General features of quantum chaos and its relevance to nuclear physics

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Some general features of the eigenenergies and the eigenwave functions of a chaotic quantum system are investigated by level dynamics in connection with nuclear physics. It is shown that the chaotic property of the eigensolutions is due to a large number of avoided level crossings. The critical strength of the perturbation for onset of quantum chaos is that the average value of the matrix elements of the perturbation equals the average level spacing of the unperturbed Hamiltonian (in the so-called strong mixing limit in nuclear physics). This kind of critical perturbation makes each level experience 4–5 avoided level crossings, and produces a spreading width of 16–32 on average. The extreme sensitivity of eigenenergies and eigenwave functions to a small change of the perturbation also originates from the avoided level crossings. The analytical expressions for a measured sensitivity are derived. More general expressions for the probability distribution (or strength function) and the spreading width of a perturbed state over a regular basis are obtained, which generalize the previous results: the strength function is still in the form of a Lorentzian function but with a spreading width consisting of the regular level contribution (the width being given by the so-called picket-fence model) and the avoided-level crossing contribution (width from level fluctuations). The relation between two chaotic bases is peculiar and the corresponding strength function shows a remarkable discontinuity, which is new and due to the statistical independence of different chaotic bases. The decay properties of nuclear ergodic collective states are discussed and explained in terms of the above results. Extensive results of computer simulations are presented to verify the level dynamical predictions.

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I. INTRODUCTION

As is well known, nuclear many-body systems show both regular and chaotic behaviors [1]. The low lying single particle states and collective states with well defined good quantum numbers are the regular aspect of nuclear motion, while the statistical properties of the level spectra of complex nuclei [2] and the chaotic dynamics in heavy-ion collisions [3] are the exhibition of nuclear chaotic behavior. Since nuclei are typical of quantum systems, one would consider the nuclear statistical behavior as a manifestation of quantum chaos [the commonly accepted definition of quantum chaos is that it is a quantum manifestation of classically chaotic systems whose quantum eigen states have the statistics of GOE (Gaussian orthogonal ensemble) [4]]. Because nuclei are conservative quantum systems, both their regular and chaotic behaviors should be understood in terms of deterministic dynamics. The dynamical theories for nuclear regular motions have been well established [5,6], while the dynamical theory for nuclear chaotic motion is still lacking. The most popular theoretical description of nuclear statistical properties is based on random matrix theory introduced by Wigner [7,8]. However, only after the link between random matrix theory and deterministic dynamics has been established can the random matrix theory as a nuclear statistical theory acquire more physical meaning.

To expose this link, it is found that the level dynamics proposed by Dyson, Pechukas, and Yukawa [9–11] is very useful. In a recent book [12], Haake has given a careful proof that random matrix theory is just an equilibrium statistical mechanics of level dynamical systems and that Dyson Brownian motion model is a rigorous consequence of level dynamics for autonomous systems. Recently, by virtue of level dynamics we found [13] that the local fluctuation in level distribution of a quantum system is generated by a large number of avoided level crossings, and that the role played by avoided level crossings in generating chaoticity in level dynamics is similar to the role played by short range collisions in causing thermalization in many-body dynamics. As will be seen soon, regular or chaotic level spectrum have their own manifestation in level dynamical equations. Quantum chaotic level spectrum is intimately related to a large number of avoided level crossings. Transition from quantum regular motion to chaotic motion occurs only after many avoided level crossings set in.

In this paper we shall study some general features of quantum chaotic systems by means of level dynamics and computer experiments. In Sec. II, the condition for onset of quantum chaos and the critical strength of perturbation are discussed and estimated. Section III studies the sensitivity of E_n and ψ_n to a small change of perturbation, and the expressions for measures of sensitivity are derived from level dynamics. In Sec. IV, some generic features of chaotic basis are investigated from both computer experiments and level dynamical analysis. Characteristics of the strength function of chaotic bases and decay property of nuclear ergodic collective states are discussed and explained in Sec. V. Section VI is a summary of this paper.

II. CONDITION FOR ONSET OF QUANTUM CHAOS AND ESTIMATE OF CRITICAL PERTURBATION

Since chaotic level distribution is generated by a large number of avoided level crossings, the condition for onset of quantum chaos is just that for full occurrence of avoided level crossings. The transition from quantum regular motion to chaotic motion can be illustrated by the following Hamiltonian [13,14]:

$$\hat{H}(\lambda)=\hat{H}_0+\lambda\hat{V}=\hat{H}_0+\lambda\hat{V}^{dia}+\lambda\hat{V}^{off}, \tag{1}$$

where $\hat{H}_0$ with good quantum numbers and dynamical symmetry describes regular motion with Poissonian energy spectrum, while $\hat{V}$ with GOE energy spectrum and destroying the dynamical symmetry mixes states with different quantum numbers and causes transition to chaos. The diagonal and off-diagonal parts of $\hat{V}$ are defined with respect to the $\hat{H}_0$ representation. The level dynamical equations are [13]

$$\frac{dE_n}{d\lambda}=V_{nn}, \tag{2}$$

$$\frac{dV_{nn}}{d\lambda}=2\sum_{m(\neq n)}\frac{V_{nm}V_{mn}}{E_n-E_m}, \tag{3}$$

$$\frac{dC_{nm}}{d\lambda}=\sum_{l(\neq n)}\frac{C_{lm}V_{ln}}{E_n-E_l}, \tag{4}$$

where $V_{nm}=\langle n(\lambda)|\hat{V}|m(\lambda)\rangle$ and $\hat{H}(\lambda)|n(\lambda)\rangle=E_n(\lambda)|n(\lambda)\rangle$. C_{nm} is the expansion coefficient of the eigenstate $|n(\lambda)\rangle$ defined by $|n(\lambda)\rangle=\Sigma_m C_{nm}|m(0)\rangle$, where $|m(0)\rangle$ is the eigenstate of $\hat{H}_0$. Let the average level spacing of $\hat{H}_0$ be D. Averaging the energy levels of $\hat{H}_0+\lambda\hat{V}^{dia}$, one has

$$\langle E_n\rangle=E_0+nD+\lambda\,\delta_n\overline{V}^{dia}, \tag{5}$$

where $\overline{V}^{dia}=|\langle\hat{V}^{dia}\rangle|$. If the sign $\delta_n=\pm$ distributes randomly, the condition for the occurrence of level crossings is

$$\langle E_{n+1}\rangle-\langle E_n\rangle=D-2\lambda\overline{V}^{dia}=0,\quad \lambda=\frac{D}{2\overline{V}^{dia}}. \tag{6}$$

Here $\overline{V}^{dia}$ playing a role of initial ''velocity,'' is very crucial to result in level crossings. As $\hat{V}^{off}$ turns on, the level crossings become avoided. Meanwhile, the interaction will generate an induced ''velocity'' which can be estimated from Eqs. (2) and (3) as follows:

$$\begin{aligned}\langle\Delta E_n\rangle&=2\int_0^\lambda dt_1\int_0^{t_1}dt_2\left[\frac{(\overline{V}^{off})^2}{\langle E_n\rangle-\langle E_{n-1}\rangle}+\frac{(\overline{V}^{off})^2}{\langle E_n\rangle-\langle E_{n+1}\rangle}\right]_{t_2}\\&=2\int_0^\lambda dt_1\int_0^{t_1}dt_2(\overline{V}^{off})^2\left[\frac{1}{D+\overline{V}^{dia}2t_2}-\frac{1}{D-\overline{V}^{dia}2t_2}\right]\\&\approx\frac{4\lambda^3}{3}\frac{(\overline{V}^{off})^2\overline{V}^{dia}}{D^2},\end{aligned} \tag{7}$$

where $\overline{V}^{off}=|\langle\hat{V}^{off}\rangle|$. From Eqs. (5) and (7), we have

TABLE I. Some numerical results of quantum chaos.

Ref.	D	$\lambda\ \overline{V}$	$\frac{\lambda\overline{V}}{D}$	Γ
[13]	1	1	1	25
[14]	1	1.15	1.15	20
[15]	2 keV	2 keV	1	40
[16]	10 keV	15 keV	1.5	30

$$\langle E_n\rangle=E_0+nD+\delta_n\left[\lambda\overline{V}^{dia}+\frac{4\lambda^3}{3}\frac{(\overline{V}^{off})^2\overline{V}^{dia}}{D^2}\right]. \tag{8}$$

If the level n and the level $n+\Delta n$ cross each other, from Eq. (8), one has

$$\Delta n=2\left[\lambda\frac{\overline{V}^{dia}}{D}+\frac{4\lambda^3}{3}\frac{(\overline{V}^{off})^2\overline{V}^{dia}}{D^3}\right]. \tag{9}$$

Assume $\overline{V}^{dia}=\overline{V}^{off}=\overline{V}$. If $\lambda=D/2\overline{V}$, each level has experienced $\Delta n=2(\frac{1}{6}+\frac{1}{2})=1.3$ avoided crossings. This is just the beginning of avoided level crossings and one cannot expect a full chaotic level spectum. As $\lambda=D/\overline{V}$, each level on average has $\Delta n=2(1+\frac{4}{3})=4.7$ avoided level crossings. Since each avoided level crossing will mix at least two levels, $\Delta n=4.7$ avoided level crossings will produce a level mixing width of $\Gamma=2^{4.7}=25.4$. In this case, one would expect a full chaotic distribution of the levels. The computer experiments confirm the above analysis, as shown in Table I.

In Ref. [17], Weidenmüller and Guhr have noticed from nuclear statistical theory that as $\lambda\overline{V}=D$, a quatum system will have GOE level statistics. From level dynamics, one can get more information on and new insight into this problem: the critical strength of the dynamical symmetry breaking perturbation for onset of quantum chaos is $\lambda\overline{V}=D$, which corresponds to a full occurrence of avoided level crossings. In this case, each level on average has experienced 4–5 avoided level crossings and the mixing width of the levels is about 25–32. It is the enormous avoided level crossings that result in the local fluctuations of the levels and make the level distribution chaotic.

III. SENSITIVITY OF E_n AND ψ_n TO A SMALL CHANGE OF PERTURBATION

Consider a small change of the perturbation,

$$\delta(\lambda\hat{V})=\delta\lambda\hat{V}+\lambda\,\delta\hat{V}, \tag{10}$$

and look at the response of eigenenergies E_n and eigenwave functions ψ_n. From the integral solutions of the level dynamical Eqs. (2) and (4), one has

$$\delta E_n(\lambda)=2\sum_{m\neq n}\int_0^\lambda dt_1\int_0^{t_1}dt_2\left[\frac{\delta V_{nm}V_{mn}+V_{nm}\delta V_{mn}}{E_n-E_m}-(\delta E_n-\delta E_m)\frac{V_{nm}V_{mn}}{(E_n-E_m)^2}\right]_{t_2}+\delta\lambda V_{nn}(0)+2\,\delta\lambda\sum_{m\neq n}\int_0^\lambda\left[\frac{V_{nm}V_{mn}}{E_n-E_m}\right]_t dt, \tag{11}$$

$$\delta C_{nm}(\lambda)=\sum_{l\neq n}\int_0^\lambda\left[\frac{V_{ln}\delta C_{lm}+\delta V_{ln}C_{lm}}{E_n-E_l}-\frac{V_{ln}C_{lm}}{(E_n-E_l)^2}(\delta E_n-\delta E_l)\right]_t dt+\delta\lambda\sum_{l\neq n}\left[\frac{C_{lm}V_{ln}}{E_n-E_l}\right]_\lambda, \tag{12}$$

where

$$\delta V_{nm}(t)=\sum_{ll'}\left[(\delta C_{nl}C_{ml'}+C_{nl}\delta C_{ml'})V_{ll'}(0)+C_{nl}C_{ml'}\langle l(0)|\delta\hat{V}|l'(0)\rangle\right]_t. \tag{13}$$

Equations (11)–(13) are iterative integral equations for $\delta E_n(\lambda)$ and $\delta C_{nm}(\lambda)$. If there is no avoided level crossing, the integrands of Eqs. (11) and (12) are well-behaved functions and the mapping from $\delta(\lambda\hat{V})$ to $\delta E_n(\lambda)$ and $\delta C_{nm}(\lambda)$ is smooth. Consequently, E_n and C_{nm} are not sensitive to a small change of $(\lambda\hat{V})$, since the summation will result in cancellation and smooth out any large variation. On the contrary, when λ reaches its critical value, there occur many avoided level crossings which make the energy denominators almost zero and consequently make the Eqs. (11) and (12) quasisingular at the strong avoided level crossing points. In this case, a small variation of perturbation could be amplified to a large scale and generate large local fluctuations in eigenenergies and eigenwave functions. Thus the summation procedure cannot smear out the large local fluctuations since other terms are relatively too small. As a result, δE_n and δC_{nm} are very sensitive to a small change of perturbation and will fluctuate violently. Thus the sensitivity of a quantum chaotic system to a small change of perturbation stems from the avoided level crossings, and the resulted quasisingularity in Eqs. (11) and (12) provides a mechanism of nonlinear amplification of the small variation of perturbation.

To study this problem quantatively, we need expressions for the measures of sensitivity of energies and wave functions. From Eqs. (2) and (4), we can derive the function $S[E,\lambda]$,

$$S[E,\lambda]=\frac{1}{N}\sum_{n=1}^{N}\left(\frac{dE_n}{d\lambda}\right)^2\approx\frac{4}{N}\sum_{m\neq n}\frac{V_{nm}^4}{(E_n-E_m)^2}, \tag{14}$$

and the variance of energies

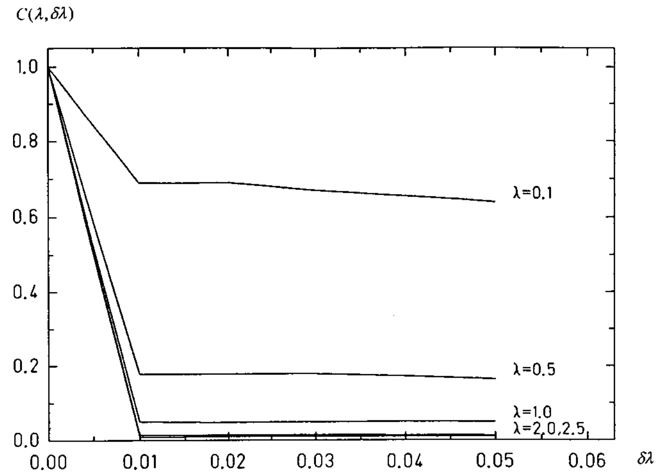

FIG. 1. Sensitivity of the correlation of wave functions $C(\lambda,\delta\lambda)$ as a function of $\delta\lambda$.

$$\delta^2\sigma(E,\lambda,\delta\lambda)=\frac{1}{N}\sum_{n=1}^{N}[E_n(\lambda+\delta\lambda)-E_n(\lambda)]^2=S[E,\lambda](\delta\lambda)^2, \tag{15}$$

as the measures of sensitivity of eigenenergies, and define the function $S[\psi,\lambda]$

$$S[\psi,\lambda]=\left\langle\frac{d\psi_c(\lambda)}{d\lambda}\middle|\frac{d\psi_c(\lambda)}{d\lambda}\right\rangle\approx\sum_{n\neq c}\frac{V_{cn}^2}{(E_n-E_c)^2}, \tag{16}$$

and the correlation of wave functions

$$C(\lambda,\delta\lambda)=|\langle\psi_c(\lambda)|\psi_c(\lambda+\delta\lambda)\rangle|^2=1-S[\psi,\lambda](\delta\lambda)^2, \tag{17}$$

to be the measures of sensitivity of eigenwave functions. Here ψ_c is a chosen eigenstate (we usually chose ψ_c at the middle of eigenspectrum to minimize the boundary effect due to finite dimension of the Hilbert space). In our numerical calculation, an ensemble average has been taken for the sensitivity measures $S[E,\lambda]$ and $S[\psi,\lambda]$. Equations (14) and (16) clearly show that the avoided level crossings play a crucial role in the sensitivity measures of the eigenenergies and eigenwave functions: in the regular region of λ, no avoided level crossing has developed, no quasisingularity appears in Eqs. (14) and (16), and both $S[E,\lambda]$ and $S[\psi,\lambda]$ are thus small. As λ increases, more and more avoided level crossings have developed, and $S[E,\lambda]$ and $S[\psi,\lambda]$ increase rapidly due to the large contribution from the quasisingularity of the inverse of the nearly zero energy denominators at the avoided level crossing points. In the chaotic region of λ, the avoided level crossings have fully developed, and both $S[E,\lambda]$ and $S[\psi,\lambda]$ will reach their maxima. Figures 1 and 2 plot the correlation of wave functions and the variance of energies as functions of $\delta\lambda$ at different values of λ, which indicate that as $\lambda\leqslant0.5$, eigenwave functions and eigenenergies are not sensitive to the small change of perturbation, while as λ reaches its critical value $\lambda_c(=2)$, both quantities are sensitive to $\delta\lambda$. Figures 3 and 4 plot $S[\psi,\lambda]$ and $S[E,\lambda]$

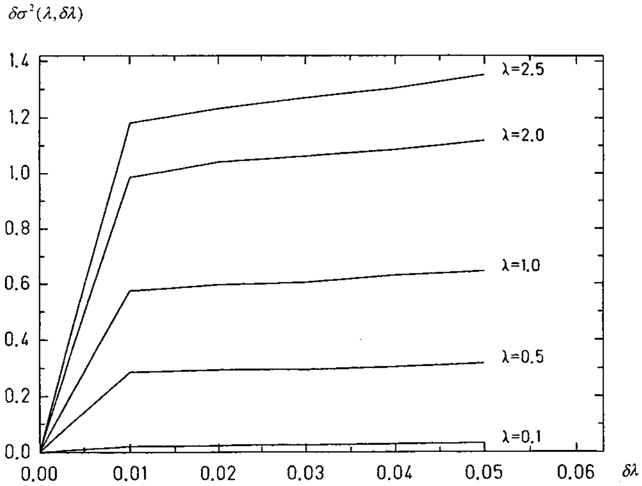

FIG. 2. Sensitivity of the variance of eigenenergies $\sigma^2(\lambda,\delta\lambda)$ as a function of $\delta\lambda$.

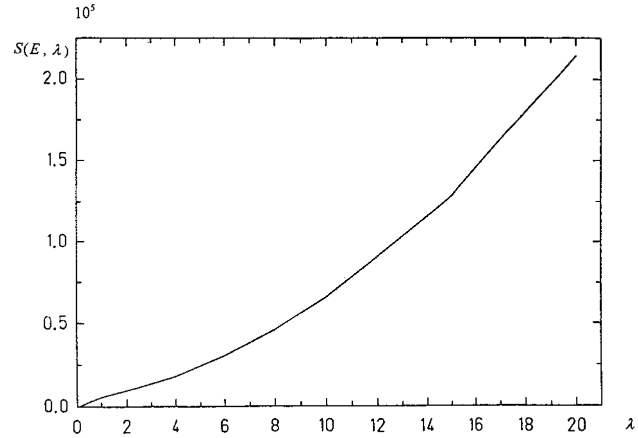

FIG. 4. Sensitivity measure of eigenenergies $S(E,\lambda)$ as a function of λ.

as functions of λ, which show a rapid rise of the curves as λ increases from the critical value.

Now consider the time dependent case that $\lambda=\lambda(t)$, is a function of time. The time-dependent Schrödinger equation,

$$i\frac{\partial\psi(t)}{\partial t}=\hat{H}(t)\psi(t), \tag{18}$$

can be rewritten in the adiabatic eigenrepresentation of $\hat{H}(t)$,

$$\hat{H}(t)|n(\lambda(t))\rangle=E_n(\lambda(t))|n(\lambda(t))\rangle, \tag{19}$$

$$\psi(t)=\sum_n D_{nn_0}(t)e^{-i\phi_n(t)}|n(\lambda(t))\rangle, \tag{20}$$

$$\phi_n(t)=\int_0^t\left\langle n(\lambda(\tau))\middle|\hat{H}(\lambda(\tau))-i\frac{\partial}{\partial\tau}\middle|n(\lambda(\tau))\right\rangle d\tau. \tag{21}$$

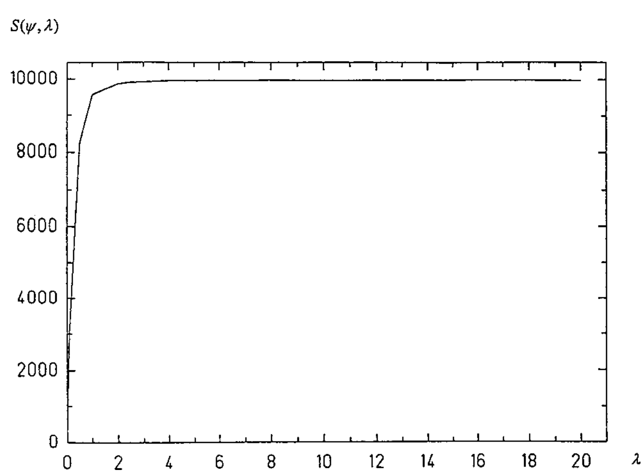

FIG. 3. Sensitivity measure of eigenfunctions $S(\psi,\lambda)$ as a function of λ.

From Eq. (18), one obtains the equation of motion for the expansion coefficient D_{nn_0} of the time-dependent wave function,

$$\frac{dD_{nn_0}}{dt}=\dot{\lambda}\sum_{m\neq n}\frac{V_{nm}}{E_n(\lambda(t))-E_m(\lambda(t))}\times\exp[i(\phi_n(t)-\phi_m(t))]D_{mn_0}. \tag{22}$$

From the similar structure of Eqs. (22) and (4), one would claim that the avoided level crossings will play a similar role in generating local fluctuations and chaoticity in the time dependent case. It is likely that the time-dependent behavior of a quantum system is determined by the level spectrum structure of the adiabatic eigenstates of the Hamiltonian, and the chaotic spectrum of the adiabatic levels will induce a chaotic time-dependent behavior. This observation has been confirmed in our recent work [18].

IV. GENERIC FEATURES OF CHAOTIC BASIS

The level dynamical Eqs. (2)–(4) are very useful for exploring the relation between two different adiabatic eigenbases of a quantum system with the $\hat{H}(\lambda)$ of Eq. (1) at λ_1 and λ_2. Consider two cases: (i) $\hat{H}(\lambda_1)$ describes a regular motion, while $\hat{H}(\lambda_2)$ can be regular or chaotic. (ii) Both $\hat{H}(\lambda_1)$ and $\hat{H}(\lambda_2)$ are chaotic.

Case (i). For simplicity, assume $\lambda_1=0$ and $\lambda_2=\lambda$. We proceed to establish the relation between the two bases. To this end, Eqs. (2)–(4) should be solved in a different way. Let

$$\hat{H}(\lambda)|n(\lambda)\rangle=E_n(\lambda)|n(\lambda)\rangle, \tag{23}$$

$$\hat{H}_0=|n(0)\rangle=E_0(0)|n(0)\rangle, \tag{24}$$

and

$$|n(\lambda)\rangle=N_n(\lambda)\sum_m A_{nm}(\lambda)|m(0)\rangle, \tag{25}$$

where the normalization factor is

$$\frac{1}{N_n^2}=\sum_m |A_{nm}|^2. \tag{26}$$

From Eqs. (1),(23–25), one obtains

$$(E_n(\lambda)-E_m(0))\langle m(0)|n(\lambda)\rangle=\langle m(0)|\lambda\hat{V}|n(\lambda)\rangle, \tag{27}$$

$$A_{nm}=\langle m(0)|n(\lambda)\rangle/N_n=\frac{\langle m(0)|\lambda\hat{V}|n(\lambda)\rangle}{N_n(E_n(\lambda)-E_m(0))}. \tag{28}$$

Let $A_{nn}=1$, namely, $\langle m(0)|\lambda\hat{V}|n(\lambda)\rangle=(E_n(\lambda)-E_m(0))N_n$ or $\langle n(0)|n(\lambda)\rangle=N_n$, we obtain

$$|A_{nm}|^2=\frac{\dfrac{|\langle m(0)|\lambda\hat{V}|n(\lambda)\rangle|^2}{(E_n(\lambda)-E_m(0))^2}}{1+\dfrac{1}{N_n^2}\displaystyle\sum_{l\neq n}\frac{|\langle l(0)|\lambda\hat{V}|n(\lambda)\rangle|^2}{(E_n(\lambda)-E_m(0))^2}}=\frac{D}{2\pi}\frac{\Gamma_{nm}^r}{(E_n(\lambda)-E_m(0))^2+(\Gamma_{nm}^f)^2}, \tag{29}$$

where the regular part of the width, Γ_{nm}^r, and the fluctuation part of the width, Γ_{nm}^f, are defined, respectively, as follows:

$$\Gamma_{nm}^r=2\pi|\langle m(0)|\lambda\hat{V}|n(\lambda)\rangle|^2/D, \tag{30}$$

$$(\Gamma_{nm}^f)^2=\lambda^2\sum_{l\neq n}|\langle l(0)|\hat{V}|n(\lambda)\rangle|^2\frac{(E_n(\lambda)-E_m(0))^2}{N_n^2(E_n(\lambda)-E_l(0))^2}. \tag{31}$$

Since $|A_{nm}(\lambda,E_m(0))|$ is a smooth function of $E_m(0)$ which has a regular spectrum without large fluctuations, we can take an average of $|A_{nm}|^2$ over the energy $E_m(0)$,

$$\begin{aligned}\langle|A_{nm}|^2\rangle_{E_m(0)}&=\frac{1}{2}\left(\frac{D}{\pi}\right)^2\frac{\Gamma_{nm}^r}{\Gamma_{nm}^f}\int d\left(\frac{E_m(0)}{D}\right)\\&\times\frac{\Gamma_{nm}^f}{(E_n(\lambda)-E_m(0))^2+(\Gamma_{nm}^f)^2}\\&\times\frac{\Gamma_{nm}^r}{(E_m(0)-E)^2+(\Gamma_{nm}^r)^2}\\&=\frac{\Gamma_{nm}^r}{2\pi\Gamma_{nm}^f}\frac{D}{\pi}\cdot\frac{(\Gamma_{nm}^r+\Gamma_{nm}^f)}{(E_n(\lambda)-E)^2+(\Gamma_{nm}^r+\Gamma_{nm}^f)^2}.\end{aligned} \tag{32}$$

Taking the ensemble average and considering the normalization, we obtain

$$P_{nm}=\langle|A_{nm}(\lambda,E)|^2\rangle=\frac{D}{2\pi}\frac{\Gamma(\lambda)}{(E_n(\lambda)-E)^2+\Gamma(\lambda)^2}, \tag{33}$$

where

$$\Gamma(\lambda)=\langle\Gamma_{nm}^r+\Gamma_{nm}^f\rangle=\Gamma^r+\Gamma^f\approx 2\pi\lambda^2\frac{\overline{V^2}}{D}+k\lambda\overline{V}\sqrt{\langle S[\psi,\lambda]\rangle}. \tag{34}$$

Here

$$\Gamma^r=\lambda^2\frac{2\pi}{D}\langle|\langle m(0)|\hat{V}|n(\lambda)\rangle|^2\rangle\approx 2\pi\lambda^2\frac{\overline{V^2}}{D}, \tag{35}$$

and

$$\begin{aligned}(\Gamma^f)^2&\approx\lambda^2\left\langle\frac{|\langle m(0)|\hat{V}|n(\lambda)\rangle|^2}{|\langle n(0)|n(\lambda)\rangle|^2}\right\rangle\left\langle\sum_{l\neq n}\frac{|\langle l(0)|\hat{V}|n(\lambda)\rangle|^2}{(E_n(\lambda)-E_l(0))^2}\right\rangle\\&\approx\lambda^2\overline{V^2}k^2\langle S[\psi,\lambda]\rangle\\&=\frac{\Gamma^r D}{2\pi}k^2\langle S[\psi,\lambda]\rangle,\end{aligned} \tag{36}$$

$$\left\langle\frac{|\langle m(0)|\hat{V}|n(\lambda)\rangle|^2}{|\langle n(0)|n(\lambda)\rangle|^2}\right\rangle\approx k^2\overline{V^2}\approx\frac{\Gamma^r D}{2\pi\lambda^2}k^2. \tag{37}$$

Equations (33) and (34) are general expressions for the probability distribution (or strength function) and the spreading width of a perturbed state over the regular basis, which are derived from level dynamics under quite general assumption: no quasisingularity appears in Eq. (28). It was derived before under the picket-fence model approximation [19] where the spreading width consists of only Γ^r. The second term in Eq. (34), Γ^f, coming from level fluctuations respect to the picket-fence (equal-distance) level distribution; is therefore neglected in the picket-fence model approximation. It is worth noting that the fluctuation width Γ^f is related to the sensitivity measure of wave functions, $\langle S[\psi,\lambda]\rangle$, which becomes larger and larger as the system undertakes the transition to chaos. Of course, in the regular region, both $\langle S[E,\lambda]\rangle$ and $\langle S[\psi,\lambda]\rangle$ are small, Γ^f is thus small. However, as the transition to chaos occurs, the fluctuation width must be taken into account carefully, since it becomes more and more important as more and more avoided level crossings set in. It is interesting to note that Γ^r is a function of λ^2, while Γ^f becomes linear in λ as $\langle S[\psi,\lambda]\rangle$ becomes constant. Figure 5 plots the computer result of Γ as a function of λ. It is seen that as $\lambda\leqslant 2$ (in the regular region of our model) $\Gamma(\lambda)$ follows a parabolic curve very well, while in the region of $\lambda=3$–5 (this is chaotic region in our model), $\Gamma(\lambda)$ is nearly a straight line with a slope of 37.5. From Fig. 3, we noticed that in this region $\langle S[\psi,\lambda]\rangle$ is almost constant and $\overline{V}\sqrt{\langle S[\psi,\lambda]\rangle}\approx 40$. Thus the computer experiment confirms the expressions (33) and (34).

The expression (36) for the fluctuation width Γ^f can be also obtained from a statistical treatment of the level dynami-

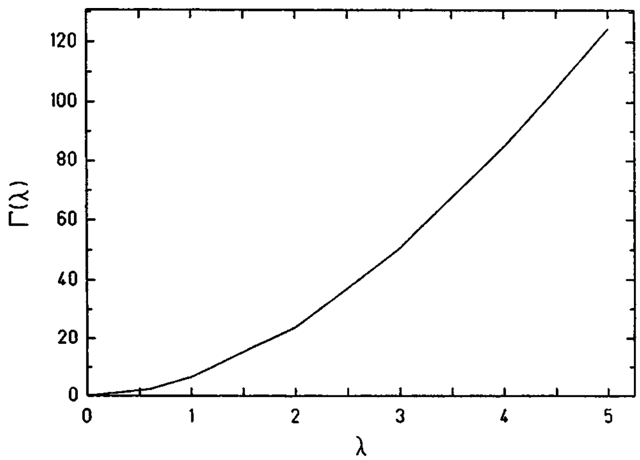

FIG. 5. Spreading width $\Gamma(\lambda)$ as a function of λ.

cal equation (4) as the system is in the chaotic region. In the following, our treatment is similar to that used in Ref. [20]. From Eq. (4), the difference of C_{nm} is

$$\Delta C_{nm}=\sum_{l\neq n}\int_t^{t+\Delta t}\left(\frac{V_{ln}C_{lm}}{E_n-E_l}\right)_{t'}dt', \tag{38}$$

and the difference of probability is

$$\begin{aligned}\Delta P_{nm}(t)&=\langle (C_{nm}(t)+\Delta C_{nm})^2-C_{nm}(t)^2\rangle\\&=\langle (\Delta C_{nm})^2+2\Delta C_{nm}C_{nm}\rangle.\end{aligned} \tag{39}$$

In the chaotic region, we assume $\langle C_{nm}\Delta C_{nm}\rangle=\langle C_{nm}\rangle\langle \Delta C_{nm}\rangle=0$. Hence

$$\begin{aligned}\Delta P_{nm}(t)&=\langle (\Delta C_{nm})^2\rangle\\&=\sum_{l,l'\neq n}\int_t^{t+\Delta t}dt_1dt_2\left\langle\left(\frac{V_{lm}C_{ln}}{E_n-E_l}\right)_{t_1}\left(\frac{V_{l'm}C_{l'n}}{E_n-E_{l'}}\right)_{t_2}\right\rangle\\&=\sum_{l\neq n}\int_t^{t+\Delta t}dt_1dt_2\left\langle\frac{V_{ln}^2}{(E_n-E_l)^2}\right\rangle_{t_1}\langle C_{lm}C_{lm}\rangle_{t_2}\\&\quad\times f(t_1-t_2)\\&=\sum_{l\neq n}R_{nl}(t)P_{lm}(t)\Delta t,\end{aligned} \tag{40}$$

where we have assumed

$$\langle V_{ln}(t_1)V_{l'n}(t_2)\rangle=\delta_{ll'}\langle (V_{ln}(t_1))^2\rangle f(t_1-t_2), \tag{41}$$

with $f(t_1-t_2)$ to be a $\delta(t_1-t_2)$-like function and $\int f(t_1-t_2)dt_2=\Gamma/\hbar=\tau$. Thus

$$R_{nl}(t)=\int_t^{t+\Delta t}\left\langle\frac{V_{ln}^2}{(E_n-E_l)^2}\right\rangle_t f(t-t')dt'=\tau\left\langle\frac{V_{ln}}{(E_n-E_l)^2}\right\rangle_t. \tag{42}$$

In the limit of $\Delta t\rightarrow dt$, Eq. (40) becomes

$$\frac{dP_{nm}}{dt}=\sum_{l\neq n}R_{nl}P_{lm}. \tag{43}$$

Reciprocity [20] implies $R_{nl}=R_{ln}$ and probability conservation, which leads to

$$\frac{dP_{nm}}{dt}=\sum_{l\neq n}R_{nl}(P_{lm}-P_{nm}). \tag{44}$$

For a very large Hilbert space, P_{nm} and R_{nl} are approximately translationally invariant so that $P_{nm}=P(n-m)$, $R_{nl}=R(n-l)=R(l-n)=R_{ln}$, and

$$\begin{aligned}P_{lm}&=P(l-m)=P((n-m)-(n-l))\\&=P(n-m)-\frac{\partial P(n-m)}{\partial(n-m)}(n-l)\\&\quad+\frac{1}{2}\frac{\partial^2P(n-m)}{\partial(n-m)^2}(n-l)^2+\cdots.\end{aligned} \tag{45}$$

From Eqs. (44) and (45), one obtains

$$\begin{aligned}\frac{dP(n-m)}{dt}&=\int R(n-l)[P(l-m)-P(n-m)]d(n-l)\\&=D_\lambda\frac{\partial^2P(n-m)}{\partial(n-m)^2},\end{aligned} \tag{46}$$

where the diffusion coefficient is defined as

$$D_\lambda=\frac{1}{2}\int R(n-l)(n-l)^2d(n-l) \tag{47}$$

$$=\frac{1}{2}\sum_{l\neq n}\left\langle\frac{V_{ln}^2}{(E_n-E_l)^2}\right\rangle(n-l)^2 \tag{48}$$

$$\approx\frac{1}{2}\langle (n-l)^2\rangle_{midvalue}\sum_{l\neq n}\left\langle\frac{V_{ln}^2}{(E_n-E_l)^2}\right\rangle \tag{49}$$

$$\approx\frac{1}{2}\langle (n-l)^2\rangle_{midvalue}\langle S[\psi,\lambda]\rangle. \tag{50}$$

In Eq. (49), the midvalue theorem of definite integral has been used and Eq. (50) follows from Eq. (16). The solution of Eq. (46) with the initial condition $P(n-m,t=0)=\delta(n-m)$ is

$$P((n-m),\lambda)=\frac{\sqrt{\pi}}{\Gamma_\lambda}\exp\left[-\left(\frac{n-m}{\Gamma_\lambda}\right)^2\right], \tag{51}$$

where the spreading width is $\Gamma_\lambda=\sqrt{D_\lambda\cdot\lambda}$. From the solution (51), we have $\langle (n-l)^2\rangle_{midvalue}=\sigma_n^2\lambda$ and $\Gamma_\lambda=(\sigma_n/\sqrt{2})\lambda\sqrt{\langle S[\psi,\lambda]\rangle}$ which is in agreement with Γ^f/D in Eq. (34). However, the computer experiment indicates that even though $\hat{H}(\lambda)$ is in chaotic region, only the central part

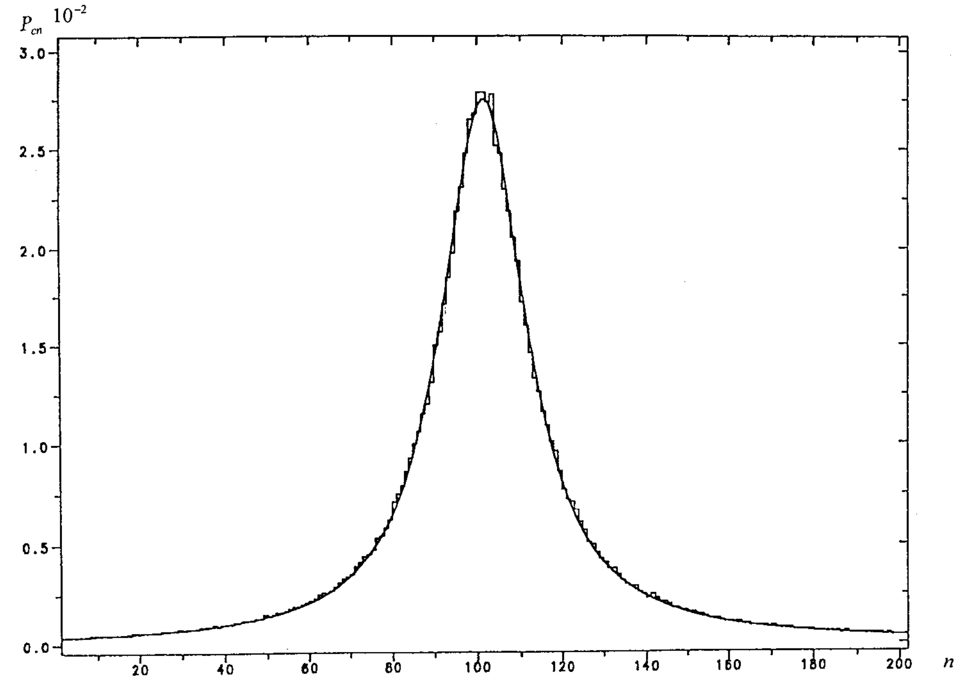

FIG. 6. Probability distribution P_{cn} of a chaotic state over a regular basis with $\lambda_1=0.1$, $\lambda_2=2.0$, and $c=101$. The Hilbert subspace is of 201 dimensions. The data (histogram) are fitted with a Lorentzian function (solid curve).

of $P(n-m)$ ($|n-m|<\Gamma_\lambda$) follows the Gaussian distribution, while the tail part of that does not. The reason is that the central part of the distribution is generated chaotically by a large number of avoided level crossings and the tail part of the distribution is contributed from the regular levels without avoided level crossings. Therefore, the statistical treatment does not apply to the tail part of $P(n-m)$. Hence the overall probability distribution should follow the Lorentzian distribution, Eq. (33), predicted from level dynamics. Figures 6 and 7 confirm the above analysis.

Case (ii). Since both $E_n(\lambda_1)$ and $E_m(\lambda_2)$ are of chaotic spectrum with violent fluctuations, the procedure of energy average used in case (i) is not appropriate. Thus the method employed in case (i) is not suitable to this case. To treat the present problem properly, we should adapt the solution of Eq. (4) to its special initial condition as follows:

$$C_{cn}(\lambda_1,\lambda_2)=C_{cn}(\lambda_1,\lambda_1)+\sum_{l\neq c}\int_{\lambda_1}^{\lambda_2}dt\left(\frac{C_{ln}V_{lc}}{E_c-E_l}\right)_t, \tag{52}$$

with the initial condition $C_{cn}(\lambda_1,\lambda_1)=\delta_{cn}$. Here $|c(\lambda_1)\rangle$ is

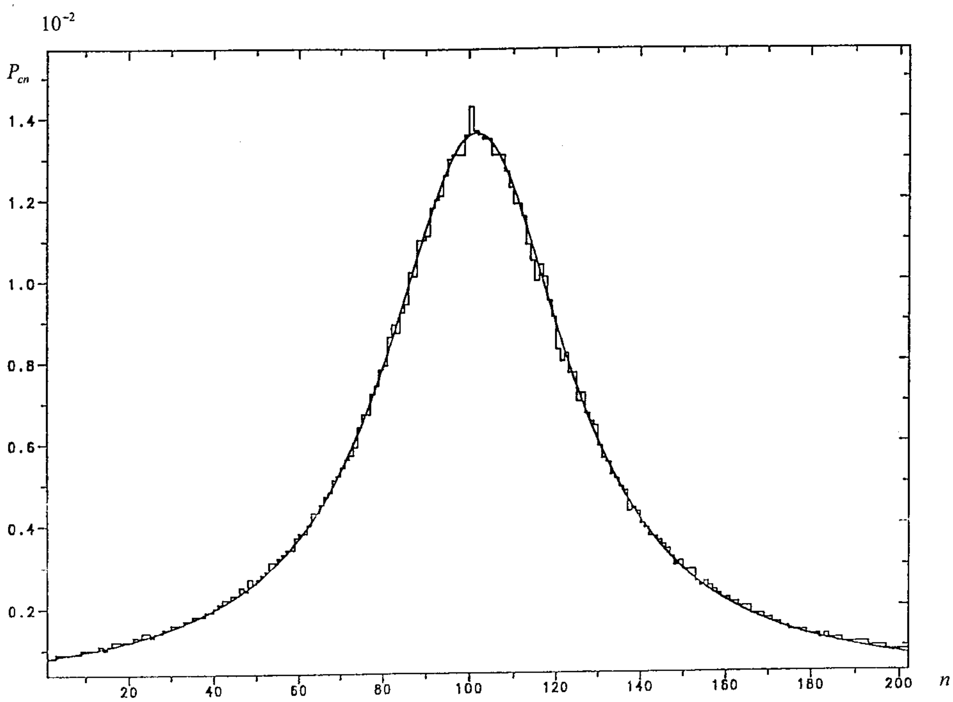

FIG. 7. Probability distribution P_{cn} of a chaotic state over a regular basis with $\lambda_1=0.1$, $\lambda_2=3.0$, and $c=101$. The Hilbert subspace is of 201 dimensions. The data (histogram) are fitted with a Lorentzian function (solid curve).

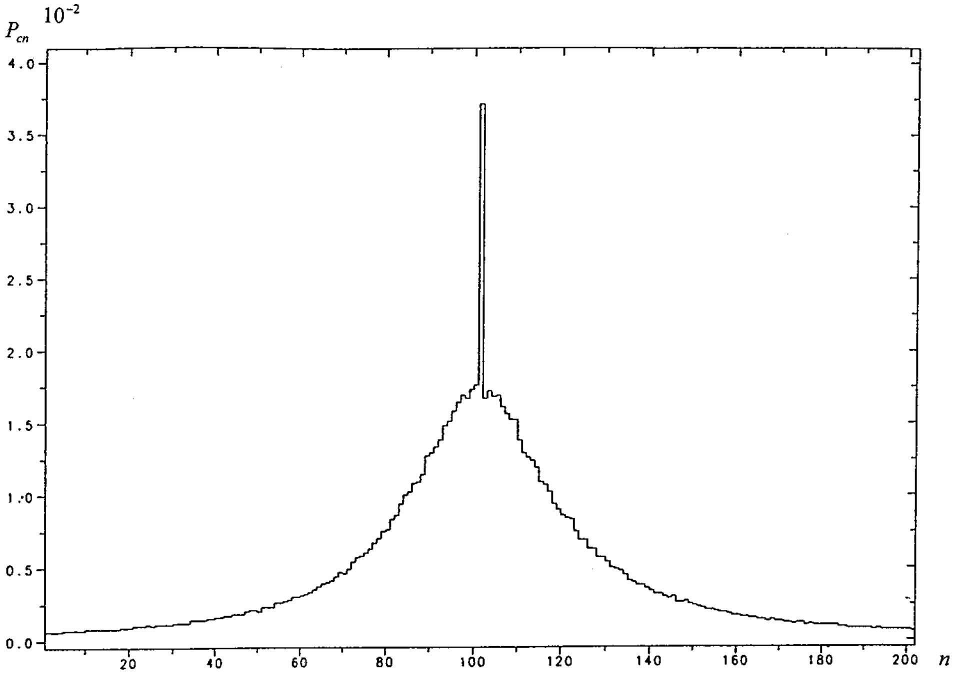

FIG. 8. Probability distribution P_{cn} of a chaotic state over another chaotic basis with $\lambda_1=1.5$, $\lambda_2=2.0$, and $c=101$. The Hilbert subspace is of 201 dimensions. The data (histogram) are fitted with a Lorentzian function (solid curve).

an eigenstate of $\hat{H}(\lambda_1)$ properly chosen [we take $|c(\lambda_1)\rangle$ to be a state at the middle of the whole spectrum studied], which is expanded in term of the eigenbasis of $\hat{H}(\lambda_2)$ with the expansion coefficient $C_{cn}(\lambda_1,\lambda_2)$. Since both $\hat{H}(\lambda_1)$ and $\hat{H}(\lambda_2)$ are in chaotic region, we can apply statistical assumption to both V_{ln} and C_{ln}. The probability distribution of the state $|c(\lambda_1)\rangle$ over the basis $|n(\lambda_2)\rangle$ is

$$\begin{aligned}P_{cn}(\lambda_1,\lambda_2)&=\langle|C_{cn}(\lambda_1,\lambda_2)|^2\rangle\\&=\delta_{cn}+2\delta_{cn}\sum_{l\neq c}\int_{\lambda_1}^{\lambda_2}\left\langle\left(\frac{C_{ln}V_{lc}}{E_c-E_l}\right)_t\right\rangle dt\\&\quad+\sum_{l_1,l_2\neq c}\int_{\lambda_1}^{\lambda_2}dt_1dt_2\\&\quad\times\left\langle\left(\frac{C_{l_1n}V_{l_1c}}{E_c-E_{l_1}}\right)_{t_1}\left(\frac{C_{l_2n}V_{l_2c}}{E_c-E_{l_2}}\right)_{t_2}\right\rangle.\end{aligned}\quad(53)$$

Iterating $C_{lc}(t)$ in P_{cn} to second order in V_{lc}, we obtain

$$P_{cc}=1-\tau\sum_{l\neq c}\int_{\lambda_1}^{\lambda_2}dt\left\langle\frac{V_{lc}^2}{(E_c-E_l)^2}\right\rangle_t=1-\tau\int_{\lambda_1}^{\lambda_2}\langle S[\psi,t]\rangle dt,\quad(54)$$

and

$$P_{c,n\neq c}=\tau\int_{\lambda_1}^{\lambda_2}\left\langle\frac{V_{nc}^2}{(E_c-E_n)^2}\right\rangle_t dt.\quad(55)$$

In the above derivation we have assumed that $\langle V_{lc}(t)V_{l'c}(t')\rangle=\delta_{ll'}\langle V_{lc}^2(t)\rangle f(t-t')$ as in case (i). A peculiar feature of the probability distribution is its two expressions: Eq. (54) for P_{cc} and Eq. (55) for the others, which describe the discontinuity of the distribution. Computer experiments indeed show this kind of discontinuity. This fact leads us to the following observation of the chaotic basis: different chaotic bases are statistically independent. The observation can be made more clearly as follows: if a chaotic state is expanded in terms of regular basis, its expansion coefficients are statistically independent; while as the chaotic bases $|n(\lambda_1)\rangle$ are expanded in terms of the other chaotic bases $|m(\lambda_2)\rangle$, because of the statistical independence, only the coherent component $m=n$ has a constructive contribution, the other components yield a weak destructive background. As a result, the probability distribution exhibits discontinuity: a sharp peak from the constructive contribution and a weak statistical background from the destructive contribution. In Figs. 8 and 9 we plot the probability distribution of one chaotic state over another chaotic basis. Just as predicted from level dynamics, the curves consist of two part: a strong δ-function like peak and a weak statistical background.

V. DECAY OF NUCLEAR ERGODIC COLLECTIVE STATES

For hot rotating nuclei, residual interactions become more effective and will mix different rotational bands. As the excitation energy above the yrast line is large enough, the mixed collective states become very complicated and can be chaotic. Bohr and Mottelson proposed the concept of ''ergodic collective states'' to describe such kind of phenomena [21]. Since then, the decay of such ergodic collective states (especially high spin states of hot nuclei) has been studied extensively by nuclear theorists [16,22] and experimentalists [23]. In contrast to the transition between two regular collective states where the strength function is sharply peaked, the strength function for the transition between ergodic high spin states are spreaded over a wide range. Experimentally, in the

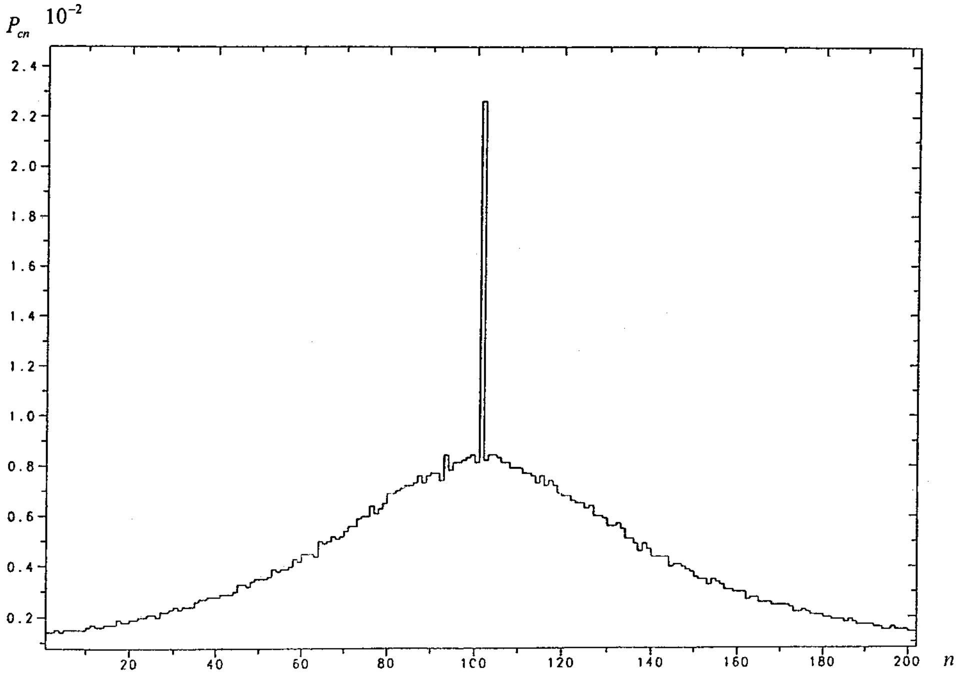

FIG. 9. Probability distribution P_{cn} of a chaotic state over another chaotic basis with $\lambda_1=2.5$, $\lambda_2=3.0$, and $c=101$. The Hilbert subspace is of 201 dimensions. The data (histogram) are fitted with a Lorentzian function (solid curve).

γ-γ coincidence measurement [22,23], one found a weak peak embedding in the statistical background, which depicts the structure of the strength function. According to Ref. [22], the strength function for the transition between ergodic collective states is proportional to P_{cn}. Let us assume that in Eq. (1), $\hat{H}_0$ describes all bands of regular collective states, while the residual interaction $\hat{V}$ mixes collective bands and produces ergodic collective states. In this case, our model provides a simplified understanding of the phenomena: Eqs. (33) and (34) and Figs. 6 and 7 provide a description of the transition from an ergodic collective state to regular collective states; while Eqs. (54) and (55) and Figs. 8 and 9 describe the transition between ergodic collective states.

VI. CONCLUSION AND DISCUSSION

The results obtained in this paper can be summarized as follows.

(1) Level dynamics is very useful for studying the transition from quantum regular motion to chaotic motion.

(2) The condition for onset of quantum chaos is that the strength of the dynamical symmetry breaking perturbation appraoches its critical value—the average energy level spacing of the regular unperturbed Hamiltonian (i.e., in the the strong mixing limit in nuclear physics), as a result, each level on average experiences 4–5 avoided level crossings and the level mixing width is about 16–32.

(3) The mechanism of transition to quantum chaos is that, as the dynamical symmetry breaking perturbation reaches its critical value, a large number of avoided level crossings have developed, which result in strong mixing and violent fluctuations of the levels.

(4) The sensitivity of eigenenergies and eigenwave functions to a small change of perturbation originates from the quasisingularity of the inverse of the energy denominator, $1/(E_n-E_m)^2$, caused by avoided level crossings.

(5) The property of the chaotic basis is that different chaotic bases or different components of one chaotic state are statistically independent. This statistical independence results in a discontinuity in the strength function for the transition between two chaotic bases. This property may explain the peculiar decay behavior of ergodic nuclear collective states observed in experiments.

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