

Connection between distribution and fragmentation functions

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We show that the quark fragmentation function $D(z)$ and the quark distribution function $q(x)$ are connected in the $z \rightarrow 1$ limit by the approximate relation $D(z)/z \approx q(2 - 1/z)$, where both quantities are in their physical regions. Predictions for proton production in inelastic e^+e^- annihilation, based on the new relation and standard parametrizations of quark distribution functions, are found to be compatible with the data.

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Inclusive deep inelastic scattering (DIS) and hadron production in e^+e^- inelastic annihilation (IA) are important sources of information on the structure of hadrons. By simple crossing it is possible to relate the structure functions of these two processes [1]. However, the relation so obtained (known as the Drell-Levy-Yan relation) is of little use, as it connects the structure functions in the *physical* region of one process to the structure functions in the *unphysical* region of the other process.

It would obviously be important to have a relation between the DIS and IA structure functions, taking both in their physical regions. We could then exploit the accurate information we already possess on the quark distribution functions of the nucleon to predict the quark fragmentation functions, which are still poorly known, or vice versa, by measuring the fragmentation functions of hadrons which cannot be used as DIS targets we could predict the quark densities inside those hadrons. An example which immediately comes to mind is the Λ : the quark dynamics inside this hyperon is highly relevant for our understanding of the spin and flavor structure of hadrons [2–4].

As a matter of fact, a relation connecting the structure functions of DIS and IA in their physical regions does exist in the literature. It is the so-called Gribov-Lipatov “reciprocity” relation. As we shall see, this relation, in its commonly used form, has no real justification. Moreover it is not supported by phenomenological evidence. The purpose of this Rapid Communication is to derive another, well founded, relation connecting DIS and IA, and to show that this relation, within its range of validity, is in good agreement with the existing data.

The DIS cross section is written in terms of two structure functions $F_1(x, Q^2)$ and $F_2(x, Q^2)$, where $x = Q^2/2p \cdot q$ is the Bjorken variable and $Q^2 = -q^2$ is the momentum transfer squared. Two analogous quantities appear in the IA cross section: they are denoted by $\bar{F}_1(z, Q^2)$ and $\bar{F}_2(z, Q^2)$, where now $z = 2p \cdot q/Q^2$ and $Q^2 = q^2$ is the center-of-mass energy

squared. In leading order QCD the Callan-Gross relations connect $F_1(\bar{F}_1)$ to $F_2(\bar{F}_2)$ as

$$F_2(x, Q^2) = 2x F_1(x, Q^2), \quad (1)$$

$$2z \bar{F}_1(z, Q^2) = -z^2 \bar{F}_2(z, Q^2), \quad (2)$$

and, at the same order, the structure functions can be expressed in terms of the distribution functions $q_{a,\bar{a}}$ and the fragmentation functions $D_{a,\bar{a}}$ as

$$2F_1(x, Q^2) = \frac{1}{x} F_2(x, Q^2) = \sum_a e_a^2 [q_a(x, Q^2) + q_{\bar{a}}(x, Q^2)], \quad (3)$$

$$\begin{aligned} 2z \bar{F}_1(z, Q^2) &= -z^2 \bar{F}_2(z, Q^2) \\ &= 3 \sum_a e_a^2 [D_a(z, Q^2) + D_{\bar{a}}(z, Q^2)], \end{aligned} \quad (4)$$

where the sums run over all flavors (the factor 3 comes from a sum over colors).

The traditional form of the Gribov-Lipatov relation reads [5]

$$\begin{aligned} z \bar{F}_1(z) &= F_1(z), \\ z^3 \bar{F}_2(z) &= -F_2(z), \end{aligned} \quad (5)$$

where $F_{1,2}(z)$ means that the DIS structure functions are evaluated at $x=z$. Phenomenological tests of this relation have been carried out [6–8] and it turns out that the IA structure functions predicted by Eq. (5) undershoot the data. But the main shortcoming of Eq. (5) is its uncertain theoretical status. In fact, what Gribov and Lipatov proved in their classical papers [9] is that the nonsinglet splitting functions for DIS and IA are equal at leading order (for a detailed and clear discussion see [10]):

$$\bar{P}_{qq}^{(0)}(z) = P_{qq}^{(0)}(z). \quad (6)$$

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Thus Eq. (5) is true at leading order if one assumes that at a nonperturbative scale μ^2 the input distribution and fragmentation functions are related to each other by

$$3D(z) = q(z). \quad (7)$$

This relation is unjustified. We will now show that the true nonperturbative relation existing between $q(x)$ and $D(z)$ in the large- z limit is the approximate relation

$$\frac{1}{z}D(z) \simeq q(2-1/z). \quad (8)$$

Since $2-1/z \simeq (1/[1-(1-1/z)]) = z$, as $z \rightarrow 1$, Eq. (8) can be approximated further by

$$\frac{1}{z}D(z) \simeq q(z). \quad (9)$$

This relation was used as a phenomenological ansatz in [4,11].

Let us start from the general definition of the quark distribution function [12]

$$q(x) = \int \frac{d\xi^-}{4\pi} e^{ixp^+ \xi^-} \langle h(p) | \bar{\psi}(0) \gamma^+ \psi(\xi^-) | h(p) \rangle, \quad (10)$$

for a hadron h with mass M and momentum $p = (p^0, \mathbf{p})$. The light-cone components are defined as $p^\pm = \frac{1}{2}(p^0 \pm p^3)$. The normalization of the states is $\langle p | p' \rangle = (2\pi)^3 2p^+ \delta(p^+ - p'^+) \delta^2(\mathbf{p}_\perp - \mathbf{p}'_\perp)$. By inserting a complete set of intermediate states $|n\rangle$ with momentum $p_n = (p_n^0, \mathbf{p}_n)$ and mass M_n and making use of the translational invariance, one obtains

$$q(x) = \frac{\sqrt{2}}{2} \sum_n \int \frac{d^4 p_n}{(2\pi)^3} \delta(p_n^2 - M_n^2) \delta(p^+ - xp^+ - p_n^+) \times |\langle n(p_n) | \psi_+(0) | h(p) \rangle|^2, \quad (11)$$

where $\psi_+ = \frac{1}{2} \gamma^- \gamma^+ \psi$ is the ‘‘good’’ component the quark field operator.

Similarly, the fragmentation function of a quark into an unpolarized hadron h is defined as [12] (a sum over the final spin of the hadron is performed)

$$\frac{1}{z}D(z) = \sum_n \int \frac{d\xi^-}{4\pi} e^{-ip^+ \xi^- / z} \int \frac{d^4 p_n}{(2\pi)^3} \delta(p_n^2 - M_n^2) \times \text{Tr}\{\gamma^+ \langle 0 | \psi(0) | h(p), n(p_n) \rangle \times \langle h(p), n(p_n) | \bar{\psi}(\xi^-) | 0 \rangle\}, \quad (12)$$

where the initial quark carries a light-cone momentum $k^+ = p^+ / z$. The normalization of $D(z)$ is such that $\sum_h \int dz z D(z) = 1$. Using translational invariance one obtains [13]

$$\frac{1}{z}D(z) = \frac{1}{2\sqrt{2}} \sum_n \int \frac{d^4 p_n}{(2\pi)^3} \delta(p_n^2 - M_n^2) \delta(p^+ / z - p^+ - p_n^+) \times |\langle 0 | \psi_+(0) | h(p), n(p_n) \rangle|^2. \quad (13)$$

Crossing symmetry means

$$\langle 0 | \psi_+(0) | h(p), n(p_n) \rangle = \langle \bar{n}(-p_n) | \psi_+(0) | h(p) \rangle. \quad (14)$$

If we make the change $p_n \rightarrow -p_n$ in the integral (13) and use Eq. (14) we get (remember that $x = 1/z$)

$$\frac{1}{z}D(z) = q(1/z). \quad (15)$$

This relation, which connects the fragmentation function in the physical region $0 \leq z \leq 1$ to the quark distribution in the unphysical region $x = 1/z \geq 1$, and vice versa, is equivalent to the Drell-Levy-Yan relation [1].

A further step is needed in order to get the relation (8). What we have to do is to establish a connection between the physical and the unphysical region of the quark distribution function (or, equivalently, of the fragmentation function).

In the physical region $0 \leq x \leq 1$ the δ function in Eq. (11) constrains $p_n^+ = (1-x)p^+$ to be positive and hence selects positive energy states in the sum over n . Equation (11) can be rewritten as

$$q(x) = \frac{\sqrt{2}}{4\pi} \sum_n \int \frac{d\mathbf{p}_n}{2|p_n^0| (2\pi)^3} \delta(p^+ - xp^+ - p_n^+) \times |\langle n(p_n) | \psi_+(0) | h(p) \rangle|^2, \quad (16)$$

where $p_n^0 = (\mathbf{p}_n^2 + M_n^2)^{1/2}$. The δ function allows the integration to be simplified giving [14]

$$\int d\mathbf{p}_n \delta[p^+ - xp^+ - p_n^+] = 2\pi \int_{p_{min}}^{\infty} d|\mathbf{p}_n| |\mathbf{p}_n|, \quad (17)$$

where

$$p_{min}(x) = \left| \frac{M^2(1-x)^2 - M_n^2}{2M(1-x)} \right|. \quad (18)$$

We observe now that p_{min} remains unchanged if we replace x by $2-x$:

$$p_{min}(x) = p_{min}(2-x). \quad (19)$$

But $p_{min}(x)$ is not the only source of x dependence in $q(x)$. After exploiting the δ function as in Eq. (17), the matrix elements appearing in Eq. (16) also depend on x . Hence Eq. (16) is in general noninvariant under the substitution $x \rightarrow 2-x$. However, in the large- x limit [which, according to Eq. (18), is equivalent to the large- $|\mathbf{p}_n|$ limit] the matrix elements in Eq. (16) tend to become x independent. The reason is simple. If we describe the quarks inside the hadron by Dirac spinors with an upper component $u(|\mathbf{p}_n|)$ and a lower component $v(|\mathbf{p}_n|)$, the x dependence of the matrix elements is contained in interference terms of the type $u(|\mathbf{p}_n|)v(|\mathbf{p}_n|) -$

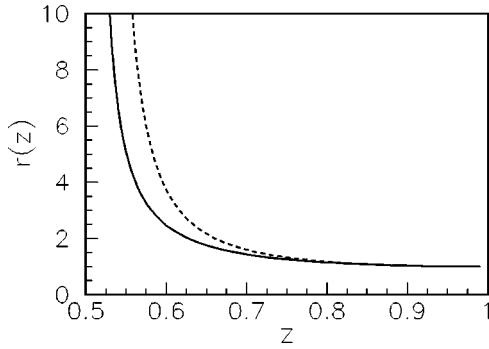


FIG. 1. The ratio $r(z)=q(1/z)/q(2-1/z)$ in the light-cone quark model. The solid and dashed curves are the results in the light-cone quark model for the Gaussian-type wave function (22) and the power-law type wave function (23), with $m_q=220$ MeV, $\beta=450$ MeV, $m_D=800$ MeV, and $a=3.5$.

see [14]. Now for large momenta u and v must behave as plane waves and their product vanishes when integrated over $|\mathbf{p}_n|$. Therefore as x gets large the quark distribution tends to become invariant with respect to the substitution $x \rightarrow 2-x$, namely, $q(x) \approx q(2-x)$, or equivalently

$$q(1/z) \approx q(2-1/z), \quad (20)$$

in the large- z limit. Incidentally, we notice that the same happens in the limit where relativistic effects can be neglected. By combining Eq. (15) with Eq. (20) we finally get the relation we have anticipated above:

$$\frac{1}{z} D(z) \approx q(2-1/z). \quad (21)$$

Notice that for $z \geq 0.5$ this relation connects (approximately) the *physical region* of DIS to the *physical region* of IA. Equation (21) is intended to be valid at a fixed and small scale $\mu^2 < 1$ GeV².

We can check the validity of Eq. (20) by an explicit model calculation. We use a quark-diquark model [15] in the framework of the light-cone approach to quark distribution functions [16,17]. In this model the probability to hit a quark of mass m_q and transverse momentum \mathbf{k}_\perp inside the nucleon, leaving a spectator-diquark with mass m_D in the state D , is $q_D(x) \sim \int d^2\mathbf{k}_\perp |\varphi_D(x, \mathbf{k}_\perp)|^2$, where $\varphi_D(x, \mathbf{k}_\perp)$ is the momentum space wave function of the quark-diquark system with invariant mass $\mathcal{M}^2 = (m_q^2 + \mathbf{k}_\perp^2)/x + (m_D^2 + \mathbf{k}_\perp^2)/(1-x)$. For the light-cone wave function $\varphi_D(x, \mathbf{k}_\perp)$ we use two different forms: the Gaussian-type wave function of the Brodsky-Huang-Lepage model [16] and a power-law type wave function

$$\varphi_D(x, \mathbf{k}_\perp) = A_{\text{BHL}} \exp(-\mathcal{M}^2/8\beta^2), \quad (22)$$

$$\varphi_D(x, \mathbf{k}_\perp) = A_{\text{PL}} (1 + \mathcal{M}^2/\beta^2)^{-a}. \quad (23)$$

In Fig. 1 we plot the ratio

$$r(z) = \frac{q(1/z)}{q(2-1/z)}. \quad (24)$$

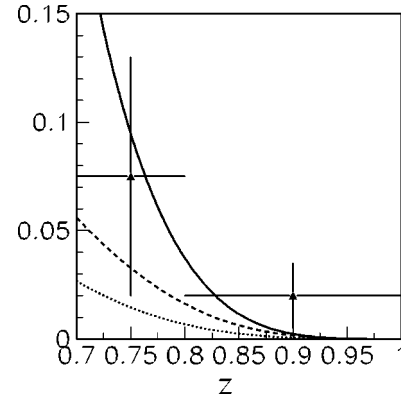


FIG. 2. The structure function $z^3 \bar{F}_2(z)$ in e^+e^- annihilation. The data are the experimental results from DASP at $Q^2 = 13$ GeV² [6,7]. The solid curve is the prediction based on (21) and (26). The dotted curve is the prediction of the traditional Gribov-Lipatov relation (5). The dashed curve is the prediction based on Eq. (9). For the quark distribution functions we used the CTEQ parametrization [18].

One can notice that $r(z)$ approaches 1 at large z , as we expected, and that the two model wave functions lead to very similar results. The sharp increase of $r(z)$ as $z \rightarrow 0.5$ is due to the vanishing of the denominator $q(2-1/z)$ when its argument tends to zero. This is an artifact of the quark-diquark light-cone model which is purely valencelike. In more sophisticated models containing a sea of quarks and antiquarks $q(x)$ does not vanish as $x \rightarrow 0$ and the increase of $r(z)$ is tamed so that no spurious singularity exists $z=0.5$.

Let us come now to the phenomenology of the new relation (21). If we stick to leading order QCD and use Eq. (6), we can translate Eq. (21) in terms of structure functions as (remember that at large x or z the evolution is dominated by the quark splitting functions)

$$\bar{F}_1(z) = 3F_1(2-1/z); \quad (25)$$

$$z\bar{F}_2(z) = -\frac{3}{2-1/z} F_2(2-1/z). \quad (26)$$

Using standard parametrizations for the DIS structure functions we can predict the IA structure functions at large z by means of Eqs. (25),(26). A caveat is in order. Since there are only few DIS experimental data for $x > 0.7$, the quark distributions in this region are not very well known. This introduces some uncertainty in our predictions.

In Fig. 2 we compare the DASP data on $z^3 \bar{F}_2(z)$ with the predictions based on the new relation (26) and on the traditional Gribov-Lipatov relation (5). For comparison, we also show the results for $z^3 \bar{F}_2(z)$ based on the approximation (9). For the DIS structure functions we used the CTEQ5L parametrization [18]. We find that the result of the new relation (26) is in better agreement with the data at $z \rightarrow 1$. Clearly, precision measurements of both $\bar{F}_2(z)$ and $F_2(x)$ at large z and x would allow a more conclusive check of Eq. (26).

We already pointed out that Eqs. (25),(26) are valid at leading order (LO) only. At next-to-leading order (NLO) the

evolution of nonsinglet distribution and fragmentation functions is different [19,20]. Since the fragmentation functions evolve more rapidly, NLO effects lead to a suppression at large z of the LO results for \bar{F}_2 shown in Fig. 2. Due to the large uncertainty of the present-day data, in this Rapid Communication we chose for simplicity to stick to a leading-order phenomenological treatment. Using NLO splitting functions for the fragmentation functions [21] and the new relation (21) as the initial condition for the evolution one can calculate the NLO corrections to Eqs. (25),(26).

In conclusion, we presented a new relation between distribution and fragmentation functions in their physical regions, which leads to simple testable relations between DIS and IA structure functions. A revised form of the Gribov-Lipatov relation with a color factor and an additional factor of z is also proved to be an approximate relation at large z .

An immediate application of the new relation connecting $q(x)$ to $D(z)$ is in the study of the Λ polarization near the Z resonance in e^+e^- annihilation and in polarized lepton DIS scattering. Using the Gribov-Lipatov relation and the QCD counting rules for the quark helicity distributions [22], it was found that the data are not satisfactorily reproduced [4]. We have checked that the situation improves if the new relation (21) is used.

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