Identical relations among transverse parts of variant Green's functions and the full vertices in gauge theories

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Identity relations among the transverse parts of variant vertex functions in gauge theories are derived by computing the curl of the time-ordered products of three-point Green's functions involving the vector, the axial-vector, and the tensor current operators, respectively. Combining these transverse relations with the normal (longitudinal) Ward-Takahashi identities forms a complete set of Ward-Takahashi relations for three-point vertex functions. As a consequence, the complete solutions for the vector, the axial-vector, and the tensor vertex functions in the momentum space are consistently and exactly obtained by solving this complete set of Ward-Takahashi relations. In the case of massless fermions, the full vector and the full axial-vector vertices are expressed in terms of the fermion propagators only.

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In quantum field theory symmetries lead to relations among the Green's functions of the theory, which are referred to as the Ward-Takahashi (WT) identities [1]. They play an important role in proving renormalizability and in providing a consistent description in the perturbation approach of any quantum field theory. But the normal WT identities specify only the longitudinal part of Green's functions, leaving the transverse part undetermined [2]. Therefore, to obtain the complete constraint on the vertex functions and then to obtain the complete expressions for the vertex functions we need to study the WT type constraint relations for the transverse parts of variant vertices, which is of great significance. In this regard, a very interesting problem relates to the Dyson-Schwinger equation (DSE) approach [3].

The Dyson-Schwinger equations embody the full structure of any field theory and consequently provide a natural way to study the dynamics such as describing the dynamical chiral symmetry breaking, confinement, and other problems of hadronic physics [4]. The structure of DSEs is such that they relate the n-point Green's function to the $(n+1)$-point function; at its simplest, propagators are related to three-point vertices, thus leading to an infinite set of coupled equations. Therefore, one has to find some way to truncate this set of equations. If we can express the full three-point vertices in terms of the two-point functions, these equations will form a closed system for the two-point functions. How to solve exactly the transverse part of the vertex and thereby the full vertex function then becomes a crucial problem [2]. Up to now this problem has not been solved. Although there have been several attempts to construct the transverse part of the vertex by an ansatz which satisfies some constraints [2,5], however, all such attempts remain *ad hoc* without considering the constraint imposed by the symmetry of the system. The latter is the key point to understand the transverse part of the vertex as in the case of the longitudinal part of the vertex. In Ref. [6], we studied the transverse WT relation for the vector vertex. It showed that in order to obtain the complete solution for the vector vertex one needs to build WT relations for the axial-vector and tensor vertices as well. In this paper we present the complete set of WT relations for the vector, the axial-vector and the tensor vertex functious in gauge theories, from which we obtain the complete solutions for these vertex functions in gauge theories in four dimensions. In particular, we find that in the chiral limit with zero fermion masses the full vector and the full axial-vector vertex functions are expressed in terms of the fermion propagators only.

We first provide the WT type identical relations among the transverse parts of variant three-point vertex functions, i.e., the transverse WT relations for three-point vertex functions in gauge theories, which are derived by computing the curl of the time-ordered products of three-point Green's functions involving the vector, the axial-vector, and the tensor current operators, respectively [7]. This approach is motivated by the fact that the normal WT identities which specify the longitudinal part of the Green's functions have been derived by computing the divergence of the time-ordered products of the corresponding Green's functions [8]. We find three sets of transverse WT relations for the vector, the axial-vector, and the tensor vertex functions, respectively, which are coupled to each other. These relations are given in coordinate space as well as in momentum space. The latter form is partically useful. Combining these transverse relations with the normal (longitudinal) WT identities for the vector, the axial-vector, and the tensor vertex functions leads to a complete set of WT type constraint relations for the fermion's three-point functions. As a consequence, the complete expressions for the vector, the axial-vector, and the tensor vertex functions in the momentum space are then consistently and exactly deduced by solving this complete set of WT relations without any ansatz.

Let us briefly describe the basic approach of computing the curl of the time-ordered products of the fermion's three-point functions involving the vector, the axial-vector, and the tensor current operators respectively. For convenience, we introduce three bilinear covariant current operators:

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$$V^{\lambda\mu\nu}(x)=\tfrac{1}{2}\overline{\psi}(x)[\gamma^{\lambda},\sigma^{\mu\nu}]\psi(x)=i[g^{\lambda\mu}j^{\nu}(x)-g^{\lambda\nu}j^{\mu}(x)], \tag{1}$$

$$V_5^{\lambda\mu\nu}(x)=\tfrac{1}{2}\overline{\psi}(x)[\gamma^{\lambda},\sigma^{\mu\nu}]\gamma_5\psi(x)$$
$$=i[g^{\lambda\mu}j_5^{\nu}(x)-g^{\lambda\nu}j_5^{\mu}(x)], \tag{2}$$

and

$$V^{\lambda\mu\nu\alpha}(x)=\tfrac{1}{4}\overline{\psi}(x)([\gamma^{\lambda},\sigma^{\mu\nu}]\gamma^{\alpha}-\gamma^{\alpha}[\gamma^{\lambda},\sigma^{\mu\nu}])\psi(x)$$
$$=g^{\lambda\mu}j^{\nu\alpha}(x)-g^{\lambda\nu}j^{\mu\alpha}(x), \tag{3}$$

where $j^{\mu}(x)=\overline{\psi}(x)\gamma^{\mu}\psi(x)$, $j_5^{\mu}(x)=\overline{\psi}(x)\gamma^{\mu}\gamma_5\psi(x)$, and $j^{\mu\nu}(x)=\overline{\psi}(x)\sigma^{\mu\nu}\psi(x)$. Thus the curl of the T products of the corresponding fermion's three-point function is given by $\partial_{\lambda}^{x}T[V^{\lambda\mu\nu}(x)\psi(x_1)\overline{\psi}(x_2)]$ or $\partial_{\lambda}^{x}T[V_5^{\lambda\mu\nu}(x)\psi(x_1)\overline{\psi}(x_2)]$ or $\partial_{\lambda}^{x}T[V^{\lambda\mu\nu\alpha}(x)\psi(x_1)\overline{\psi}(x_2)]$, where ∂_{λ}^{x} denotes the derivative operator with respect to the argument x. In terms of the definition for the time-ordered products and the equal-time anticommutation relations for fermion fields, it is not difficult to carry out the above differential operations. The procedure of deriving the transverse WT identity for the vector vertex in gauge theory was given already in Ref. [6]. With the similar procedure we derive the transverse WT relations for the axial-vector and the tensor vertices. We find the following covariant identical relations in the operator form:

$$\begin{aligned}&\partial_x^{\mu}T[j^{\nu}(x)\psi(x_1)\overline{\psi}(x_2)]-\partial_x^{\nu}T[j^{\mu}(x)\psi(x_1)\overline{\psi}(x_2)]\\&=i\sigma^{\mu\nu}T[\psi(x_1)\overline{\psi}(x_2)]\delta^4(x_1-x)+iT[\psi(x_1)\overline{\psi}(x_2)]\sigma^{\mu\nu}\delta^4(x_2-x)\\&\quad+T[\overline{\psi}(x)(\sigma^{\mu\nu}i\vec{D\!\!\!/}_x-i\overleftarrow{D\!\!\!/}_x\sigma^{\mu\nu})\psi(x)\psi(x_1)\overline{\psi}(x_2)]\\&\quad+\lim_{x'\to x}i(\partial_{\lambda}^{x}-\partial_{\lambda}^{x'})T[\overline{\psi}(x')\varepsilon^{\lambda\mu\nu\rho}\gamma_{\rho}\gamma_5U_P(x',x)\psi(x)\psi(x_1)\overline{\psi}(x_2)],\end{aligned}\tag{4}$$

$$\begin{aligned}&\partial_x^{\mu}T[j_5^{\nu}(x)\psi(x_1)\overline{\psi}(x_2)]-\partial_x^{\nu}T(j_5^{\mu}(x)\psi(x_1)\overline{\psi}(x_2))\\&=i\sigma^{\mu\nu}\gamma_5T[\psi(x_1)\overline{\psi}(x_2)]\delta^4(x_1-x)-iT[\psi(x_1)\overline{\psi}(x_2)]\sigma^{\mu\nu}\gamma_5\delta^4(x_2-x)\\&\quad-T[\overline{\psi}(x)(i\overleftarrow{D\!\!\!/}_x\sigma^{\mu\nu}\gamma_5+i\sigma^{\mu\nu}\gamma_5\vec{D\!\!\!/}_x)\psi(x)\psi(x_1)\overline{\psi}(x_2)]\\&\quad+\lim_{x'\to x}i(\partial_{\lambda}^{x}-\partial_{\lambda}^{x'})T[\overline{\psi}(x')\varepsilon^{\lambda\mu\nu\rho}\gamma_{\rho}U_P(x',x)\psi(x)\psi(x_1)\overline{\psi}(x_2)],\end{aligned}\tag{5}$$

and

$$\begin{aligned}&\partial_x^{\mu}T[j^{\nu\alpha}(x)\psi(x_1)\overline{\psi}(x_2)]-\partial_x^{\nu}T[j^{\mu\alpha}(x)\psi(x_1)\overline{\psi}(x_2)]\\&=\varepsilon^{\mu\nu\alpha\rho}\gamma_{\rho}\gamma_5T[\psi(x_1)\overline{\psi}(x_2)]\delta^4(x_1-x)-T[\psi(x_1)\overline{\psi}(x_2)]\varepsilon^{\mu\nu\alpha\rho}\gamma_{\rho}\gamma_5\delta^4(x_2-x)\\&\quad-T[\overline{\psi}(x)\varepsilon^{\mu\nu\alpha\rho}(\overleftarrow{D\!\!\!/}_x\gamma_{\rho}\gamma_5+\gamma_{\rho}\gamma_5\vec{D\!\!\!/}_x)\psi(x)\psi(x_1)\overline{\psi}(x_2)]\\&\quad-\lim_{x'\to x}(\partial_{\lambda}^{x}-\partial_{\lambda}^{x'})T[\overline{\psi}(x')\varepsilon^{\lambda\mu\nu\alpha}\gamma_5U_P(x',x)\psi(x)\psi(x_1)\overline{\psi}(x_2)]\\&\quad-\lim_{x'\to x}(\partial_x^{\alpha}+\partial_{x'}^{\alpha})T[\overline{\psi}(x')\sigma^{\mu\nu}U_P(x',x)\psi(x)\psi(x_1)\overline{\psi}(x_2)],\end{aligned}\tag{6}$$

where $\vec{D}_{\mu}=\vec{\partial}_{\mu}+igA_{\mu}$ and $\overleftarrow{D}_{\mu}=\overleftarrow{\partial}_{\mu}-igA_{\mu}$ are covariant derivatives. In the QED case, $g=e$ and A_{μ} is the photon field. In the QCD case, $g=g_c$ and $A_{\mu}=A_{\mu}^{a}T^{a}$ $(a=1,\ldots,8)$, A_{μ}^{a} are the color gluon fields and T^{a} are the generators of $SU(3)_c$ group. The Wilson line $U_P(x',x)=P\exp[-ig\int_x^{x'}dy^{\rho}A_{\rho}(y)]$ is introduced in order that the current operators in the last term of Eqs. (4),(5) and in the last two terms of Eq. (6) are locally gauge invariant. Note that such expressions are also useful for studying the Adler-Bell-Jackiw anomaly [9,10] contribution in the present case.

Taking into account the equations of motion for fermions with mass m, $(i\vec{D\!\!\!/}-m)\psi=0$, and $\overline{\psi}(i\overleftarrow{D\!\!\!/}+m)=0$, which have the same form for both QED and QCD, we arrive at the identical relations among the transverse parts of the fermion's three-point functions, i.e., the transverse WT relations for the fermion's vertex functions in gauge theories (in coordinate space):

$$\begin{aligned}
&\partial_x^\mu \langle 0|Tj^\nu(x)\psi(x_1)\bar{\psi}(x_2)|0\rangle - \partial_x^\nu \langle 0|Tj^\mu(x)\psi(x_1)\bar{\psi}(x_2)|0\rangle \\
&\quad = i\sigma^{\mu\nu}\langle 0|T\psi(x_1)\bar{\psi}(x_2)|0\rangle \delta^4(x_1-x) + i\langle 0|T\psi(x_1)\bar{\psi}(x_2)|0\rangle \sigma^{\mu\nu}\delta^4(x_2-x) \\
&\qquad + 2m\langle 0|T\bar{\psi}(x)\sigma^{\mu\nu}\psi(x)\psi(x_1)\bar{\psi}(x_2)|0\rangle \\
&\qquad + \lim_{x'\to x} i(\partial_\lambda^x - \partial_\lambda^{x'})\varepsilon^{\lambda\mu\nu\rho}\langle 0|T\bar{\psi}(x')\gamma_\rho\gamma_5 U_P(x',x)\psi(x)\psi(x_1)\bar{\psi}(x_2)|0\rangle, \qquad (7)
\end{aligned}$$

$$\begin{aligned}
&\partial_x^\mu \langle 0|Tj_5^\nu(x)\psi(x_1)\bar{\psi}(x_2)|0\rangle - \partial_x^\nu \langle 0|Tj_5^\mu(x)\psi(x_1)\bar{\psi}(x_2)|0\rangle \\
&\quad = i\sigma^{\mu\nu}\gamma_5\langle 0|T\psi(x_1)\bar{\psi}(x_2)|0\rangle \delta^4(x_1-x) - i\langle 0|T\psi(x_1)\bar{\psi}(x_2)|0\rangle \sigma^{\mu\nu}\gamma_5\delta^4(x_2-x) \\
&\qquad + \lim_{x'\to x} i(\partial_\lambda^x - \partial_\lambda^{x'})\varepsilon^{\lambda\mu\nu\rho}\langle 0|T\bar{\psi}(x')\gamma_\rho U_P(x',x)\psi(x)\psi(x_1)\bar{\psi}(x_2)|0\rangle, \qquad (8)
\end{aligned}$$

and

$$\begin{aligned}
&\partial_x^\mu \langle 0|Tj^{\nu\alpha}(x)\psi(x_1)\bar{\psi}(x_2)|0\rangle - \partial_x^\nu \langle 0|Tj^{\mu\alpha}(x)\psi(x_1)\bar{\psi}(x_2)|0\rangle \\
&\quad = \varepsilon^{\mu\nu\alpha\rho}\gamma_\rho\gamma_5\langle 0|T\psi(x_1)\bar{\psi}(x_2)|0\rangle \delta^4(x_1-x) - \langle 0|T\psi(x_1)\bar{\psi}(x_2)|0\rangle \varepsilon^{\mu\nu\alpha\rho}\gamma_\rho\gamma_5\delta^4(x_2-x) \\
&\qquad - \lim_{x'\to x} (\partial_\lambda^x - \partial_\lambda^{x'})\varepsilon^{\lambda\mu\nu\alpha}\langle 0|T\bar{\psi}(x')\gamma_5 U_P(x',x)\psi(x)\psi(x_1)\bar{\psi}(x_2)|0\rangle \\
&\qquad - \lim_{x'\to x} (\partial_x^\alpha + \partial_{x'}^\alpha)\langle 0|T\bar{\psi}(x')\sigma^{\mu\nu}U_P(x',x)\psi(x)\psi(x_1)\bar{\psi}(x_2)|0\rangle, \qquad (9)
\end{aligned}$$

where the vacuum expectation values are used. Equations (7)–(9) are valid for both QED and QCD.

The transverse WT relations can be written in more clear and elegant form in the momentum space. By computing the Fourier transformation of Eqs. (7)–(9) and using the standard definition for the three-point functions in momentum space [6], we get the transverse WT relations for the three-point vertex functions in the momentum space

$$\begin{aligned}
&iq^\mu\Gamma_V^\nu(p_1,p_2) - iq^\nu\Gamma_V^\mu(p_1,p_2) \\
&\quad = S_F^{-1}(p_1)\sigma^{\mu\nu} + \sigma^{\mu\nu}S_F^{-1}(p_2) + 2m\Gamma_T^{\mu\nu}(p_1,p_2) \\
&\qquad + (p_{1\lambda}+p_{2\lambda})\varepsilon^{\lambda\mu\nu\rho}\Gamma_{A\rho}(p_1,p_2), \qquad (10)
\end{aligned}$$

$$\begin{aligned}
&iq^\mu\Gamma_A^\nu(p_1,p_2) - iq^\nu\Gamma_A^\mu(p_1,p_2) \\
&\quad = S_F^{-1}(p_1)\sigma^{\mu\nu}\gamma_5 - \sigma^{\mu\nu}\gamma_5 S_F^{-1}(p_2) \\
&\qquad + (p_{1\lambda}+p_{2\lambda})\varepsilon^{\lambda\mu\nu\rho}\Gamma_{V\rho}(p_1,p_2), \qquad (11)
\end{aligned}$$

and

$$\begin{aligned}
&q^\mu\Gamma_T^{\nu\alpha}(p_1,p_2) + q^\nu\Gamma_T^{\alpha\mu}(p_1,p_2) + q^\alpha\Gamma_T^{\mu\nu}(p_1,p_2) \\
&\quad = -S_F^{-1}(p_1)\varepsilon^{\mu\nu\alpha\rho}\gamma_\rho\gamma_5 + \varepsilon^{\mu\nu\alpha\rho}\gamma_\rho\gamma_5 S_F^{-1}(p_2) \\
&\qquad + (p_{1\lambda}+p_{2\lambda})\varepsilon^{\lambda\mu\nu\alpha}\Gamma_5(p_1,p_2), \qquad (12)
\end{aligned}$$

where $q=p_1-p_2$, Γ_V^μ, Γ_A^μ, $\Gamma_T^{\mu\nu}$, and Γ_5 are the vector, the axial-vector, the tensor and the psudoscalar vertex functions in momentum space, respectively. $S_F(p_1)$ is the complete propagator of fermion. The third term in left-hand side of Eq. (12) comes from the Fourier transformation of the last term in Eq. (9).

To understand the physics implication of Eqs. (10),(11) more clearly, we multiply both sides of Eqs. (10) and (11) by iq_ν and move the terms proportional to $q_\nu\Gamma_V^\nu$ and $q_\nu\Gamma_A^\nu$ into the right-hand side of the equations, we then have

$$\begin{aligned}
q^2\Gamma_V^\mu(p_1,p_2) &= q^\mu[q_\nu\Gamma_V^\nu(p_1,p_2)] + iS_F^{-1}(p_1)q_\nu\sigma^{\mu\nu} \\
&\quad + iq_\nu\sigma^{\mu\nu}S_F^{-1}(p_2) + 2imq_\nu\Gamma_T^{\mu\nu}(p_1,p_2) \\
&\quad + i(p_{1\lambda}+p_{2\lambda})q_\nu\varepsilon^{\lambda\mu\nu\rho}\Gamma_{A\rho}(p_1,p_2), \qquad (13)
\end{aligned}$$

$$\begin{aligned}
q^2\Gamma_A^\mu(p_1,p_2) &= q^\mu[q_\nu\Gamma_A^\nu(p_1,p_2)] \\
&\quad + iS_F^{-1}(p_1)q_\nu\sigma^{\mu\nu}\gamma_5 - iq_\nu\sigma^{\mu\nu}\gamma_5 S_F^{-1}(p_2) \\
&\quad + i(p_{1\lambda}+p_{2\lambda})q_\nu\varepsilon^{\lambda\mu\nu\rho}\Gamma_{V\rho}(p_1,p_2). \qquad (14)
\end{aligned}$$

Writing the full vertices Γ_V^μ and Γ_A^μ as

$$\Gamma_V^\mu(p_1,p_2) = \Gamma_{V(L)}^\mu(p_1,p_2) + \Gamma_{V(T)}^\mu(p_1,p_2), \qquad (15)$$

$$\Gamma_A^\mu(p_1,p_2) = \Gamma_{A(L)}^\mu(p_1,p_2) + \Gamma_{A(T)}^\mu(p_1,p_2), \qquad (16)$$

we then obtain from Eqs. (13) and (14)

$$\Gamma_{V(L)}^\mu(p_1,p_2) = q^{-2}q^\mu[q_\nu\Gamma_V^\nu(p_1,p_2)], \qquad (17)$$

$$\Gamma^{\mu}_{V(T)}(p_1,p_2)=q^{-2}q_{\nu}[iS_F^{-1}(p_1)\sigma^{\mu\nu}+i\sigma^{\mu\nu}S_F^{-1}(p_2)+2im\Gamma_T^{\mu\nu}(p_1,p_2)+i(p_{1\lambda}+p_{2\lambda})\varepsilon^{\lambda\mu\nu\rho}\Gamma_{A\rho}(p_1,p_2)], \quad (18)$$

and

$$\Gamma^{\mu}_{A(L)}(p_1,p_2)=q^{-2}q^{\mu}[q_{\nu}\Gamma^{\nu}_{A}(p_1,p_2)], \quad (19)$$

$$\Gamma^{\mu}_{A(T)}(p_1,p_2)=q^{-2}q_{\nu}[iS_F^{-1}(p_1)\sigma^{\mu\nu}\gamma_5-i\sigma^{\mu\nu}\gamma_5S_F^{-1}(p_2)+i(p_{1\lambda}+p_{2\lambda})\varepsilon^{\lambda\mu\nu\rho}\Gamma_{V\rho}(p_1,p_2)]. \quad (20)$$

By using the antisymmetry property of $\sigma^{\mu\nu}$ and $\varepsilon^{\lambda\mu\nu\rho}$, it is easy to check that $q_{\mu}\Gamma^{\mu}_{V(T)}=0$ and $q_{\mu}\Gamma^{\mu}_{A(T)}=0$, which show that $\Gamma^{\mu}_{V(T)}$ and $\Gamma^{\mu}_{A(T)}$ are indeed the transverse components of the corresponding vertex functions. Note that $\Gamma^{\mu}_{V(T)}$ and $\Gamma^{\mu}_{A(T)}$ correspond, respectively, to the right-hand side of Eqs. (10) and (11) except the factor $iq^{-2}q_{\nu}$. Therefore, Eqs. (10) and (11) [and the corresponding expressions in coordinate space, Eqs. (7) and (8)] describe the relations among the transverse parts of the vector and the axial-vector vertex functions and other Green's functions. Equations (7) and (10) show that the transverse part of the vector vertex function is related to the inverse of the fermion propagator, the tensor and the axial-vector vertex functions, while Eqs. (8) and (11) show that the transverse part of the axial-vector vertex function is related to the inverse of the fermion propagator and the vector vertex function. Thus, the transverse parts of variant vertex functions are coupled to each other. As a result, the full vector and the full axial-vector vertex functions are also coupled to each other and form a set of coupled equations, which is described by Eqs. (13) and (14). The WT relation (12) can be similarly discussed.

In Eqs. (13)–(19), $q_{\mu}\Gamma^{\mu}_{V}$ and $q_{\mu}\Gamma^{\mu}_{A}$ satisfy the well-known Ward-Takahashi identities

$$q_{\mu}\Gamma^{\mu}_{V}(p_1,p_2)=S_F^{-1}(p_1)-S_F^{-1}(p_2), \quad (21)$$

$$q_{\mu}\Gamma^{\mu}_{A}(p_1,p_2)=S_F^{-1}(p_1)\gamma_5+\gamma_5S_F^{-1}(p_2)-2im\Gamma_5(p_1,p_2). \quad (22)$$

In addition to these two identities, the WT identity for the tensor vertex is also needed. By the procedure similar to that of deriving Eqs. (21) and (22), we find

$$iq_{\nu}\Gamma^{\mu\nu}_{T}(p_1,p_2)=S_F^{-1}(p_1)\gamma^{\mu}+\gamma^{\mu}S_F^{-1}(p_2)+2m\Gamma^{\mu}_{V}(p_1,p_2)+(p_1^{\mu}+p_2^{\mu})\Gamma_S(p_1,p_2), \quad (23)$$

where Γ_S is the scalar vertex function.

Now we have the normal Ward-Takahashi identities (21)–(23), describing the longitudinal part of the three-point vertex functions, and the transverse WT relations (10)–(12), describing the identical relations among transverse parts of variant three-point vertex functions. They form a complete set of WT type constraint relations for the fermion's three-point vertex functions in gauge theories. As a consequence, we can consistently derive the complete solusions for the vertex functions Γ^{μ}_{V}, Γ^{μ}_{A}, and $\Gamma^{\mu\nu}_{T}$, by solving this complete set of relations without any ansatz.

In fact, by substituting Eq. (14) into Eq. (13) and using Eqs. (21)–(23), it is not difficult to obtain the complete expression for the vector-vertex function in the case of massless fermions

$$\Gamma^{\mu}_{V}(p_1,p_2)=\Gamma^{\mu}_{V(L)}(p_1,p_2)+\Gamma^{\mu}_{V(T)}(p_1,p_2) \quad (24)$$

with

$$\Gamma^{\mu}_{V(L)}(p_1,p_2)=q^{-2}q^{\mu}[S_F^{-1}(p_1)-S_F^{-1}(p_2)], \quad (25)$$

$$\begin{aligned}\Gamma^{\mu}_{V(T)}(p_1,p_2)=&\{q^2+(p_1+p_2)^2-[(p_1+p_2)\cdot q]^2q^{-2}\}^{-1}\\&\times\{[S_F^{-1}(p_1)-S_F^{-1}(p_2)][q^{\mu}((p_1+p_2)\cdot q)^2q^{-2}-(p_1^{\mu}+p_2^{\mu})(p_1+p_2)\cdot q]q^{-2}\\&+[S_F^{-1}(p_1)+S_F^{-1}(p_2)][p_1^{\mu}+p_2^{\mu}-q^{\mu}(p_1+p_2)\cdot qq^{-2}]\\&+iS_F^{-1}(p_1)\sigma^{\mu\nu}q_{\nu}+i\sigma^{\mu\nu}q_{\nu}S_F^{-1}(p_2)+i[S_F^{-1}(p_1)\sigma^{\mu\lambda}-\sigma^{\mu\lambda}S_F^{-1}(p_2)](p_{1\lambda}+p_{2\lambda})\\&+i[S_F^{-1}(p_1)\sigma^{\lambda\nu}-\sigma^{\lambda\nu}S_F^{-1}(p_2)]q_{\nu}(p_{1\lambda}+p_{2\lambda})q^{\mu}q^{-2}\\&-i[S_F^{-1}(p_1)\sigma^{\mu\nu}-\sigma^{\mu\nu}S_F^{-1}(p_2)]q_{\nu}(p_1+p_2)\cdot qq^{-2}\}. \quad (26)\end{aligned}$$

Similarly, substituting Eqs. (24)–(26) into Eq. (14) and using Eq. (22), we can write the full axial-vector vertex function in the massless fermion case as

$$\Gamma^{\mu}_{A}(p_1,p_2)=\Gamma^{\mu}_{A(L)}(p_1,p_2)+\Gamma^{\mu}_{A(T)}(p_1,p_2) \quad (27)$$

with

$$\Gamma^{\mu}_{A(L)}(p_1,p_2)=q^{-2}q^{\mu}[S_F^{-1}(p_1)\gamma_5+\gamma_5S_F^{-1}(p_2)], \quad (28)$$

$$\Gamma^{\mu}_{A(T)}(p_1,p_2)=\{q^2+(p_1+p_2)^2-[(p_1+p_2)\cdot q]^2q^{-2}\}^{-1}$$
$$\times\{i[S_F^{-1}(p_1)\gamma_5\sigma^{\mu\nu}-\sigma^{\mu\nu}\gamma_5S_F^{-1}(p_2)]q_\nu-i[S_F^{-1}(p_1)\gamma_5\sigma^{\mu\nu}$$
$$+\sigma^{\mu\nu}\gamma_5S_F^{-1}(p_2)]q_\nu(p_1+p_2)\cdot qq^{-2}+i[S_F^{-1}(p_1)\gamma_5\sigma^{\mu\lambda}+\sigma^{\mu\lambda}\gamma_5S_F^{-1}(p_2)](p_{1\lambda}+p_{2\lambda})$$
$$+i[S_F^{-1}(p_1)\gamma_5\sigma^{\lambda\nu}+\sigma^{\lambda\nu}\gamma_5S_F^{-1}(p_2)]q_\nu(p_{1\lambda}+p_{2\lambda})q^\mu q^{-2}$$
$$+i[S_F^{-1}(p_1)\gamma_5\sigma^{\lambda\nu}-\sigma^{\lambda\nu}\gamma_5S_F^{-1}(p_2)]q_\nu(p_{1\lambda}+p_{2\lambda})[p_1^\mu+p_2^\mu-q^\mu(p_1+p_2)\cdot qq^{-2}]q^{-2}\}. \quad (29)$$

We see that in the chiral limit with zero fermion masses the full vector and the full axial-vector vertex functions are now expressed in terms of fermion propagators only.

Finally, by using Eqs. (12) and (23) we can get the complete expression for the tensor vertex function:

$$q^2\Gamma_T^{\mu\nu}(p_1,p_2)=iS_F^{-1}(p_1)(q^\mu\gamma^\nu-q^\nu\gamma^\mu)$$
$$+i(q^\mu\gamma^\nu-q^\nu\gamma^\mu)S_F^{-1}(p_2)+2im[q^\mu\Gamma_V^\nu(p_1,p_2)-q^\nu\Gamma_V^\mu(p_1,p_2)]$$
$$+i[q^\mu(p_1^\nu+p_2^\nu)-q^\nu(p_1^\mu+p_2^\mu)]\Gamma_S(p_1,p_2)-S_F^{-1}(p_1)\varepsilon^{\mu\nu\alpha\rho}q_\alpha\gamma_\rho\gamma_5$$
$$+\varepsilon^{\mu\nu\alpha\rho}q_\alpha\gamma_\rho\gamma_5S_F^{-1}(p_2)+(p_{1\lambda}+p_{2\lambda})q_\alpha\varepsilon^{\lambda\alpha\mu\nu}\Gamma_5(p_1,p_2), \quad (30)$$

where $\Gamma_V^{\mu(\nu)}$ is given by Eqs. (24)–(26).

Before concluding, I would like to give the following comments.

(i) The transverse WT relations for the vector vertex function, Eqs. (7) and (10), involve the mass term arising from the equations of motion. This is similar to the normal Ward-Takahashi identity for the axial-vector vertex [see Eq. (22)]. On the contrary, the transverse WT relations for the axial-vector vertex function, Eqs. (8) and (11), have no mass term, i.e., they are independent of the dynamics, which is similar to the WT identity for the vector vertex function [see Eq. (21)].

(ii) The transverse WT relations for the three-point functions, Eqs. (7)–(12), have been derived in QED and the classical QCD (without Faddeev-Popov ghost fields). It remains to show if these identical relations will be modified by higher-order correction terms in perturbation theory. It is well-known that the normal WT identity for the axial-vector vertex, Eq. (22), is modified due to the Adler-Bell-Jackiw anomaly [9]. As a result, the WT identity for axial-vector vertex function, Eq. (22), should add the anomaly term contribution. By applying the approach of deriving ABJ anomaly [9,10] to the present case, we find that the ABJ anomaly does not contribute to the transverse WT relations for the vector vertex, Eqs. (7) and (10). For Eqs. (7) and (10), the modification by higher-order correction happens only to the tensor vertex term due to the renormalization of the tensor current operator, which leads to the appearance of anomalous dimension in the tensor vertex term. But such modification does not affect the transverse WT relation for the vector vertex function in the chiral limit with zero fermion masses. Is there the contribution from anomaly to other transverse WT relations? This problem needs to be studied further.

(iii) In the chiral limit with zero fermion masses, the complete expressions for the vertex functions Γ_V^μ and Γ_A^μ, are expressed in terms of the fermion's two-point functions (the fermion propagators) only. Applying these results to the Dyson-Schwinger equations will lead to that these equations form a closed system for the fermion propagators in QED and classical QCD. In QCD we usually consider the vertex function involving the current operator $j^\mu(x)=\bar{\psi}(x)\gamma^\mu(\lambda^a/2)\psi(x)$ or $j_5^\mu(x)=\bar{\psi}(x)\gamma^\mu\gamma_5(\lambda^a/2)\psi(x)$, where λ^a are flavor generators. In such case, the results of present work will be modified simply just by putting $\lambda^a/2$ into the suitable position in each term of the correspoding identical relations. For the case of effective QCD with Faddeev-Popov ghost fields, there seems to be more vertices. The transverse WT type relations for these new vertices and the full vertices need to be studied further.

In summary, we have derived the transverse Ward-Takahashi relations for the fermion's three-point vertex functions in coordinate space as well as in momentum space. These transverse WT relations together with the normal (longitudinal) Ward-Takahashi identities form a complete set of WT type constraint relations for three-point vertex functions. As a consequence, the complete expressions for the vector, the axial-vector, and the tensor vertex functions in the momentum space have been consistently and exactly deduced by solving this complete set of WT relations. In particular, in the case of massless fermions, the full vector, and the full axial-vector vertex functions are expressed in terms of fermion propagators only. Applying these expressions of the full vertex functions to the Dyson-Schwinger equations will lead to that these equations form a closed system for the fermion propagators. It shows that these full vertex functions will be very useful to the nonperturbative study of gauge field theories by using the Dyson-Schwinger equation approach and its application to hadronic physics.

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