# Noncentral interactions in inelastic scattering of nucleons on nuclei: The case of $^{12}\mathrm{C}(p,p')^{12}\mathrm{C}^*(1^+)$

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An irreducible tensor formalism to analyze inelastic scattering of nucleons on nuclei with arbitrary spins is outlined and the nature of noncentral interactions involved therein is studied. In the particular case of inelastic scattering of nucleons on $^{12}$C leading to the $(1^+)$ excited state at 15.11 MeV, the relative importance of vector and tensor amplitudes is examined using the DW81 and DREX codes.

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## I. INTRODUCTION

The primary motivation behind experimental studies of proton-nucleus and meson-nucleus collisions was, to start with, to study the hadronic structure of nuclei in contrast to the information derived from electron-scattering studies on their electromagnetic structure. As pointed out by Hoffmann in the historical survey at the LAMPF workshop [1]: ''The pioneering cross-section measurements suggested that the pre-existing non-relativistic microscopic theories were adequate to describe the data and that the data could be analyzed to obtain the details of the underlying structure. But when high quality cross-section and analyzing power data were obtained with the intention of using it to deduce new nuclear structure information, systematic discrepancies surfaced. Experiment then turned to providing data which fully tested the theories, and the systematic disagreement between these data and theoretical predictions led to a questioning of the fundamental validity of the NR theories themselves.'' The advent of the Dirac optical model and relativistic impulse approximation remedied the situation to some extent but the successes reported prompted Negele at the same workshop [2] to remark, ''There is no clear evidence yet as to whether or not these models do the right deed for the wrong reasons.''

This assessment highlights the need for developing formalisms through which inelastic scattering of nucleons on nuclei can be analyzed in a model-independent way. In particular, Piekarewicz, Amado, and Sparrow [3] have expressed the scattering matrix for inelastic scattering of protons on $^{12}$C leading to the $1^+$ excited state at 15.11 MeV in the form

$$\mathcal{M}=A_{n0}(\mathbf{\Sigma}\cdot\hat{\mathbf{n}})+A_{nn}(\mathbf{\Sigma}\cdot\hat{\mathbf{n}})(\boldsymbol{\sigma}\cdot\hat{\mathbf{n}})+A_{KK}(\mathbf{\Sigma}\cdot\hat{\mathbf{K}})(\boldsymbol{\sigma}\cdot\hat{\mathbf{K}})$$
$$+A_{Kq}(\mathbf{\Sigma}\cdot\hat{\mathbf{K}})(\boldsymbol{\sigma}\cdot\hat{\mathbf{q}})+A_{qK}(\mathbf{\Sigma}\cdot\hat{\mathbf{q}})(\boldsymbol{\sigma}\cdot\hat{\mathbf{K}})$$
$$+A_{qq}(\mathbf{\Sigma}\cdot\hat{\mathbf{q}})(\boldsymbol{\sigma}\cdot\hat{\mathbf{q}}), \quad (1.1)$$

where

$$\hat{\mathbf{n}}=\frac{\mathbf{p}_i\times\mathbf{p}_f}{|\mathbf{p}_i\times\mathbf{p}_f|};\quad \hat{\mathbf{K}}=\frac{\mathbf{p}_i+\mathbf{p}_f}{|\mathbf{p}_i+\mathbf{p}_f|};\quad \hat{\mathbf{q}}=\hat{\mathbf{n}}\times\hat{\mathbf{K}} \quad (1.2)$$

to provide an orthogonal coordinate system with $\mathbf{p}_i$ and $\mathbf{p}_f$ being incident and outgoing c.m. momenta, $\boldsymbol{\sigma}$ denote the Pauli spin matrices of the nucleon, and

$$\mathbf{\Sigma}_M\equiv|1^+M\rangle\langle 0^+| \quad (1.3)$$

connects the excited nuclear state to the ground state.

This approach is similar to the well-known discussion of the more basic $NN$ scattering in terms of the five Wolfenstein amplitudes [4]. These amplitudes are in turn expressible in terms of the phase shifts and mixing parameters associated with partial waves [5]. They can also be discussed in terms of effective potentials [6] representing central as well as noncentral interactions. Likewise, the role of noncentral interactions in the case of elastic scattering of particles with arbitrary spin $s$ on spin-zero targets was discussed by Johnson [7], by expressing the scattering matrix in terms of irreducible tensors $\tau^k_q(\mathbf{S})$ of rank $k$ constructed out of the spin operators, $\mathbf{S}$. This was generalized more recently [8] to the discussion of noncentral interactions in the case of elastic scattering of particles where both projectiles and targets have arbitrary spins $s_1$ and $s_2$, using the irreducible tensor formalism developed earlier [9] for hadron scattering and reactions with arbitrary spins.

The purpose of this paper is to fully develop the model-independent approach, to inelastic scattering of nucleons on nuclei, considering in general nuclear transitions $J_i^{\pi_i}\to J_f^{\pi_f}$, expressing the amplitudes in terms of partial waves and identifying the nature of noncentral interactions involved. The case of $^{12}\mathrm{C}(p,p')^{12}\mathrm{C}^*(1^+)$ is then utilized for purposes of illustration and the relative importance of vector and tensor amplitudes is discussed in comparison with experiment. This example is particularly interesting since the isovector $1^+$ excited state at 15.11 MeV is a $\gamma$ emitter. Piekarewicz, Rost, and Shepard [10] point out that there are $8J+3$ independent quantities to determine in the case of a parity conserving $0\to J$ transition and the singles $(p,p')$ observables are themselves inadequate to perform the empirical reconstruction of the scattering matrix, except when $J=0$. They say ''coincidence measurements may then provide the only practical way to completely determine the scattering amplitude and therefore isolate those quantities that are particularly sensitive to differences between various theoretical models.'' Even after leaving out photon polarization, which is difficult to measure at this energy, one can find [11,12] many sets of coincidence observables to complement the singles $(p,p')$ measurements and the coincidence cross section. Wells *et al.* [13], who have reported simultaneous measurements of $(\vec{p},\vec{p}')$ and $(\vec{p},p'\gamma)$ coincidence observables at 200 MeV, point out that $^{12}$C$(p,p')^{12}$C*$(1^+)$ has been found to be an excellent candidate from the experimental point of view, since it has a large enough branching ratio to the ground state and its excitation energy is sufficiently high to make events of interest easy to identify in a photon spectrum. Based on the data obtained in [13], Wells and Wissink [14] have recently made a model-independent determination of the complete scattering amplitude for the 15.11 MeV, $1^+$ state of $^{12}$C following [10–12]. The relevance of observables based on photon polarization have also been discussed theoretically [15], which would come in handy, when the appropriate techniques are developed. In fact, Wells *et al.* [13] say, ''Understanding the behavior of the 15.11 MeV state in $^{12}$C has therefore often been viewed as a critical test of our models of the nucleon-nucleus (NA) interactions.''

The plan of the paper is as follows. In Sec. II, we outline the irreducible tensor formalism for inelastic scattering of nucleons on nuclei involving nuclear transitions from an initial state with spin-parity $J_i^{\pi_i}$, isospin $I_i$, and its projection $\nu_i$ to a final excited state with spin-parity $J_f^{\pi_f}$, isospin $I_f$, and its projection $\nu_f$. Explicit partial wave expansions are obtained for the irreducible tensor amplitudes. Matrix elements in spin space are discussed in terms of these amplitudes with reference to several well-known coordinate systems. In Sec. III, we identify the rich variety of noncentral interactions. In Sec. IV, we consider the important particular case of $^{12}$C$(p,p')^{12}$C*$(1^+)$ and illustrate the results using the well-known computer codes DW81 and DREX, based, respectively, on the nonrelativistic distorted wave impulse approximation [16] and its relativistic counterpart, taking explicit knock on exchange terms into consideration [17]. Summary and conclusions are presented in Sec. V.

## II. FORMALISM

We consider, in general, the inelastic scattering of nucleons on a nucleus with initial spin-parity $J_i^{\pi_i}$, isospin $I_i$, and its projection $\nu_i$ leading to a final excited state with spin-parity $J_f^{\pi_f}$, isospin $I_f$, and its projection $\nu_f$. Conservation of isospin implies that the on-energy-shell transition matrix $T$ for this process is given by

$$T=\sum_I C(I_f\,\tfrac{1}{2}\,I;\nu_f\nu'\nu_I)T^I C(I_i\,\tfrac{1}{2}\,I;\nu_i\nu\nu_I) \tag{2.1}$$

in terms of the transition matrices $T^I$, characterized by channel isospin, $I$. The Clebsch-Gordon coefficients are denoted by $C$ [18]. The isospin projections for the projectile and scattered nucleons are denoted by $\nu$ and $\nu'$, respectively. The matrix, $\mathcal{M}$ in spin space for the inelastic scattering may then be defined in terms of $T$ given by Eq. (2.1) through

$$\langle \tfrac{1}{2}\,\mu' J_f m_f|\mathcal{M}|\tfrac{1}{2}\mu J_i m_i\rangle = \sqrt{\frac{2\pi D}{v}}\langle \tfrac{1}{2}\,\mu' J_f m_f;\mathbf{p}_f|T|\mathbf{p}_i;\tfrac{1}{2}\,\mu J_i m_i\rangle, \tag{2.2}$$

where $\mathbf{p}_f$ and $\mathbf{p}_i$ are the final and initial c.m. momenta, respectively. The density of final states and the magnitude of the relative velocity in the initial state are denoted by $D$ and $v$, respectively. Introducing channel spins $s_i$ and $s_f$, the matrix elements may be expressed [19,7–9] in the form

$$\begin{aligned}\langle s_f\mu_f;\mathbf{p}_f|T|\mathbf{p}_i;s_i\mu_i\rangle &= \sum_{l_f,l_i,j,k}(-1)^{l_i+s_i+l_f-j}W(s_il_is_fl_f;jk)[j]^2[k]\\ &\quad\times[s_f]^{-1}T^j_{l_fs_f;l_is_i}C(s_iks_f;\mu_iq\mu_f)(-1)^q(Y_{l_f}(\hat{\mathbf{p}}_f)\\ &\quad\otimes Y_{l_i}(\hat{\mathbf{p}}_i))^k_{-q},\end{aligned} \tag{2.3}$$

where the notations are same as in [8,9] and $T^j_{l_fs_f;l_is_i}$ are the partial wave amplitudes [20]. Defining $M^j_{l_fs_f;l_is_i}=\sqrt{(2\pi D/v)}T^j_{l_fs_f;l_is_i}$ and noting that $|s_f\mu_f\rangle$ transforms under rotations like an irreducible tensor $K^{s_f}_{\mu_f}$ of rank $s_f$; while $\langle s_i\mu_i|$ does so like $(i)^{2\mu_i}B^{s_i}_{-\mu_i}$, an irreducible tensor of rank $s_i$ [21], we now express $\mathcal{M}$ in the form

$$\mathcal{M}=\sum_{s_f,s_i,k}[\mathcal{S}^k(s_f,s_i)\cdot M^k(s_f,s_i)], \tag{2.4}$$

where

$$\mathcal{S}^k_q(s_f,s_i)=(i)^{2s_i}[s_f](K^{s_f}\otimes B^{s_i})^k_q \tag{2.5}$$

connects the spin spaces of $s_i$ and $s_f$ and

$$\begin{aligned}M^k_q(s_f,s_i)&=\sum_{l_f,l_i,j}(-1)^{l_i+s_i+l_f-j}W(s_il_is_fl_f;jk)\\ &\quad\times M^j_{l_fs_f;l_is_i}[j]^2[s_f]^{-1}(Y_{l_f}(\hat{\mathbf{p}}_f)\otimes Y_{l_i}(\hat{\mathbf{p}}_i))^k_q.\end{aligned} \tag{2.6}$$

Defining spin operators,

$$\sigma_0^0=1;\quad \sigma_0^1=\sigma_z;\quad \sigma_{\pm1}^1=\mp\frac{1}{\sqrt{2}}(\sigma_x\pm i\sigma_y) \tag{2.7}$$

for the nucleons and $\mathcal{S}_q^k(J_f,J_i)$ for the nuclear excitation on the same lines as in Eq. (2.5), we may express the spin tensors of Eq. (2.4) in terms of $\sigma_{q_1}^{k_1}$ and $\mathcal{S}_{q_2}^{k_2}(J_f,J_i)$, so that we arrive at the general form for the spin structure of $\mathcal{M}$ as

$$\mathcal{M}=\sum_{k_1,k_2,k}[(\sigma^{k_1}\otimes\mathcal{S}^{k_2}(J_f,J_i))^k\cdot\mathcal{T}^k(k_1,k_2)], \tag{2.8}$$

for inelastic scattering of nucleons on a nuclear target with arbitrary spin $J_i$ leading to an excited state with spin $J_f$. The irreducible tensor amplitudes $\mathcal{T}_q^k(k_1,k_2)$ are made explicit through

$$\mathcal{T}_q^k(k_1,k_2)=\frac{1}{\sqrt{2}}\sum_{l_i,l_f,s_i,s_f,j}(-1)^{l_i+l_f-j+s_i}W(s_il_is_fl_f;jk)\times[j]^2\begin{Bmatrix}\frac{1}{2} & J_f & s_f\\ \frac{1}{2} & J_i & s_i\\ k_1 & k_2 & k\end{Bmatrix}\frac{[s_f][s_i][k_1][k_2]}{[J_f]}\times M^j_{l_fs_f;l_is_i}(Y_{l_f}(\hat{\mathbf{p}}_f)\otimes Y_{l_i}(\hat{\mathbf{p}}_i))^k_q, \tag{2.9}$$

where $M^j_{l_fs_f;l_is_i}$ contain completely the dependence on energy, while $(Y_{l_f}(\hat{\mathbf{p}}_f)\otimes Y_{l_i}(\hat{\mathbf{p}}_i))^k_q$ decide completely the angular dependence.

The elegant structure of Eq. (2.8), with each of its terms being a scalar product of two irreducible tensors, demonstrates explicitly the rotational invariance. The summation over $l_i,l_f$ in Eqs. (2.6) and (2.9) needs to be limited to

$$(-1)^{l_f}\pi_f=(-1)^{l_i}\pi_i \tag{2.10}$$

due to parity conservation. Clearly Eq. (2.8) provides a natural generalization of Eq. (1.1) for arbitrary nuclear spin transitions $J_f^{\pi_f}\leftarrow J_i^{\pi_i}$. When $J_i=0,J_f=1$, it is also clear that $k_2$ can take only one value $k_2=1$ and the operator given by Eq. (1.3) is identical to $(1/\sqrt{3})\mathcal{S}_q^1(1,0)$ with $q=M$. We can readily establish a connection between the amplitudes in Eq. (1.1) and our irreducible tensor amplitudes $\mathcal{T}_q^k(k_1,1)$ as will be shown in Sec. IV. Moreover, we have a bonus here in that Eq. (2.9) provides the partial wave expansions for the amplitudes.

In general, the angular dependence of $\mathcal{T}_q^k$ can be made more explicit by choosing a convenient coordinate system. If we choose traditionally the $z$ axis parallel to the beam, i.e., along $\mathbf{p}_i$ and the $y$ axis along $\mathbf{p}_i\times\mathbf{p}_f$ so that $z$-$x$ is the scattering plane (we shall refer to this frame as the conventional frame, CF), we note that

$$Y_{l_im_i}(\hat{\mathbf{p}}_i)=\delta_{m_i0}(4\pi)^{-1/2}[l_i] \tag{2.11}$$

so that

$$(Y_{l_f}(\hat{\mathbf{p}}_f)\otimes Y_{l_i}(\hat{\mathbf{p}}_i))^k_q=(4\pi)^{-1/2}C(l_fl_ik;q0q)[l_i]Y_{l_fq}(\theta,0). \tag{2.12}$$

Consequently, in the CF,

$$\mathcal{T}_{-q}^k(k_1,k_2)=(-1)^{k-q}\mathcal{T}_q^k(k_1,k_2)\pi_f\pi_i. \tag{2.13}$$

On the other hand, if the quantization axis or $z$ axis is chosen parallel to $\mathbf{p}_i\times\mathbf{p}_f$ and the $x$ axis is chosen parallel to $\mathbf{p}_i$ (which we shall refer to as the transverse frame, TF), we have

$$Y_{l_fm_f}(\hat{\mathbf{p}}_f)=Y_{l_fm_f}\left(\frac{\pi}{2},\theta\right);\quad Y_{l_im_i}(\hat{\mathbf{p}}_i)=Y_{l_im_i}\left(\frac{\pi}{2},0\right). \tag{2.14}$$

Using the parity constraint Eq. (2.10) and noting [22] that $Y_{lm}(\pi/2,\phi)=0$ for odd $(l-m)$, we obtain the conditions

$$\mathcal{T}_q^k(k_1,k_2)=0\quad\text{for all odd}\quad q\quad\text{if}\quad \pi_f=\pi_i \tag{2.15}$$

and

$$\mathcal{T}_q^k(k_1,k_2)=0\quad\text{for all even}\quad q\quad\text{if}=\quad \pi_f=-\pi_i, \tag{2.16}$$

in TF. A further constraint is obtained in the case of forward scattering $(\theta=0)$, necessitating $k$ to be even when condition (2.15) is satisfied and odd when Eq. (2.16) holds. This fact can be of considerable significance in the analysis of forward-scattering data.

The irreducible tensor amplitudes $\mathcal{T}_q^k(k_1,k_2)$ in any frame (AF) may be expressed in terms of $\mathcal{T}_q^k(k_1,k_2)$ in some standard frames (SF), say CF or TF using the standard rotation matrices $D^k_{q'q}(\alpha,\beta,\gamma)$ with the appropriate choice of the Euler angles $(\alpha,\beta,\gamma)$. In particular, $\mathcal{T}_q^k(k_1,k_2)_{TF}$ can be expressed in terms of $\mathcal{T}_q^k(k_1,k_2)_{CF}$ with $(\alpha,\beta,\gamma)=(\pi/2,\pi/2,\pi)$.

Further, from Eq. (2.4), the matrix elements of $\mathcal{M}$ between initial- and final-channel spin states are readily given by

$$\mathcal{M}_{\mu_f\mu_i}=\langle s_f\mu_f|\mathcal{M}|s_i\mu_i\rangle=\sqrt{\frac{2\pi D}{v}}\sum_k[k](-1)^qC(s_iks_f,\mu_iq\mu_f)T^k_{-q}(s_f,s_i). \tag{2.17}$$

The magnetic quantum numbers $\mu_f=\mu'+m_f$ and $\mu_i=\mu+m_i$ in Eq. (2.17) are all measured with respect to the same $z$ axis or quantization axis. Choosing the TF, i.e., the $z$ axis parallel to $\mathbf{p}_i\times\mathbf{p}_f$, the constraints (2.15) and (2.16) imply that the matrix elements satisfy

$$\mathcal{M}_{\mu_f\mu_i}=\eta(-1)^q\mathcal{M}_{\mu_f\mu_i}, \tag{2.18}$$

where $\eta=\pi_f\pi_i$ and $q=\mu_f-\mu_i$. This result, known to be valid in the so-called transversity frame [23,24] is usually derived by first of all establishing the symmetry constraints due to parity conservation on the helicity amplitudes fol-

TABLE I. Euler angles affecting transformation from the transverse frame to the helicity and transversity frames.

| Frame | $(\alpha_1,\beta_1,\gamma_1)$ | $(\alpha_2,\beta_2,\gamma_2)$ | $(\alpha_3,\beta_3,\gamma_3)$ | $(\alpha_4,\beta_4,\gamma_4)$ |
|---|---|---|---|---|
| Helicity [25] | $\left(0,\frac{\pi}{2},\frac{\pi}{2}\right)$ | $\left(0,-\frac{\pi}{2},\frac{\pi}{2}\right)$ | $\left(\theta,\frac{\pi}{2},\frac{\pi}{2}\right)$ | $\left(\theta,-\frac{\pi}{2},\frac{\pi}{2}\right)$ |
| Transversity [23] | $\left(\frac{\pi}{2},0,0\right)$ | $\left(-\frac{\pi}{2},-\pi,0\right)$ | $\left(\frac{\pi}{2}+\theta,0,0\right)$ | $\left(-\frac{\pi}{2},-\pi,-\theta\right)$ |
| Transversity [24] | $(0,0,0)$ | $(\pi,0,0)$ | $(\theta,0,0)$ | $(\pi+\theta,0,0)$ |

lowed by Lorentz transformations (or boosts) and rotations in three dimensions. The elegant derivation given here of Eq. (2.18) following [9] shows that it is enough if the quantization axis is chosen parallel to $\mathbf{p}_i\times\mathbf{p}_f$. Any further rotation with respect to the $z$ axis adds only a phase and does not change the above conclusion that have the effect of reducing the number of amplitudes from $4(2J_i+1)(2J_f+1)$ to $2(2J_i+1)(2J_f+1)$ linearly independent, nonzero amplitudes. Also, the matrix elements of $\mathcal{M}$ with respect to the nucleon- and nuclear-spin projections along a common $z$ axis of quantization say, in TF, are readily obtained through

$$\mathcal{M}_{\mu' m_f,\mu m_i}=\langle \tfrac{1}{2}\,\mu' J_f m_f|\mathcal{M}|\tfrac{1}{2}\,\mu J_i m_i\rangle = \sum_{s_f,s_i} C(\tfrac{1}{2}\,J_f s_f;\mu' m_f\mu_f) \times C(\tfrac{1}{2}\,J_i s_i;\mu m_i\mu_i)\mathcal{M}_{\mu_f\mu_i}. \tag{2.19}$$

Having identified the nonzero matrix elements in the TF, the matrix elements in any other frame (AF) can be obtained through

$$\mathcal{M}_{\mu' m_f,\mu m_i}(AF) = \sum_{\mu''',m_f',\mu'',m_i'} D^{(1/2)*}_{\mu'''\mu'}(\alpha_3,\beta_3,\gamma_3) \times D^{J_f^*}_{m_f' m_f}(\alpha_4,\beta_4,\gamma_4) D^{(1/2)}_{\mu''\mu}(\alpha_1,\beta_1,\gamma_1) \times D^{J_i}_{m_i' m_i}(\alpha_2,\beta_2,\gamma_2)\mathcal{M}_{\mu''' m_f',\mu'' m_i'}(TF) \tag{2.20}$$

as linear combinations of the elements in TF. Thus the number of linearly independent amplitudes remain the same in any frame. The Euler angles for transformation from the TF to some of the well-known frames are as in Table I.

## III. NONCENTRAL INTERACTIONS

The nature of noncentral interactions in elastic scattering of particles with arbitrary spins $s_1$ and $s_2$ was investigated in [8] by introducing projection-cum-spin-orbit flip operator

$$S_{s_1,s_2}(l_f s_f;j;l_i s_i)=\sum_{k_1=0}^{2s_1}\sum_{k_2=0}^{2s_2}\sum_k G_{k_1k_2k}(l_f s_f;j;l_i s_i) \times[\mathcal{S}^k(l_f,l_i)\cdot\tau^{(k_1k_2)k}(\mathbf{S}_1,\mathbf{S}_2)], \tag{3.1}$$

where

$$\tau_q^{(k_1k_2)k}(\mathbf{S}_1,\mathbf{S}_2)=(\tau^{k_1}(\mathbf{S}_1)\otimes\tau^{k_2}(\mathbf{S}_2))^k_q \tag{3.2}$$

and the geometrical factors are explicitly given by

$$G_{k_1k_2k}(l_f s_f;j;l_i s_i)=(-1)^{l_f-s_f-j}\frac{[j]^2[s_i][s_f]}{[l_f][s_1][s_2]} \times W(l_i l_f s_i s_f;kj)(-1)^{k_1+k_2+k}[k_1][k_2] \times\begin{Bmatrix} s_1 & s_2 & s_i\\ s_1 & s_2 & s_f\\ k_1 & k_2 & k\end{Bmatrix}, \tag{3.3}$$

so that $\mathcal{M}$ could be expressed in the form

$$\mathcal{M}=\sum_{l_f,s_f,j,l_i,s_i}\mathcal{M}^j_{l_f s_f;l_i s_i}S_{s_1,s_2}(l_f s_f;j;l_i s_i). \tag{3.4}$$

Comparison of Eq. (2.8) with Eq. (3.4) reveals that they are very much similar and we can in general define the effective interaction in the case of inelastic scattering through

$$\langle\mathbf{r}_f|V_{\text{eff}}|\mathbf{r}_i\rangle=\sum_{k_1,k_2,k}(\sigma^{k_1}\otimes\mathcal{S}^{k_2}(J_f,J_i))^k\cdot\langle\mathbf{r}_f|V^{(k_1k_2)k}|\mathbf{r}_i\rangle, \tag{3.5}$$

where

$$\langle\mathbf{r}_f|V_q^{(k_1k_2)k}|\mathbf{r}_i\rangle=\sum_{l_f,s_f,j,l_i,s_i}G_{k_1k_2k}(l_f s_f;j;l_i s_i) \times\langle r_f|V^j_{l_f s_f;l_i s_i}|r_i\rangle\langle\hat{\mathbf{r}}_f|\mathcal{S}^k_q(l_f,l_i)|\hat{\mathbf{r}}_i\rangle. \tag{3.6}$$

Since $\mathbf{p}_f\neq\mathbf{p}_i$ in the case of inelastic scattering, the radial nonlocal terms in Eq. (3.6) are given by

$$\langle r_f|V^j_{l_f s_f;l_i s_i}|r_i\rangle=\frac{2}{\pi}(i)^{l_f-l_i}\int dE\,E^2 p_f p_i\, j_{l_f}(pr_f)\times\mathcal{M}^j_{l_f s_f;l_i s_i} j_{l_i}(pr_i) \tag{3.7}$$

following [25,26], where $j_l$ denote spherical Bessel functions and the integration is with respect to the c.m. energy $E$, in terms of which $p_f$ and $p_i$ are readily known. In general,

$$\langle\hat{\mathbf{r}}_f|\mathcal{S}^k_q(l_f,l_i)|\hat{\mathbf{r}}_i\rangle=(-1)^{l_i}[l_f](Y_{l_f}(\hat{\mathbf{r}}_f)\otimes Y_{l_i}(\hat{\mathbf{r}}_i))^k_q \tag{3.8}$$

for diagonal ($l_f=l_i$) as well as off-diagonal ($l_f\neq l_i$) terms. For $l_i=l_f=l$ one can identify

$$\mathcal{S}^k_q(l,l)=\tau^k_q(\mathbf{L}), \tag{3.9}$$

where $\mathbf{L}$ denotes the orbital-angular momentum operator. For example, in the case of $NN$ scattering the choice for $k=2$ leads to the well-known tensor interaction, whereas Eq. (3.9) leads to a spin-orbit tensor force.

Thus the term $k_1=k_2=k=0$ in Eq. (3.5) defines the spin-independent central interaction, while terms with $k_1=k_2$ but $k\neq 0$ lead to spin-dependent central interactions. All the rest of the terms correspond to spin-dependent noncentral interactions and these include spin-orbit interactions when the choice (3.9) is made. Thus, it is interesting to note that the irreducible tensor amplitudes $\mathcal{T}^k_q(k_1,k_2)$ readily admit interpretation in terms of a variety of central and noncentral interactions including spin-orbit interactions, so that the sensitivity of various spin observables to the different forms of interaction can be studied by expressing these observables in terms of the irreducible tensor amplitudes.

In the particular case of $^{12}$C$(p,p')^{12}$C*$(1^+)$, the effective interaction may be expressed in the notation of [3] as

$$V_{\rm eff}=V_1\,\boldsymbol{\sigma}\cdot\boldsymbol{\Sigma}+V_2\,\boldsymbol{\Sigma}\cdot\mathbf{L}+V_3(\boldsymbol{\sigma}\times\boldsymbol{\Sigma})\cdot\mathbf{L}+V_4\times((\boldsymbol{\sigma}\otimes\boldsymbol{\Sigma})^2\cdot(\hat{\mathbf{r}}\otimes\hat{\mathbf{r}})^2)+V_5((\boldsymbol{\sigma}\otimes\boldsymbol{\Sigma})^2\cdot(\mathbf{L}\otimes\mathbf{L})^2), \tag{3.10}$$

where the first three terms corresponding to $(k_1,k_2,k)$ are $(1,1,0),(0,1,1),(1,1,1)$, respectively; while the last two terms correspond to $(1,1,2)$; there is no spin-independent central interaction, since $k_2=1$.

## IV. INELASTIC NUCLEON SCATTERING ON $^{12}$C

The inelastic scattering of nucleons on $^{12}$C leading to the $1^+$ excited state is a process of the type $\frac{1}{2}+0\to\frac{1}{2}+1$ and following Sec. II, the scattering matrix is given by

$$\mathcal{M}=\sum_{k_1,k}[(\sigma^{k_1}\otimes\mathcal{S}^1(1,0))^k\cdot\mathcal{T}^k(k_1,1)]. \tag{4.1}$$

Choosing to work in the transverse frame (TF), the six nonzero, independent, irreducible tensor amplitudes associated with the process are presented in Table II, where the constraint due to parity conservation, Eq. (2.10), has been applied. We next establish connection with the work of Piekarewicz *et al.* [10] wherein the most general rotational and parity invariant scattering matrix for $(p,p')$ processes, with specific reference to $0^+\to1^+$ transition, has been written as in Eq. (1.1) employing the right-handed Cartesian coordinate system given by Eq. (1.2) with the quantization axis chosen along $\hat{\mathbf{q}}$, and $x$ and $y$ axis along $\hat{\mathbf{n}}$ and $\hat{\mathbf{K}}$, respectively, so that $\hat{\mathbf{p}}_i=(\pi/2+\theta/2,\pi/2)$ and $\hat{\mathbf{p}}_f=(\pi/2-\theta/2,\pi/2)$. In this frame, which we refer to as $nKq$, the irreducible tensor amplitudes satisfy

TABLE II. The nonzero irreducible tensor amplitudes $\mathcal{T}^k_q(k_1,1)$ for $^{12}$C$(p,p')^{12}$C*$(1^+)$ in TF.

| $k_1$ | $k$ | $q$ | Amplitude |
|---|---|---|---|
| 1 | 0 | 0 | $\mathcal{T}^0_0(1,1)$ |
| 0 | 1 | 0 | $\mathcal{T}^1_0(0,1)$ |
| 1 | 1 | 0 | $\mathcal{T}^1_0(1,1)$ |
| 1 | 2 | 0 | $\mathcal{T}^2_0(1,1)$ |
| 1 | 2 | 2 | $\mathcal{T}^2_2(1,1)$ |
| 1 | 2 | $-2$ | $\mathcal{T}^2_{-2}(1,1)$ |

$$\mathcal{T}^k_{-q}(k_1,k_2)_{nKq}=(-1)^k\mathcal{T}^k_q(k_1,k_2)_{nKq}. \tag{4.2}$$

On using Eq. (4.2), we note that the process is described by ten nonzero irreducible tensor amplitudes in the $nKq$ frame, out of which only six are independent. By making a transformation from the spherical to the Cartesian coordinates, we express the irreducible tensor amplitudes in the $nKq$ frame in terms of the Piekarewicz *et al.* amplitudes of Eq. (1.1) through

$$\mathcal{T}^0_0(1,1)_{nKq}=-\frac{1}{3}[A_{nn}+A_{KK}+A_{qq}],$$

$$\mathcal{T}^2_0(1,1)_{nKq}=\frac{\sqrt{2}}{3}\left[A_{qq}-\frac{1}{2}(A_{nn}+A_{KK})\right],$$

$$\mathcal{T}^1_1(0,1)_{nKq}=-\mathcal{T}^1_{-1}(0,1)_{nKq}=-\frac{1}{\sqrt{6}}A_{n0},$$

$$\mathcal{T}^1_1(1,1)_{nKq}=-\mathcal{T}^1_{-1}(1,1)_{nKq}=-\frac{i}{2\sqrt{3}}[A_{Kq}-A_{qK}],$$

$$\mathcal{T}^2_{-1}(1,1)_{nKq}=\mathcal{T}^2_1(1,1)_{nKq}=-\frac{i}{2\sqrt{3}}[A_{Kq}+A_{qK}],$$

$$\mathcal{T}^2_{-2}(1,1)_{nKq}=\mathcal{T}^2_2(1,1)_{nKq}=\frac{1}{2\sqrt{3}}[A_{nn}-A_{KK}]. \tag{4.3}$$

A connection between the irreducible tensor amplitudes in the TF and those of Piekarewicz *et al.* is easily facilitated by noting that $\mathcal{T}^k_q(TF)=\Sigma_{q'}D^k_{q'q}(0,\pi/2,\pi/2-\theta/2)\mathcal{T}^k_{q'}(nKq)$, where $\theta$ is the scattering angle. We thus have

$$A_{n0}=\sqrt{3}\mathcal{T}^1_0(0,1)_{TF},$$

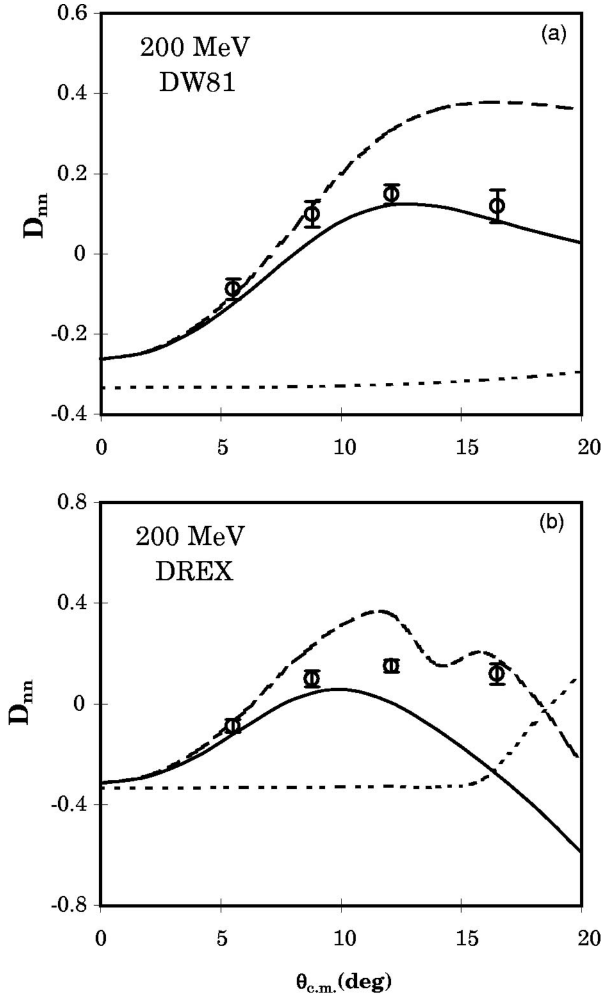


FIG. 1. $D_{nn}$ plotted against $\theta_{\mathrm{c.m.}}$(deg). The solid line shows the calculations using DW81 (a) and DREX (b). The short-dashed (long-dashed) line shows calculations with tensor (vector) amplitudes set to zero.

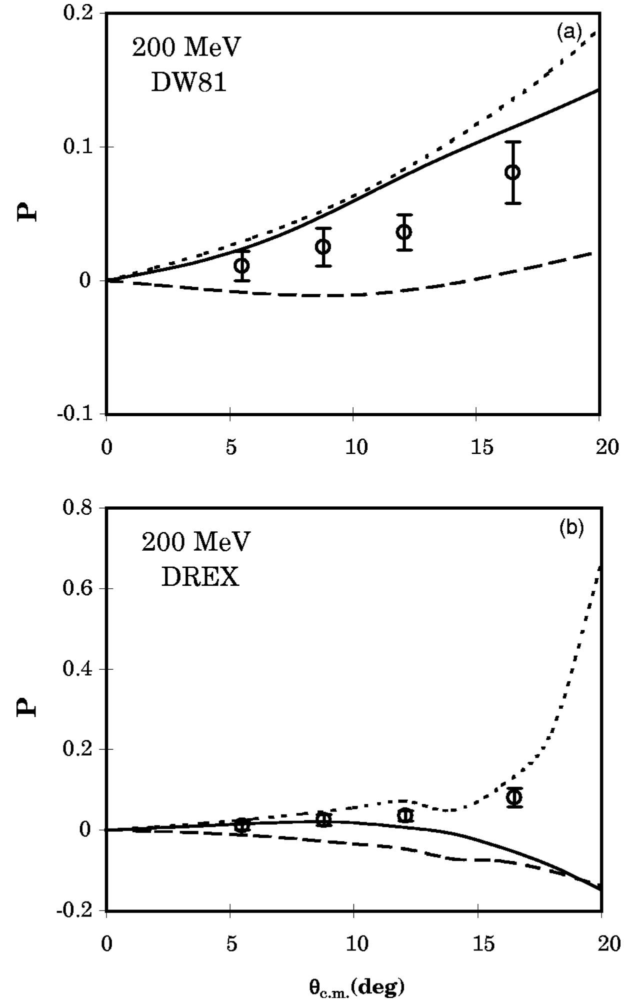


FIG. 2. $P$ plotted against $\theta_{\mathrm{c.m.}}$(deg). The solid line shows the calculations using DW81 (a) and DREX (b). The short-dashed (long-dashed) line shows calculations with tensor (vector) amplitudes set to zero.

$$A_{nn}=\sqrt{2}\mathcal{T}_0^2(1,1)_{TF}-\mathcal{T}_0^0(1,1)_{TF},$$

$$A_{qq}=-\left\{\mathcal{T}_0^0(1,1)_{TF}+\frac{1}{\sqrt{2}}\mathcal{T}_0^2(1,1)_{TF}+\sqrt{\frac{3}{2}}[e^{-i\theta}\mathcal{T}_2^2(1,1)_{TF}+e^{i\theta}\mathcal{T}_{-2}^2(1,1)_{TF}]\right\},$$

$$A_{KK}=-\left\{\mathcal{T}_0^0(1,1)_{TF}+\frac{1}{\sqrt{2}}\mathcal{T}_0^2(1,1)_{TF}-\sqrt{\frac{3}{2}}[e^{-i\theta}\mathcal{T}_2^2(1,1)_{TF}+e^{i\theta}\mathcal{T}_{-2}^2(1,1)_{TF}]\right\},$$

$$A_{Kq}=-i\sqrt{\frac{3}{2}}[e^{-i\theta}\mathcal{T}_2^2(1,1)_{TF}-e^{i\theta}\mathcal{T}_{-2}^2(1,1)_{TF}+\sqrt{2}\mathcal{T}_0^1(1,1)_{TF}],$$

$$A_{qK}=-i\sqrt{\frac{3}{2}}[e^{-i\theta}\mathcal{T}_2^2(1,1)_{TF}-e^{i\theta}\mathcal{T}_{-2}^2(1,1)_{TF}-\sqrt{2}\mathcal{T}_0^1(1,1)_{TF}]. \tag{4.4}$$

We next express the inelastic nucleon-spin observables in terms of the irreducible tensor amplitudes through

$$\frac{d\sigma}{d\Omega}D_{\alpha,\beta}=\frac{1}{2}\sum_{n=0}^{1}\sum_{n'=0}^{1}\sum_{\kappa=|n-n'|}^{(n+n')}[(P^n(\alpha)\otimes P^{n'}(\beta))^{\kappa}\cdot B^{\kappa}(n,n')]; \tag{4.5}$$

$$\alpha,\beta=0,x,y,z,$$

where

$$B_Q^{\kappa}(n,n') = 3 \sum_{k_1,k_1',k,k'} [k][k'][k_1][k_1'][n] \times [n'] W(k_1 1 \kappa k'; k k_1') \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & k_1 \\ \frac{1}{2} & \frac{1}{2} & k_1' \\ n & n' & \kappa \end{Bmatrix} \times (-1)^{k+n'-\kappa} (\mathcal{T}^k(k_1,1) \otimes \mathcal{T}^{\dagger k'}(k_1',1))_Q^{\kappa}, \tag{4.6}$$

and the irreducible tensors, $P_\mu^n(\alpha)$ are defined through

$$P_\mu^n(\alpha) = \mathrm{Tr}(\sigma_\alpha \sigma_\mu^n). \tag{4.7}$$

We note that the nonzero $P_\mu^n(\alpha)$ are

$$P_0^0(0) = 2; \quad P_0^1(z) = 2; \quad P_{\pm 1}^1(x) = \mp \sqrt{2}; \quad P_{\pm 1}^1(y) = -i\sqrt{2}. \tag{4.8}$$

Due to conditions imposed by parity and rotational invariance, out of the possible 16 spin transfer observables only eight are nonzero. The spin observables in the $nKq$ and TF are themselves related through

$$D_{n0} \equiv P = D_{z0},$$

$$D_{0n} \equiv A_y = D_{0z},$$

$$D_{nn} = D_{zz},$$

$$D_{KK} = \cos^2\frac{\theta}{2} D_{xx} + \sin^2\frac{\theta}{2} D_{yy} + \frac{\sin\theta}{2}(D_{yx} + D_{xy}),$$

$$D_{qq} = \cos^2\frac{\theta}{2} D_{yy} + \sin^2\frac{\theta}{2} D_{xx} - \frac{\sin\theta}{2}(D_{yx} + D_{xy}),$$

$$D_{Kq} = \cos^2\frac{\theta}{2} D_{xy} - \sin^2\frac{\theta}{2} D_{yx} + \frac{\sin\theta}{2}(D_{yy} - D_{xx}),$$

$$D_{qK} = \cos^2\frac{\theta}{2} D_{yx} - \sin^2\frac{\theta}{2} D_{xy} + \frac{\sin\theta}{2}(D_{yy} - D_{xx}). \tag{4.9}$$

The irreducible-tensor formalism outlined above lends itself conveniently to the study of the relative importance of the vector and second-rank tensor amplitudes. We use the DW81 and DREX computer codes that formulate the NA scattering in the nonrelativistic and relativistic impulse approximation respectively. The DW81 code uses the elementary $NN$ interaction as parameterized by Franey and Love [27] and distorted waves from an optical potential derived from the 400-MeV data of Jones *et al.* [28]. The DREX code uses the $NN$ interaction of Horowitz [29] and distorted waves from an optical potential using this $NN$ interaction and a nucleon density derived from electron scattering. Both codes take into consideration the direct+exchange terms and both the $NN$ interactions [27,29] are fitted to reproduce the $NN$ phase shift solutions of [30]. The Cohen and Kurath [31] nuclear structure amplitudes are used in both cases.

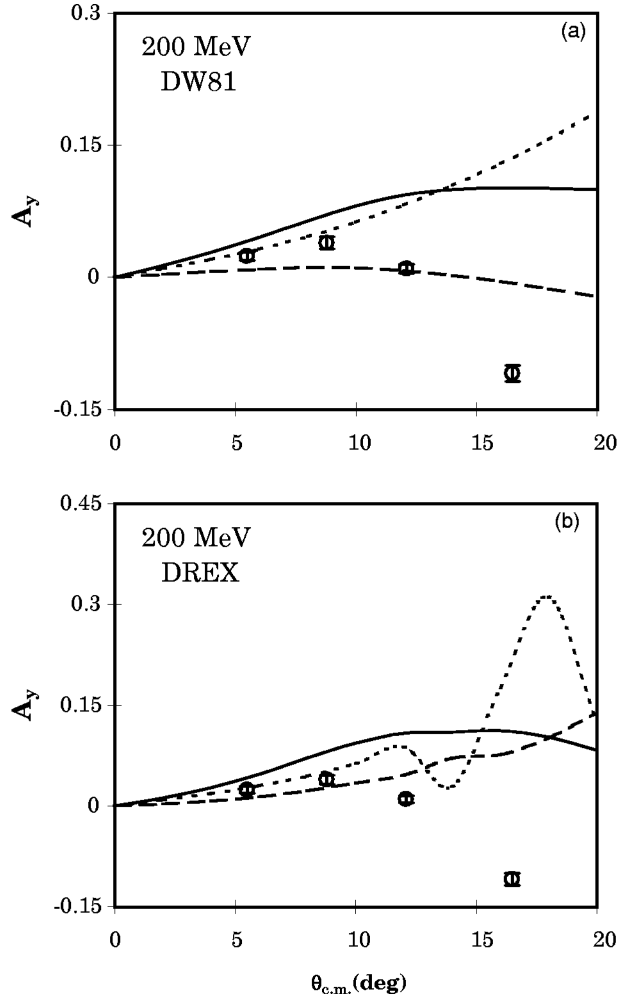


FIG. 3. $A_y$ plotted against $\theta_{\mathrm{c.m.}}$(deg). The solid line shows the calculations using DW81 (a) and DREX (b). The short-dashed (long-dashed) line shows calculations with tensor (vector) amplitudes set to zero.

A comparison is made of the theoretical estimates for the spin observables with the experimental data reported by Wells *et al.* [13] for 200-MeV measured at four angles, $\theta = 5.5°, 8.8°, 12.1°$, and $16.5°$. We next proceed to calculate the spin observables by (i) setting all the second-rank irreducible-tensor amplitudes to zero and (ii) setting all the vector amplitudes to zero, to facilitate an analysis of the relative importance of the vector and tensor terms. In the former case, the calculations are represented by short-dashed lines and in the latter, by long-dashed lines.

The plots (solid line) for the normal component observables, $D_{nn}$, $P$, and $A_y$ are shown in Figs. 1, 2, and 3. Comparing the experimental data with calculations based on DW81 and DREX codes, we notice that both the calculations account for the measured observables only partially. The DW81 calculation for $D_{nn}$ provides a closer fit in comparison with the DREX calculation, while the measured $P$ and $A_y$ are

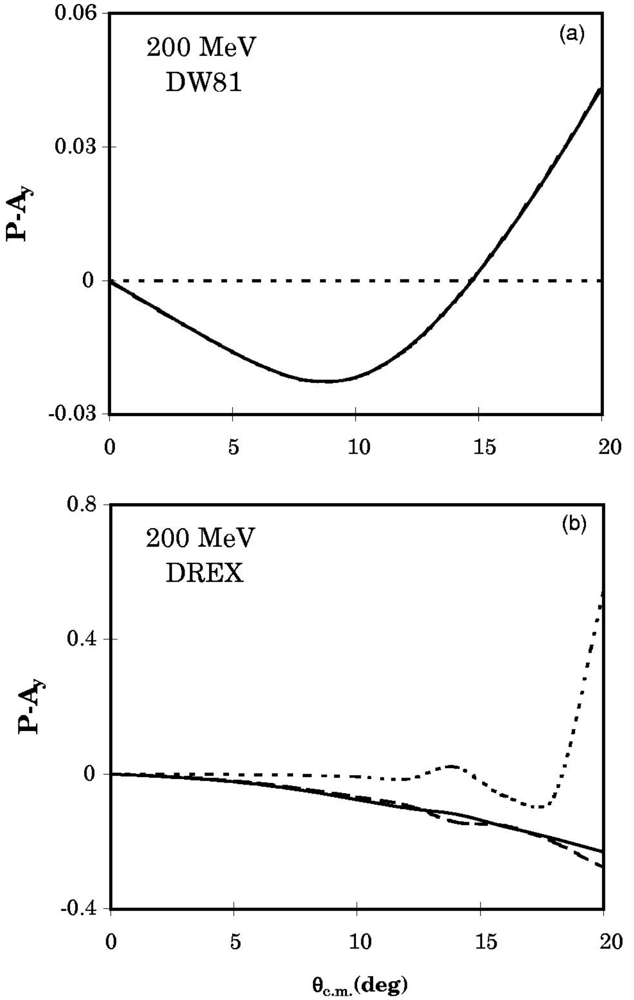


FIG. 4. $P$-$A_y$ plotted against $\theta_{\text{c.m.}}$(deg). The solid line shows the calculations using DW81 (a) and DREX (b). The short-dashed (long-dashed) line shows calculations with tensor (vector) amplitudes set to zero.

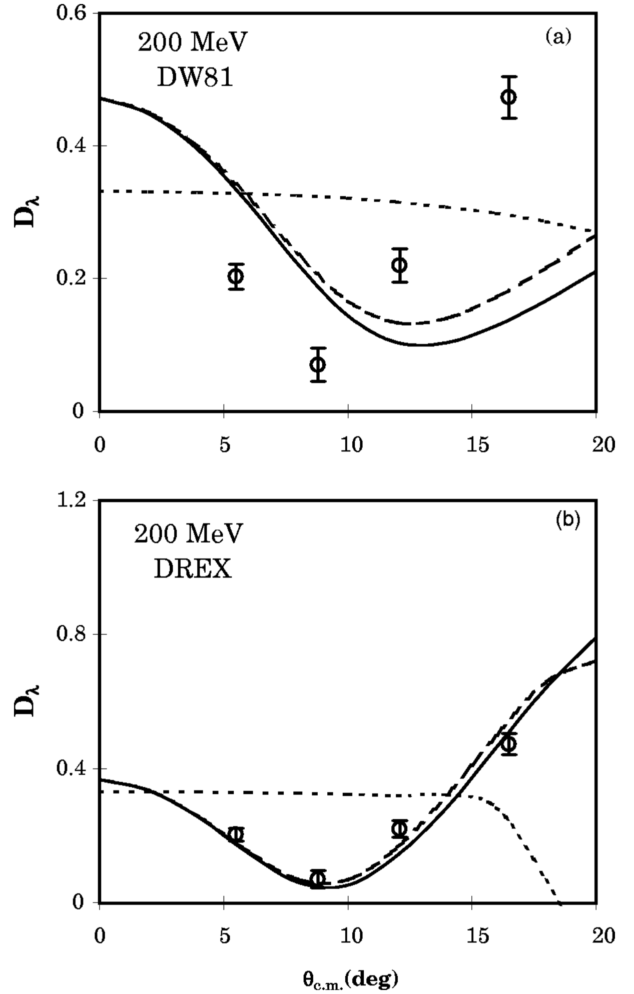


FIG. 5. $D_\lambda$ plotted against $\theta_{\text{c.m.}}$(deg). The solid line shows the calculations using DW81 (a) and DREX (b). The short-dashed (long-dashed) line shows calculations with tensor (vector) amplitudes set to zero.

not satisfactorily described by either DREX or DW81. It is interesting to note that in the absence of vector terms, the calculations for $D_{nn}$ (Fig. 1) are overestimated. Calculations with only the scalar and tensor terms compare well with the estimates for the $D_{nn}$ only at forward angles. The DW81 calculations for $P$ (Fig. 2) show that the vector amplitudes are relatively more important than the tensor amplitudes. The 200-MeV DREX calculations demonstrate the same in the range 0–8°. Similarly, the importance of vector amplitudes in $A_y$ (Fig. 3) is quite pronounced. Much attention has been evinced in accounting for the nonvanishing combination $P$-$A_y$ [32]. In terms of irreducible tensor amplitudes in the TF, we have,

$$P-A_y=6[|\mathcal{T}^2_{-2}(1,1)|^2-|\mathcal{T}^2_2(1,1)|^2]+4\sqrt{3}\mathrm{Re}[\sqrt{2}\mathcal{T}^0_0(1,1)\mathcal{T}^{1*}_0(1,1)+\mathcal{T}^2_0(1,1)\mathcal{T}^{1*}_0(1,1)], \tag{4.10}$$

which clearly indicates that the $P$-$A_y$ is sensitive to tensor amplitudes. This fact is explicitly demonstrated in Fig. 4. In fact calculations with the tensor terms set to zero yield $P$-$A_y=0$ in the range under consideration.

Figures 5 and 6 show comparison of experiment with theory for $D_\lambda$ and $D_\sigma$, respectively. The linear combination of in-plane component observables, $D_\lambda$ and $D_\sigma$, are defined in [13] as

$$D_\lambda \equiv D_{L'L}\sin\alpha + D_{S'L}\cos\alpha,$$

$$D_\sigma \equiv D_{L'S}\sin\alpha + D_{S'S}\cos\alpha, \tag{4.11}$$

where $\alpha \approx 264°$ is the angle of (horizontal) spin precision experienced by the scattered proton flux in the dipole field of K600 spectrometer. $L$ and $S$ stand for the longitudinal and sideward components and the prime indicates that the frame of reference of the scattered nucleon is rotated with respect to that of the incident nucleon by $\theta$. In terms of the amplitudes defined by Piekarewicz *et al.* [10],

$$D_{L'L}=\cos^2\frac{\theta}{2}D_{KK}-\sin^2\frac{\theta}{2}D_{qq}+\frac{\sin\theta}{2}(D_{qK}-D_{Kq}),$$

$$D_{S'S}=\cos^2\frac{\theta}{2}D_{qq}-\sin^2\frac{\theta}{2}D_{KK}+\frac{\sin\theta}{2}(D_{qK}-D_{Kq}),$$

$$D_{L'S}=\cos^2\frac{\theta}{2}D_{Kq}+\sin^2\frac{\theta}{2}D_{qK}+\frac{\sin\theta}{2}(D_{KK}+D_{qq}),$$

$$D_{S'S}=\cos^2\frac{\theta}{2}D_{qK}+\sin^2\frac{\theta}{2}D_{Kq}-\frac{\sin\theta}{2}(D_{KK}+D_{qq}). \tag{4.12}$$

The calculations show fairly good fit for these observables except in the case of DW81 calculations for $D_\lambda$. The figures also explicitly demonstrate that the vector amplitudes have a negligible role to play in the case of in-plane component observables, especially at forward angles.

### V. SUMMARY AND CONCLUSIONS

We have outlined a model-independent irreducible tensor formalism to discuss inelastic scattering of nucleons on nuclei. The formalism is valid at all energies and for arbitrary spin-parity transitions $J_i^{\pi_i}\to J_f^{\pi_f}$ of the nucleus. The irreducible tensor amplitudes have also been explicitly expressed in terms of the partial wave amplitudes. This facilitates in particular to obtain partial wave expansions for the amplitudes defined by Piekarewicz, Amado, and Sparrow [3] for inelastic scattering of protons on $C^{12}$ leading to $1^+$ excited state at 15.11 MeV, which has attracted considerable experimental work.

The formalism in terms of irreducible tensors has an additional advantage in that it facilitates the discussion of the central and noncentral interactions that come into play in inelastic scattering of nucleons on nuclei. In the particular case under consideration, there is no spin-independent central interaction and the noncentral interactions are limited to second rank. We find that the nonvanishing combination $P$-$A_y$ is highly sensitive to the second-rank tensor amplitudes. The numerical calculations show that if the tensor amplitudes are set to zero, then $P$-$A_y$ is zero in the range under consideration.

In view of the general applicability of the formalism, we advocate empirical analyses of data on inelastic scattering of nucleons on various nuclei and leading to different excited states in terms of irreducible tensor amplitudes so that one can compare the empirical values so determined with those deduced from model-dependent codes. This may facilitate the change of inputs into the models selectively so as to bring the theoretical estimates closer to experiments.

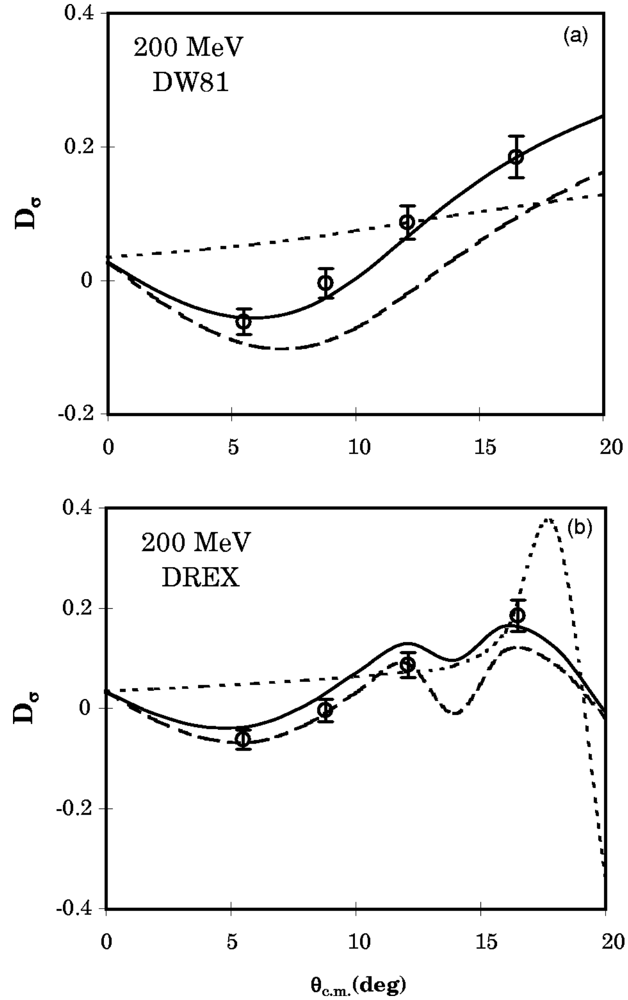


FIG. 6. $D_\sigma$ plotted against $\theta_{c.m.}$(deg). The solid line shows the calculations using DW81 (a) and DREX (b). The short-dashed (long-dashed) line shows calculations with tensor (vector) amplitudes set to zero.

### ACKNOWLEDGMENTS

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