11}(p,k')]}{E_1(k')}. \tag{3.36}$$

In this work we employ a pure scalar linear interaction, $\alpha_s = 1, \alpha_{ps} = \alpha_v = 0$, but in later calculations the coefficients α_i will be determined empirically. The one-gluon exchange and constant interactions will be pure vector

$$\mathcal{V}_g(q) = \gamma_{\mu 1} \gamma_{\nu 2} V_g^{\mu\nu}(q),$$
$$\mathcal{V}_c(q) = \gamma_{\mu 1} \gamma_2^{\mu} C, \tag{3.37}$$

where

$$V_g^{\mu\nu}(q) \equiv -\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right) V_g(q) = -\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right) \frac{1}{q^2 - \Lambda^2} \frac{d16\pi^2/3}{\ln(\tau + |q^2|/\lambda_{QCD}^2)}, \tag{3.38}$$

where $d = 12/(33 - 2N_f) = 12/27$, the color factor of 4/3 has been included, $\Lambda = 1$ GeV, $\tau = 2$, and $\lambda_{QCD} = 200$ MeV. In previous work [2] quark propagators with constant masses were used. In this work we parametrize the quark propagator by

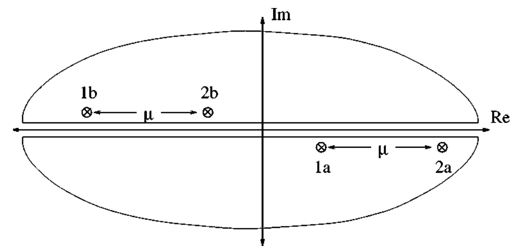

FIG. 7. Propagator poles in the complex k_0 plane.

$$S(p) = \frac{1}{m(p) - \not{p}}, \tag{3.39}$$

where $m(p)$ is a mass function for the quark, to be defined later.

If the constituents are identical or close in mass and the equations are to be applied to the description of nearly massless bound states, the *four-channel* equation should be used. Numerical solutions of the four-channel equation will be presented in this work.

The four channels are defined by the constraints in the four-momenta k_1 and k_2 arising from the requirement that *both* the quark and the antiquark be constrained to *both* their positive and negative energy mass-shells. A formal way to obtain the equations is to integrate over the internal energy k_0 by averaging the contributions from the quark and antiquark poles in *both* the upper and lower half k_0 complex plane, as illustrated in Fig. 7. This averaging is needed to ensure charge conjugation (particle-antiparticle) symmetry, and leads to four coupled equations. However, even though the form of the equations is obtained in this way, we emphasize that the equations are theoretically justified by the argument that the singularities in the interaction kernel omitted in this procedure tend to be cancelled by other higher order terms that would otherwise have been neglected, and that this leads to covariant equations with the correct nonrelativistic limit. The inclusion of the negative energy poles, neglected in other applications of the symmetrized equations [12], is required in cases where $P \to 0$ [1].

The four constraints are conveniently identified by the notation

$$k_j^s = \{ sE(k), (-)^{j+1} \mathbf{k} \}, \tag{3.40}$$

which generalizes that introduced in Eq. (3.17). Here the superscript $s = \pm$ denotes either the positive or negative energy mass shell constraints. Then, introducing the operators

$$\Lambda(k) = m(k) + \not{k}, \tag{3.41}$$

and defining the four channel vertex functions

$$\Gamma_1^s(\mathbf{p}, M_B) = \Gamma(p_1^s, p_2),$$
$$\Gamma_2^s(\mathbf{p}, M_B) = \Gamma(p_1, p_2^s), \tag{3.42}$$

and wave functions

$$\Psi_1^s(\mathbf{p},M_B)=\frac{\Lambda(p_1^s)\Gamma_1^s(\mathbf{p},M_B)\Lambda(-p_2)}{m(p_2)^2-p_2^2},$$

$$\Psi_2^s(\mathbf{p},M_B)=\frac{\Lambda(p_1)\Gamma_2^s(\mathbf{p},M_B)\Lambda(-p_2^s)}{m(p_1)^2-p_1^2}, \tag{3.43}$$

permits us to write the four-channel spectator equation in the following compact form:

$$\begin{aligned}\Gamma_i^s(\mathbf{p},M_B)=&-\frac{1}{2}\sum_{jr}\int\frac{d^3k}{(2\pi)^3\,2E(k)}\{V_{ij}^{sr}(p,k)\\&\times[\Psi_j^r(\mathbf{k},M_B)-\Psi_i^s(\mathbf{p},M_B)]\\&-\delta_{ij}\delta_{sr}V_g^{\mu\nu}(p-k)\gamma_\mu\Psi_j^r(\mathbf{k},M_B)\gamma_\nu\}\\&-C\,\gamma_\mu\Psi_i^s(\mathbf{p},M_B)\gamma^\mu,\end{aligned} \tag{3.44}$$

where the rhs of the equation now sums over both positive and negative energy contributions ($r=\pm$) from *each* quark ($j=\pm$). The Kronecker $\delta_{ij}\delta_{sr}$ functions restrict the one gluon exchange interaction to the diagonal channels (where the same particle is on the same mass shell before and after the interaction). Inclusion of the one-gluon exchange in off-diagonal channels leads to numerical instabilities, which in principle can be handled by using more grid points in numerical integrations. Restricting this interaction to diagonal channels eliminates these singularities from the gluon propagator.

I. Charge conjugation invariance

The final task is to show that Eq. (3.44) is invariant under the charge conjugation operation

$$\Gamma^{\mathcal{C}}(p_1,p_2)=\mathcal{C}\Gamma^{\mathrm{T}}(p_2,p_1)\mathcal{C}^{-1}. \tag{3.45}$$

This is done by proving that both Γ and $\Gamma^{\mathcal{C}}$ satisfy the same equation.

First note that, when particle 1 is on shell, interchange of p_1 and p_2 gives

$$\Gamma_1^s(\mathbf{p},M_B)=\Gamma(p_1^s,p_2)\to\Gamma(p_2,p_1^s)=\Gamma_2^s(-\mathbf{p},M_B) \tag{3.46}$$

and is equivalent to $1\leftrightarrow2$ and $\mathbf{p}\to-\mathbf{p}$. Then

$$\Psi_1^{s\,\mathcal{C}}(\mathbf{p},M_B)=\mathcal{C}\Psi_2^{s\,\mathrm{T}}(-\mathbf{p},M_B)\mathcal{C}^{-1},$$

$$\Psi_2^{s\,\mathcal{C}}(\mathbf{p},M_B)=\mathcal{C}\Psi_1^{s\,\mathrm{T}}(-\mathbf{p},M_B)\mathcal{C}^{-1}. \tag{3.47}$$

Finally, the Dirac direct products $\mathbb{1}\otimes\mathbb{1}$, $\gamma_\mu\otimes\gamma^\mu$, and $\gamma_5\otimes\gamma_5$ are invariant under $\mathcal{C}$. Hence, changing $\mathbf{k}\to-\mathbf{k}$ and performing the transformations (3.45) and (3.46), shows that Eq. (3.44) is also invariant. Therefore the charge conjugation eigenstates, labeled by $\eta=\pm$

$$\Gamma_\eta^s(\mathbf{p},M_B)=\Gamma_1^s(\mathbf{p},M_B)+\eta\,\Gamma_2^{s\,\mathcal{C}}(\mathbf{p},M_B), \tag{3.48}$$

are solutions of the equation and charge conjugation symmetry is proved.

J. Dynamical quark mass

The dynamical quark mass function is the solution of the Dyson-Schwinger equation. In NJL-type models, this *one-body* equation for the spontaneous generation of quark mass and the *two-body* bound state equation for a state of zero mass become identical in the chiral limit (when the bare quark mass is zero). In this limit the quark mass function and the bound state wave function for a massless pseudoscalar bound state are identical, and spontaneous symmetry breaking assures the existence of a massless pseudoscalar bound state.

In this paper, we adopt a slightly different approach. We will first choose a convenient mass function, and then *require* that the *two-body* equation for a massless pseudoscalar bound state automatically has a solution when the bare quark mass is zero. In this case the quark mass function and the wave function for the massless Goldstone boson will not be identical, but at least the existence of the Goldstone boson in the chiral limit is assured. We will *define* the quark mass function of flavor f by

$$m_f(p)\equiv m_f^0+c(m_f^0)\,f(p), \tag{3.49}$$

where m_f^0 is the current quark mass of flavor f, and $f(p)$ is a universal function defined by

$$f(p)\equiv\frac{1}{|p^2|+\Lambda^2}. \tag{3.50}$$

The function $c(m_f^0)$ can be thought of as a polynomial in powers of m_f^0. This is the typical structure of the mass function that is usually obtained from the solution of the one body equation.

The reason for not solving the one-body equation, in our case, is twofold. The first problem is the difficulty of incorporating one-gluon exchange into the one-body equation. Because of the on-shell constraint in the loop momenta, the one-gluon exchange interaction leads to an ultraviolet divergence. The second problem is associated with our choice of infrared regularization of the linear interaction. The infrared singularities are regulated by the $P\cdot q$ term in the denominator of the linear interaction equation (3.21), and this would imply that the resultant mass function is a function of two arguments, i.e., $m=m(p^2,\mathbf{p}^2)$. This is unacceptable, and rather than forsaking important features of the model such as confinement and asymptotic freedom, we choose to model the quark mass functions.

The form (3.49) guarantees that at large momenta, quark masses go to their current quark mass values as dictated by asymptotic freedom [3]. In the chiral limit the quark mass function reduces to

$$m_\chi(p)=c(0)\,f(p), \tag{3.51}$$

which has a solution [that is zero when $c(0)=0$]

$$m_\chi=\left(\sqrt{\frac{\Lambda^6}{27}+\frac{c(0)^2}{4}}+\frac{c(0)}{2}\right)^{1/3}-\left(\sqrt{\frac{\Lambda^6}{27}+\frac{c(0)^2}{4}}-\frac{c(0)}{2}\right)^{1/3}. \tag{3.52}$$

We fix the chiral mass m_χ by requiring that the pion bound state equation has a massless solution when the quark mass is m_χ. This insures that a massless pion exists in the chiral limit when $m_f^0=0$. Next we *choose* a value for the light current quark mass $m_u^0=m_d^0$ and fix $c(m_u^0)$ so that the two-body equation gives the correct value for the physical pion mass. This also fixes the value of the on-shell quark mass away from the chiral limit. Similarly, we *choose* m_s^0 and fix $c(m_s^0)$ by fitting the kaon mass. For three flavors it is therefore sufficient to have a function $c(m_f^0)$ that is a polynomial of order two in m_f^0. As new flavors are introduced the order of the polynomial accordingly can be increased.

To summarize, we have six mass parameters: $m_u^0, m_s^0, c(0), c(m_u^0), c(m_s^0)$, and Λ. In practice we fix Λ at 1 GeV and *choose* the current quark masses m_u^0 and m_s^0 to be near the values expected by current theory. We then adjust the c's to give a zero mass pion in the chiral limit, and a real pion and kaon with the observed masses. This process is repeated for different values of the current quark masses and the potential parameters σ and C until satisfactory values for the constituent quark masses and the spectrum of excited pions is obtained. The final values of the parameters will be given in the next section.

Having outlined the features of the model, we now turn our attention to the details of the pseudoscalar bound state equation with spin.

IV. PSEUDOSCALAR CHANNEL

The bound state vertex function has the following structure

$$\chi=\chi_{\rm color}\otimes\chi_{\rm flavor}\otimes\chi_{\rm spin}. \tag{4.1}$$

The color space vertex function is a Kronecker delta function δ_{cd}, which reflects the color singlet nature of the bound state. The flavor space vertex function is the matrix λ^i_{fg} in SU(3) matrix space, which chooses the right flavor combination of the meson under consideration. Indices f,g refer to up down and strange quark entries $(u,d,s=1,2,3)$ of λ^i. For example, $[\lambda^+]_{ud}=[\lambda^+]_{12}$. For a general meson type *i*, the bound state vertex function is

$$\chi^i_{\alpha\beta,fg,cd}(k_1,k_2)\equiv\delta_{cd}\,\lambda^i_{fg}\,\Gamma_{\alpha\beta}(k_1,k_2), \tag{4.2}$$

where α and β are Dirac indices (to be suppressed in the following discussion). The most general form for the spin-space part of the vertex function for pseudoscalar mesons is

$$\Gamma(k_1,k_2)=\gamma_5\{\Gamma_0+\not{P}\,\Gamma_1+\not{k}\,\Gamma_2+[\not{k},\not{P}]\,\Gamma_3\}, \tag{4.3}$$

where $\Gamma_i=\Gamma_i(k_1,k_2)$ are scalar functions. The dominant contribution to the bound state vertex function comes from the first term of Eq. (4.3),

$$\Gamma(k_1,k_2)\approx\gamma_5\,\Gamma_0(k_1,k_2). \tag{4.4}$$

This approximation, which is exact in the chiral limit when $P=0$ and $m_1=m_2$, will be used for the pion and kaon bound states in this work.

Assuming Eq. (4.4), multiplying the four-channel equations for pseudoscalar mesons by γ_5, and taking the trace gives the following approximate coupled equations for pseudoscalar states:

$$\begin{aligned}\Gamma_i^s(\mathbf{p},M_B)=&-\frac{1}{2}\sum_{jr}\int\frac{d^3k}{(2\pi)^3\,2E_j(k)}\{V_{ij}^{sr}(p,k)\\&\times[F_j(k_j^r)\Gamma_j^r(\mathbf{k},M_B)-F_i(p_i^s)\Gamma_i^s(\mathbf{p},M_B)]\\&+6\,\delta_{ij}\delta_{sr}V_g(p-k)F_j(k_j^r)\Gamma_j^r(\mathbf{k},M_B)\}\\&+2F_i(p_i^s)C\Gamma_i^s(\mathbf{p},M_B),\end{aligned} \tag{4.5}$$

where the four-channel wave functions $\Gamma_i^s(\mathbf{p},M_B)$ are obtained from Γ_0 as shown in Eq. (3.42), and

$$F_1(k_1^r)\equiv\frac{m_1m_2(k_2)+k_1^r\cdot k_2}{m_2^2(k_2)-k_2^2},$$

$$F_2(k_2^r)\equiv\frac{m_1(k_1)m_2+k_1\cdot k_2^r}{m_1^2(k_2)-k_1^2}, \tag{4.6}$$

where $m_i(k_i^r)=m_i(-k_i^r)=m_i$. For future reference we record the four-momentum $q_{ij}^{rs}\equiv(p_1-k_1)_{ij}^{rs}$ exchanged between the two quarks. This depends on the initial and final channel. The distinct cases are

$$q_{11}^{rs}=q_{22}^{-r,-s}=(rE(p)-sE(k),\,\mathbf{p}-\mathbf{k}),$$

$$q_{12}^{rs}=(rE(p)+sE(k)-M_B,\,\mathbf{p}-\mathbf{k}),$$

$$q_{21}^{rs}=(M_B-rE(p)-sE(k),\,\mathbf{p}-\mathbf{k}). \tag{4.7}$$

The solution of Eq. (4.5) for a realistic choice of the parameters will be discussed in the next section.

Before turning to this discussion, look at the coupled equations in the chiral limit, when $P=0$ and the dynamical quark masses are equal, so that $m_1(k)=m_2(k)=m(k)$. In this limit, $k_1=-k_2$, and expanding to order $P\cdot k_1^r$ gives

$$\begin{aligned}F_1(k_1^r)&=\frac{m(k_1^r)\,m(P-k_i^r)+k_1^r\cdot P-k_i^{r2}}{m^2(P-k_i^r)-(P-k_i^r)^2}\\&=\frac{1-2mm'}{2-4mm'}\\&=\frac{1}{2}=F_2(k_2^r),\end{aligned} \tag{4.8}$$

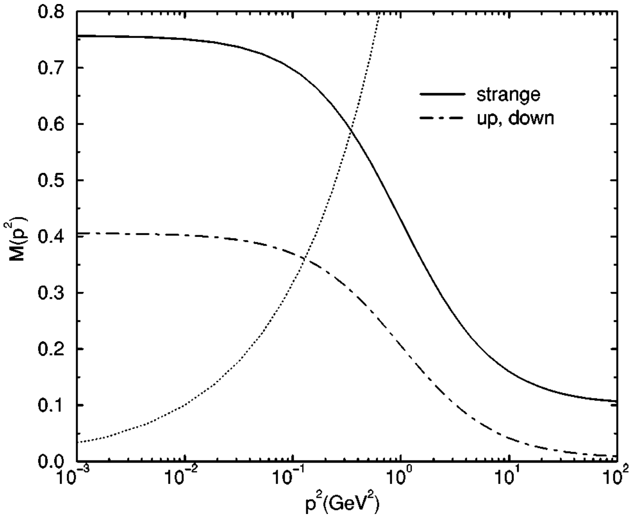

FIG. 8. Quark mass functions $m_f(p) \equiv M(p^2)$ are shown for up/down, and strange quarks. On-shell quark masses are $m_{u,d} = 360$ MeV, and $m_s = 588$ MeV. At large momenta quark mass values approach $m^0_{u,d} = 5$ MeV, and $m^0_s = 100$ MeV.

where $m \equiv m(\pm k_i^s)$ and $m' \equiv dm(\pm k)/dk^2|_{(k^2=m^2)}$. Note that, in the chiral limit, a zero in the numerator cancels a zero in the denominator, as in the NJL mechanism. Here, since one of the two propagators in the bound state equation has already been removed by the mass shell condition, this cancellation removes the all the poles from the propagator. Hence, using charge conjugation symmetry (3.45), the four coupled equations (4.5) reduce to only *two* equations in the chiral limit. These coupled equations are

$$\Gamma^+_\chi(\mathbf{p},0) = -\int \frac{d^3k}{(2\pi)^3\, 2E(k)} \{V_+(p,k) \times[\Gamma^+_\chi(\mathbf{k},0) - \Gamma^+_\chi(\mathbf{p},0)] + V_-(p,k)[\Gamma^-_\chi(\mathbf{k},0) - \Gamma^+_\chi(\mathbf{p},0)] + 6V_g(p-k)\Gamma^+_\chi(\mathbf{k},0)\} + 2C\Gamma^+_\chi(\mathbf{p},0),$$

$$\Gamma^-_\chi(\mathbf{p},0) = -\int \frac{d^3k}{(2\pi)^3\, 2E(k)} \{V_+(p,k) \times[\Gamma^-_\chi(\mathbf{k},0) - \Gamma^-_\chi(\mathbf{p},0)] + V_-(p,k)[\Gamma^+_\chi(\mathbf{k},0) - \Gamma^-_\chi(\mathbf{p},0)] + 6V_g(p-k)\Gamma^-_\chi(\mathbf{k},0)\} + 2C\Gamma^-_\chi(\mathbf{p},0), \tag{4.9}$$

TABLE I. Summary of results.

Observable	Calculated (MeV)	Experimental (MeV)
m_π	140	139.6
m_η	320	
m^*_π	1118	1300±100
m_K	495	495
m_χ	376	
$m_u = m_d$	360	
m_s	588	

TABLE II. Values of the parameters.

Parameter	Value
m^0_u	5 MeV
m^0_s	100 MeV
$c(0)$	0.429 GeV^3
$c(m^0_u)$	0.400 GeV^3
$c(m^0_s)$	0.657 GeV^3
σ	0.4 GeV^2
C	0.4929
Λ	1 GeV

where

$$V_\pm(p,k) = \frac{8\pi\sigma}{(\mathbf{p}-\mathbf{k})^4 + [E(p) \mp E(k)]^4}. \tag{4.10}$$

Note that these two equations are symmetric under the interchange

$$\Gamma^+_\chi \leftrightarrow \pm\Gamma^-_\chi, \tag{4.11}$$

and hence reduce to one equation for $\Gamma_\chi \equiv \Gamma^+_\chi = \pm\Gamma^-_\chi$,

$$\Gamma_\chi(\mathbf{p},0) = -\int \frac{d^3k}{(2\pi)^3\, 2E(k)} \{[V_+(p,k) \pm V_-(p,k) + 6V_g(p-k)]\Gamma_\chi(\mathbf{k},0) - [V_+(p,k) + V_-(p,k)]\Gamma_\chi(\mathbf{p},0)\} + 2C\Gamma_\chi(\mathbf{p},0), \tag{4.12}$$

where the sign of the V_- term depends on the sign in the relation (4.11). Since the π_0 is even under charge conjugation symmetry, the plus sign is the correct one to use.

Recalling Eq. (3.52), the energies E in Eq. (4.12) depend on m_χ. In the chiral limit, the energy is

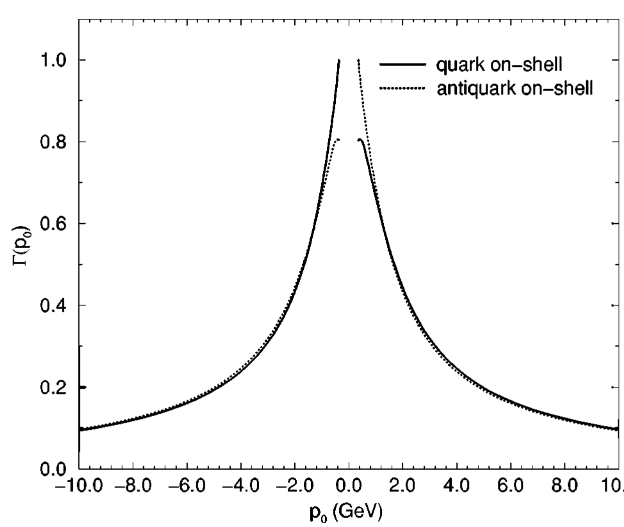

FIG. 9. The four-channel vertex functions for the ground state of the pion.

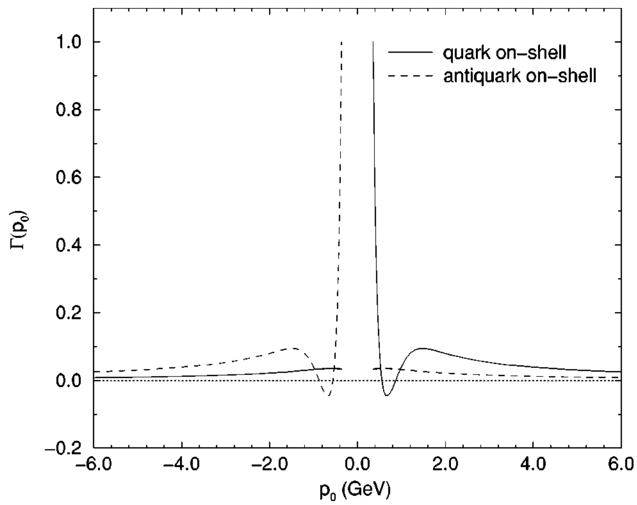

FIG. 10. The four-channel vertex functions for the first excited state of the pion.

$$E(p)=\sqrt{m_\chi^2+\mathbf{p}^2}, \tag{4.13}$$

and m_χ is adjusted to ensure that Eq. (4.12) has a solution. Once m_χ [and hence $c(0)$] has been fixed, Eq. (4.5) is solved for various values of the bare quark masses m_f^0 and the "mass functions" $c(m_f^0)$, and all parameters are adjusted to give a reasonable spectrum.

We now present some numerical results for the quark mass functions and the bound state vertex functions.

V. RESULTS

Before presenting our numerical results, we emphasize that the purpose of this paper is *not* to fit the light quark spectrum. We have far too many parameters and too few predictions to justify that. Our purpose here is to see how this model would work in practice, and to show that reasonable numerical results can be obtained. A fit to the spectrum,

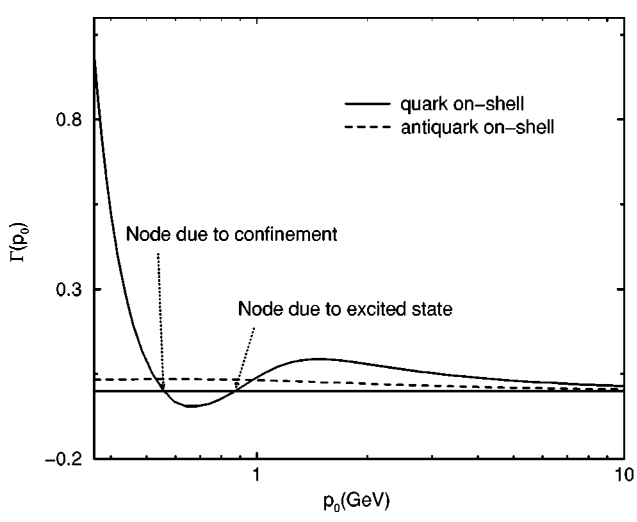

FIG. 11. The two positive energy vertex functions for the first excited state of the pion. The second node is due to the excited state, and the first node assures that the bound state does not decay.

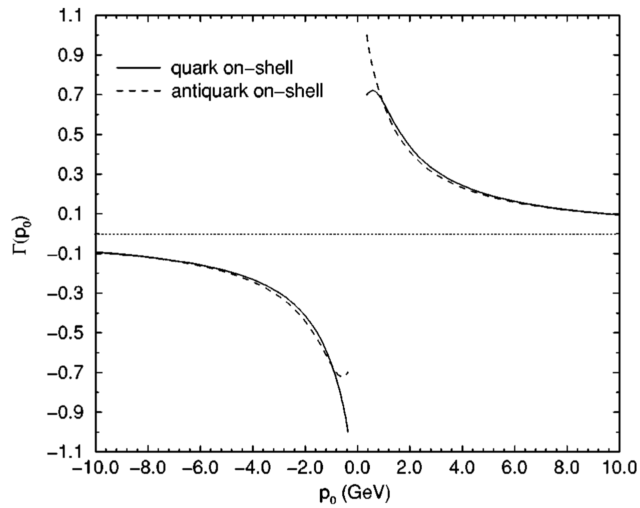

FIG. 12. The four-channel vertex functions for the nonstrange η.

meson decays, and electromagnetic interactions is postponed for a future work.

Illustrative quark mass functions are shown in Fig. 8. The on-shell quark masses m_f are given in Table I. At large momenta, the quark mass alues approach the bare quark masses m_f^0 shown in Table II. The other mass parameters and bound state parameters are also shown in Table II. The parameter Λ that determines the scale of mass function was fixed at $\Lambda=1$ GeV and not adjusted during the fits. The third line in Fig. 8 is the momentum p, and the intersection of this line with the quark mass function gives the constituent quark mass.

In Figs. 9 and 10 the ground and first excited state vertex functions of the pion are shown. Here we show the vertex functions as a function of the variable $p_j^s=sE(p)\equiv sp_0$. Note that p_0 is positive for positive energy states $(s=+)$ and negative for negative energy states $(s=-)$. Because of the symmetrization, the positive energy quark vertex function is the same as the negative energy antiquark vertex func-

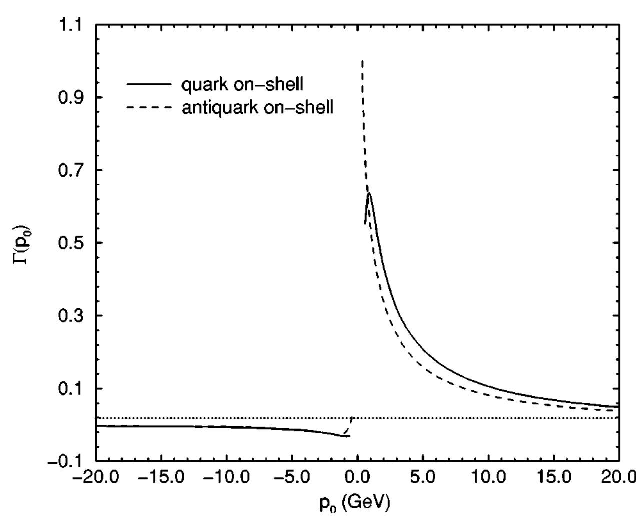

FIG. 13. The four-channel vertex functions for the ground state of the kaon.

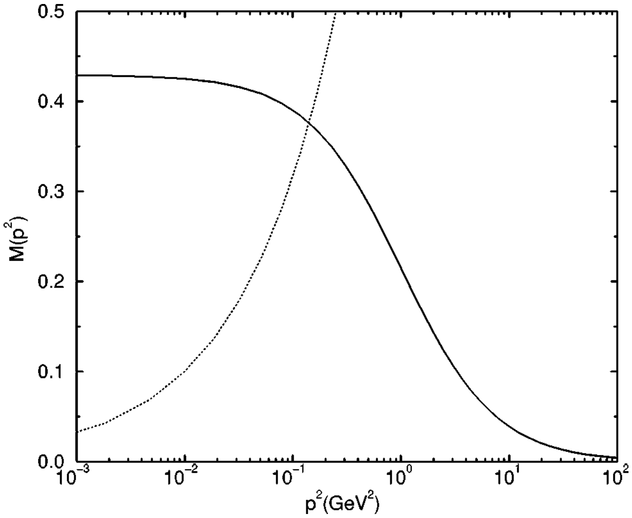

FIG. 14. The chiral limit of the quark mass function $M(p^2) \equiv m_\chi(p)$. The on-shell quark mass is $m_\chi = 376$ MeV. At large momenta quark mass function approaches 0.

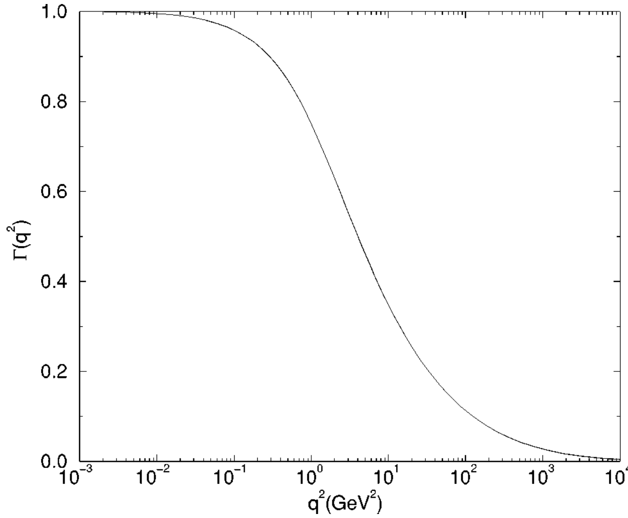

FIG. 15. The chiral limit of the pion ground state vertex function.

tion up to an overall phase ($+$ for states even under charge conjugation and $-$ for odd states). Also note that the curves are not continuous because the argument p_0 cannot take values between $(-m,+m)$. In Fig. 11 we present the excited state vertex functions on a logarithmic scale. The location of the first node is exactly where both quarks are simultaneously on shell. Therefore, although kinematically allowed, the excited state of the pion cannot decay into a free quark-antiquark pair. This numerical result is a consequence of the confinement condition (3.26).

In Fig. 12 we present the non-strange-eta (the isospin zero $u\bar{u}+d\bar{d}$ combination) ground state vertex functions. Note that these are odd under charge conjugation. The kaon vertex functions are shown in Fig. 13. Since the kaon is formed from a quark and antiquark of unequal masses, the particle-antiparticle symmetry is lost and the negative and positive energy solutions have a different shape and size.

The mass function and the pion wave function in the chiral limit are shown in Figs. 14 and 15.

VI. CONCLUSION

We have shown that a relativistic generalization of the Schrödinger equation with linear interaction leads to the Gross equation. It is not possible to write a Bethe-Salpeter equation that gives the correct linear interaction in the nonrelativistic limit. We have proved that the relativistic generalization of the linear interaction leads to vanishing vertex amplitudes when both of the constituents are on-shell. This guarantees that the bound state does not decay to its constituents. This mechanism of confinement follows from insisting on the correct nonrelativistic limit. The model incorporates asymptotic freedom through the inclusion of a vector one gluon exchange interaction, and quark mass functions that approach the current quark values at infinite momentum. There are no cutoffs or *ad hoc* form factors involved, and the linear interaction involves only one coupling parameter. The approach gives a good description of the pion, kaon, and eta.

It remains to use this formalism to describe the full meson spectrum.

ACKNOWLEDGMENT

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APPENDIX: MORE ABOUT CONFINEMENT

While Eq. (2.9) was derived for the linear potential with the specific choice of V_A given in Eq. (2.4), it is instructive to consider it in its most general form where V_A is an arbitrary function. From this point of view, the role of the second term in square brackets in Eq. (2.9) (which arises from the subtraction term) is to ensure that the coordinate space potential $\tilde{V}_A(r)$ is redefined so that it is zero at the origin, i.e., Eq. (2.9) is a standard Schrödinger equation for the potential

$$\tilde{V}_L(r) = \tilde{V}_A(r) - \tilde{V}_A(0). \tag{A1}$$

Looking at it this way, we see that any potential $\tilde{V}_A(r)$ for which $\tilde{V}_A(r_o) - \tilde{V}_A(0) = \infty$, for some r_o, gives a confined system when used with Eq. (2.9). For example, even the choice of a pure Coulomb-type interaction for $\tilde{V}_A$,

$$\tilde{V}_A(r) = -\frac{1}{r}, \tag{A2}$$

would give confinement. The subtraction term forces the interaction to vanish at the origin, which requires an infinite shift in the energy (just as in the case of the linear interaction) forcing the interaction to go to infinity at large distances. The role of the subtraction is an essential part of introducing confinement. This trivial point is worth emphasizing because it is just as crucial for the relativistic equations as it is for the nonrelativistic Schrödinger equation.

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