Radiative corrections to electron-proton scattering

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The radiative corrections to elastic electron-proton scattering are analyzed in a hadronic model including the finite size of the nucleon. For initial electron energies above 8 GeV and large scattering angles, the proton vertex correction in this model increases by at least 2% of the overall factor by which the one-photon exchange cross section must be multiplied. In addition, we refine the mathematical treatment, removing many of the approximations made in the generally used expressions previously obtained by Mo and Tsai. In particular, the contribution of soft photon emission is calculated exactly. Results are presented for some kinematics at high momentum transfer and compared with the expressions of Mo and Tsai.

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I. INTRODUCTION

Electron scattering at intermediate and high energies has been one of the most useful means of investigating nuclear structure for over 40 years. With the advent of CW accelerators and high resolution detectors such as MAMI and TJNAF it has become clear that one must have an accurate estimate of the radiative corrections if meaningful cross sections are to be obtained from the experimental measurements. Depending on the experimental conditions—initial beam energy, momentum transfer, and detector resolution or missing mass for the observed particles—the radiative corrections can be as large as 30% of the uncorrected cross section. To obtain cross sections which are accurate to 1%, one must then know the radiative correction to 3%.

The theoretical expression for the radiative correction which has been used in the analysis of almost all single arm elastic electron scattering experiments with beam energies below approximately 25 GeV (for which W and Z exchange are in general not significant) is that given originally by Tsai [1,2] in connection with experiments at Stanford, SLAC, and CEA. That work involved approximations that were both purely mathematical (made in performing the integrations needed to evaluate the inelastic cross section) and approximations denoted here as ''soft-photon approximations'' that are directly related to the physics in that the effect of proton structure was neglected; in considering the proton legs, only the soft virtual (infrared) photon contribution is calculated exactly—approximations are made in the hard virtual photon (noninfrared) contribution. In particular, the proton structure is neglected by setting the photon momentum square $k^2=0$ in the proton form factor $F(k^2)$, thus simplifying the calculation considerably.

The purpose of the present paper is twofold. First, to consider the contribution of the internal structure of the nucleon in the radiative correction to elastic electron-proton scattering. For this we have considered a simple model in which the proton current is taken to have the usual on-shell form. The model dependence of the radiative correction is clearly an important question for the analysis of electron scattering experiments at the 1% level. Second, to give a more refined mathematical treatment than that presented in Refs. [1,2] and used in essentially all experimental analyses. This refinement is needed to achieve the accuracy attainable in current experiments and to get a better handle on the errors in the expressions used for the radiative correction.

The present calculation differs from that of Tsai [1,2] in three substantive aspects. First, we evaluate the soft bremsstrahlung cross section without any approximation; the relevant integrals have been given in closed form by 't Hooft and Veltman [3]. In fact, the exact expressions are simpler in form than the approximate ones given in Refs. [1,2]. We note in particular that in the limit of the target mass $M\to\infty$, corresponding to a static Coulomb potential, we obtain exactly the result first given by Schwinger [4]. Second, in the evaluation of the contribution of the box and crossed box diagrams to the elastic cross section we make a less drastic approximation than that made in [1]. Specifically, in the integrands corresponding to the relevant matrix elements, M_2 and M_3 [Eqs. (3.20),(3.21)], we make a soft photon approximation (setting $k=0$ or $k=q$) in the *numerator* (as in Ref. [1]), but not in the denominators. Again, the required integrals (scalar four-point functions) have been given in Ref. [3]; the resulting expressions are again considerably simpler than those obtained in Ref. [1], where the soft-photon approximation is also made in the denominators of M_2 and M_3. Finally, in evaluating the proton vertex correction, we have made no soft photon approximation for the virtual photon (as was done in Ref. [1]) and have included form factors for the proton, taking the proton current to be that indicated below in Eq. (2.1).

The organization of the paper is as follows. In Sec. II we discuss questions concerning the electromagnetic nuclear current operator used in this analysis. In Sec. III we give details of the calculation of the matrix elements and cross section for elastic scattering, retaining terms of order α relative to the Rosenbluth (one photon exchange) cross section for elastic scattering. Integrals needed for the evaluation of the various matrix elements are written explicitly and expressed in closed form in terms of Spence functions (dilogarithms). Details are given in the Appendices.

In Sec. IV we consider the soft bremsstrahlung cross section in detail; as with the elastic cross section given in Sec. III, the result is expressed in closed form in terms of Spence functions. In Sec. V we add the elastic and inelastic cross sections, giving both an analytic expression and a numerical evaluation of the radiative correction for various values of the pertinent parameters (initial beam energy, final electron detector resolution, and target nucleus). We compare the values of the radiative correction calculated here with those given in Refs. [1,2].

II. ELECTROMAGNETIC NUCLEON CURRENT OPERATOR

We follow in this paper the convention of Bjorken and Drell [5]. The metric used is defined by $p_i \cdot p_j = \epsilon_i \epsilon_j - \mathbf{p}_i \cdot \mathbf{p}_j$. Further, $\alpha = e^2/4\pi = 1/137.036$, m is the electron rest mass, M is the target nucleus rest mass, Z the charge of the target nucleus, κ the anomalous magnetic moment of the proton, p_1 and p_3 the initial and final electron four-momenta, respectively, p_2 and p_4 the initial and final target nucleus four-momenta, respectively, and $q = p_1 - p_3 = p_4 - p_2$ is the four-momentum transfer to the target nucleus for elastic scattering. In the lab system we have $p_1 = (\epsilon_1, \mathbf{p}_1)$, $p_3 = (\epsilon_3, \mathbf{p}_3)$, $p_2 = (M, 0)$, $p_4 = (M + \omega, \mathbf{q})$, $\omega = -q^2/2M$. We define, in addition, $\rho = p_4 + p_2$ and $x = (\rho + \rho_1)/(\rho - \rho_1) = (\rho + \rho_1)^2/4M^2$, with $\rho_1^2 = -q^2$. Finally, η is the lab system recoil factor: For $\epsilon_1 \gg m$, $\epsilon_3 \gg m$, $\eta \cong \epsilon_1/\epsilon_3 \cong 1 + (\epsilon_1/M)(1 - \cos\theta)$ where θ is the electron scattering angle. We note, in particular, that $1 \leqslant \eta \leqslant x$.

With the aim of presenting expressions which correspond to the experimental conditions of high-energy electron scattering, we neglect, in the final expressions given in this paper, terms of relative orders m^2/ϵ^2, $m^2/(-q^2)$, and m^2/M^2. Neglect of these terms defines our high energy approximation. No assumption is made, however, with regard to the magnitudes of M/ϵ_1, M/ϵ_3, or $M^2/(-q^2)$.

At low momentum transfer the internal structure of the nucleon can safely be neglected in the determination of the radiative corrections in electron-nucleus scattering. However, with increasing energies and momenta this is in general no longer the case. One of the objectives of this paper is to investigate this in a model for the e.m. interaction of a nonpointlike nucleon. The most general e.m. off-shell nucleon vertex can be characterized by six invariant functions [6,7]. As the most simple model we may consider a vector dominancelike model for the nucleon current, characterized by only two form factors which depend only on the four-momentum square of the photon. It is given by

$$\Gamma_\mu = F_1(q^2)\gamma_\mu + \kappa F_2(q^2)\frac{i\sigma_{\mu\nu}q^\nu}{2M}, \tag{2.1}$$

where the form factors $F_1(q^2)$ and $F_2(q^2)$ are taken to have a monopole or dipole form

$$F_1(q^2) = F_2(q^2) = \left(\frac{-\Lambda^2}{q^2 - \Lambda^2}\right)^n, \quad n = 1 \text{ or } 2 \tag{2.2}$$

with Λ being a constant of the order of 1 GeV/c. Although the quantitative predictions of the radiative corrections are expected in general to be dependent on the details of the nucleon model assumed, one should already be able to see most of the salient features in the present model study. In particular, identifying regions in phase space where the finite size of the nucleon may play an important role in the size of radiative corrections can be important. In this way one may hope to get some feeling for the reliability of neglecting the internal structure of the nucleon as is usually done. The present study is intended as a first exploration of the sensitivity on the nonpointlike nature of the e.m. hadronic current. As in Ref. [1], although we are primarily interested in electron-proton scattering, the radiative corrections studied here can also be applied to electron-nucleus scattering, with appropriate changes in F_1, F_2, κ, and M. However, even in the case of electron-proton scattering, the factor Z is convenient for identifying the contributions from the various diagrams.

It should be noted that the dressed vertex function $\tilde{\Lambda}_\mu$, with Eq. (2.1) as e.m. current operator containing the form factors F_n, satisfies the identity

$$q^\mu \tilde{\Lambda}_\mu = F_1(q^2)[S^{-1}(p') - S^{-1}(p)], \tag{2.3}$$

where S is the dressed nucleon propagator and $q = p' - p$. As a direct consequence of Eq. (2.3), one gets for on-mass-shell nucleons, the current conservation

$$q^\mu \langle p' | \tilde{\Lambda}_\mu | p \rangle = 0. \tag{2.4}$$

Obviously, the radiative corrections will in general be sensitive to the choice of the e.m. nucleon current. In general, contact terms have to be introduced to satisfy gauge invariance. Due to our assumption (2.1) that the nucleon form factor is dependent only on the photon momentum square, gauge invariance is trivially satisfied. Although interesting in its own right, we will not address in this paper the issue of the sensitivity of the predictions on the choice of e.m. nucleon current.

In the study of radiative corrections we may distinguish between the elastic and inelastic contributions, the latter being the real soft photon emission processes from both the electron and hadron. The elastic electron cross section can be determined immediately from the total scattering amplitude $\mathcal{M}$ through the well-known expression

$$d\sigma = \frac{mM}{\sqrt{(p_1 \cdot p_2)^2 - m^2M^2}} \sum_{\text{spins}} \int |\mathcal{M}|^2 (2\pi)^4 \times \delta^4(p_4 + p_3 - p_2 - p_1)\frac{m\, d^3p_3}{(2\pi)^3\epsilon_3}\, \frac{M\, d^3p_4}{(2\pi)^3\epsilon_4}. \tag{2.5}$$

For single-arm experiments with unpolarized electrons in which the final proton is not observed, $d\sigma$ must be averaged

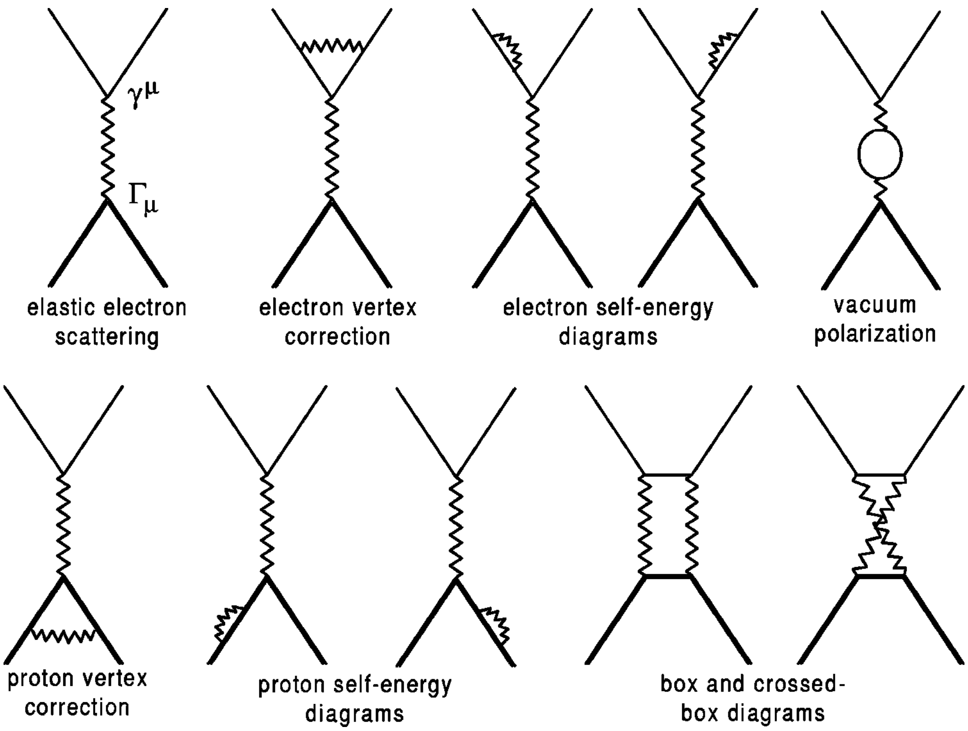

FIG. 1. Feynman diagrams for elastic amplitudes.

over initial spins, summed over final spins, and integrated over the final proton four-momentum. Up to order α^2 we have for the total scattering amplitude

$$\mathcal{M}=\sum_{n=1}^{6} M_n, \tag{2.6}$$

where the various terms correspond to the Feynman graph contributions shown in Fig. 1. Here M_1 is the matrix element for the one-photon exchange diagram

$$M_1=Ze^2\overline{u}(p_3)\gamma_\mu u(p_1)\frac{(-i)}{q^2+i\epsilon}\overline{u}(p_4)\Gamma^\mu(q^2)u(p_2). \tag{2.7}$$

M_2 and M_3 are the matrix elements for the box and crossed box (two-photon exchange) diagrams. M_4 is the vacuum polarization matrix element (only an electron-positron loop is indicated in the figure, but the contribution from higher mass lepton loops can be included without difficulty). M_5 is the electron vertex correction, and M_6 is the proton vertex correction.

III. ELASTIC CROSS SECTION

To evaluate the various one-loop corrections to Eq. (2.6) some tedious algebra has to be carried out. We outline the procedure used to evaluate the matrix elements needed for the radiative correction to the elastic cross section, M_2 through M_6.

A. Proton vertex correction

We begin with the matrix element for the proton vertex correction M_6 given by

$$M_6=Z^3e^2\overline{u}(p_3)\gamma_\mu u(p_1)\frac{(-i)}{q^2+i\epsilon}\overline{u}(p_4)\Lambda^\mu(p_4,p_2)u(p_2), \tag{3.1}$$

where

$$\Lambda^\mu(p_4,p_2)=-ie^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2-\lambda^2+i\epsilon}\Gamma^\nu(k^2)\times\frac{1}{(\not p_4-\not k-M+i\epsilon)}\Gamma^\mu(q^2)\frac{1}{(\not p_2-\not k-M+i\epsilon)}\times\Gamma_\nu(k^2). \tag{3.2}$$

In Eq. (3.2), each of the three Γ's, given by Eq. (2.1), contains a term with γ_μ (which we denote by g) and a term with $\sigma_{\mu\nu}$ (which we denote by s). The proton vertex correction $\Lambda^\mu(p_4,p_2)$ then consists of eight terms, which we represent symbolically by ggg, gsg, gss, etc. As may be seen after rationalizing the propagators, the k dependence of the numerators for $ggg,gsg,\ldots,$ is such that there are at most four factors of the form $\not k$. Moreover, the terms with three or four factors $\not k$ may, with only a minimum of algebra, be written so that two of these factors are adjacent, giving $\not k\not k=k^2$. Although the calculation can equally well be carried out with F_1 and F_2 distinct functions, we assume $F_1=F_2=F$, which simplifies the algebra. The terms $ggg,gsg,\ldots,$ can then be expressed in terms of the integrals

$$\{I_0;I_\mu;I_{\mu\nu};J_0;J_\mu;J_{\mu\nu};K_0\}=\int\frac{d^4k}{(2\pi)^4}F^2(k^2)\times\{1;k_\mu;k_\mu k_\nu;k^2;k_\mu k^2;k_\mu k_\nu k^2;(k^2)^2\}/D(\lambda^2), \tag{3.3}$$

where

$$D(\lambda^2)=(k^2-\lambda^2+i\epsilon)(k^2-2k\cdot p_2+i\epsilon)(k^2-2k\cdot p_4+i\epsilon). \tag{3.4}$$

For form factors having the form given in Eq. (2.2), the integrals in Eq. (3.3) could all be evaluated as indicated for three-point functions in Ref. [3], Sec. 5, and Ref. [8], Appendix E. However, in the interest of obtaining a relatively compact analytic expression in closed form, we have used an alternative procedure. As given here in Appendix A, the integrals may be expressed in terms of their moments, defined by Eqs. (A4)–(A6) and (A13). After straightforward though somewhat tedious algebra, the terms $ggg, gsg, \ldots,$ are then expressed in terms of these moments. Next, for form factors of the form given in Eq. (2.2), we show that all of the moments may be expressed in terms of three functions ϕ_k, which obey a three-term inhomogenous recursion, and this is used for their evaluation. Finally, we note from Eqs. (A32)–(A37) that the terms $ggg, gsg, \ldots,$ may be usefully grouped by writing them in the form

$$(g+s)g(g+s)=F(q^2)\left[G_1(q^2)\gamma_\mu+G_2(q^2)\frac{i\sigma_{\mu\nu}q^\nu}{2M}\right] \tag{3.5}$$

and

$$(g+s)s(g+s)=\kappa F(q^2)\left[X_1(q^2)\gamma_\mu+X_2(q^2)\frac{i\sigma_{\mu\nu}q^\nu}{2M}\right]. \tag{3.6}$$

We note in the expressions for $ggg, gsg, \ldots,$ in Appendix A that the infrared divergent terms are all contained solely within ggg and gsg. These are the terms with a factor $\phi_1(\lambda^2)$ in Eqs. (A32),(A33). Since these are precisely the terms which are retained in the proton vertex correction in Ref. [1], we separate them for the purpose of comparison with that work, writing M_6 in the form

$$M_6=M_6^{(0)}+M_6^{(1)}, \tag{3.7}$$

where

$$M_6^{(0)}=-\frac{\alpha Z^2}{2\pi}(2M^2-q^2)\phi_1(\lambda^2)M_1. \tag{3.8}$$

The function $\phi_1(\lambda^2)$, defined by Eq. (A20), is simply related to the function $K(p_2,p_4)$ defined in Ref. [1] by

$$K(p_i,p_j)=\frac{2p_i\cdot p_j}{-i\pi^2}\int\frac{d^4k}{(k^2-\lambda^2+i\epsilon)(k^2-2k\cdot p_i+i\epsilon)(k^2-2k\cdot p_j+i\epsilon)},$$

viz.,

$$K(p_2,p_4)=2p_2\cdot p_4\phi_1(\lambda^2). \tag{3.9}$$

In addition to Eq. (3.1) we also have to include the contribution of the proton self-energy diagrams. It is given by Σ' where

$$\Sigma'=\frac{1}{4}\mathrm{Tr}\left[\frac{\partial\Sigma}{\partial\not p}\right] \tag{3.10}$$

in which Σ is the lowest order self-energy contribution

$$\Sigma=-ie^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2-\lambda^2+i\epsilon}\Gamma^\nu(k)\times\frac{1}{(\not p-\not k-M+i\epsilon)}\Gamma_\nu(k). \tag{3.11}$$

The addition of this contribution to the lowest order vertex correction modifies the expressions for $(g+s)g(g+s)$ and $(g+s)s(g+s)$ given above in Eqs. (3.5),(3.6) so that we now have

$$\overline{(g+s)g(g+s)}=F(q^2)\times\left[(G_1(q^2)-G_1(0))\gamma_\mu+G_2(q^2)\frac{i\sigma_{\mu\nu}q^\nu}{2M}\right] \tag{3.12}$$

and

$$\overline{(g+s)s(g+s)}=\kappa F(q^2)\times\left[X_1(q^2)\gamma_\mu+(X_2(q^2)-G_1(0))\frac{i\sigma_{\mu\nu}q^\nu}{2M}\right]. \tag{3.13}$$

Here use has been made of the identity

$$\Sigma'=G_1(0)$$

which follows from Eq. (2.3). We then have, for the matrix element including self-energy diagrams

$$\bar{M}_6=\bar{M}_6^{(0)}+\bar{M}_6^{(1)}, \tag{3.14}$$

where

$$\bar{M}_6^{(0)}=\frac{\alpha Z^2}{2\pi}[-K(p_2,p_4)+K(p_2,p_2)]M_1 \tag{3.15}$$

which is the expression given in Ref. [1], Eq. (II.12). The infrared divergent part of these terms is cancelled exactly by the infrared divergent terms in the inelastic cross section. This is in accordance with the general result found in Ref. [9].

B. Electron vertex and vacuum polarization corrections

These contributions have been extensively studied in the literature. Here for completeness give the results valid at high momentum transfer. The electron vertex correction can immediately be obtained from the previous section by retaining only the term ggg and taking the limit $\Lambda\to\infty$. For $-q^2\gg m^2$ we have, after adding the contribution of the electron self-energy diagrams

$$\bar{M}_5=\frac{\alpha}{2\pi}\left\{-K(p_1,p_3)+K(p_1,p_1)+\frac{3}{2}\ln\left(\frac{-q^2}{m^2}\right)-2\right\}M_1 \tag{3.16}$$

which is the expression given in Ref. [1], Eq. (II.5). We note that the infrared divergence is contained entirely within the terms $-K(p_1,p_3)+K(p_1,p_1)$.

The matrix element M_4 for vacuum polarization is, after charge renormalization, related simply to the matrix element M_1. If we include the vacuum polarization amplitudes from particle-antiparticle fermion loops of different masses, as has been done in several experimental analyses [15], then

$$M_4=M_1\sum_i \Pi^f(q^2/m_i^2). \tag{3.17}$$

For a fermion loop in the photon propagator, $\Pi^f(q^2/m_i^2)$ is given in Ref. [5] in terms of an integral which can be evaluated in closed form [10], giving

$$\Pi^f(q^2/m_i^2)=\frac{\alpha}{3\pi}\left\{\left(1-\frac{u}{2}\right)\sqrt{1+u}\times\ln\left(\frac{\sqrt{1+u}+1}{\sqrt{1+u}-1}\right)+u-\frac{5}{3}\right\} \tag{3.18}$$

in which m_i is the mass of the fermion and $u=4m_i^2/(-q^2)$. For $-q^2/m_i^2\gg 1$ this gives

$$\Pi^f(q^2/m_i^2)=\frac{\alpha}{\pi}\left\{\frac{1}{3}\ln\left(\frac{-q^2}{m_i^2}\right)-\frac{5}{9}\right\}. \tag{3.19}$$

In principle, once one includes particle-antiparticle pairs of mass greater than the electron mass, bosons as well as fermions should be considered. The matrix elements for vacuum polarization for a pair of structureless spin zero bosons in the closed loop, first given by Feynman [11], may be found in a more accessible form in a paper of Tsai [10].

A more complete discussion of vacuum polarization should include a consideration of pion structure as well as the contribution of spin-one bosons, in particular the ρ meson. A detailed discussion of the hadronic contribution to vacuum polarization may be found in connection with calculations of the anomalous magnetic moment of the muon [12] and in connection with radiative corrections to high-energy electron-positron collider experiments [13].

C. Box and crossed-box diagrams

The matrix elements for the box and crossed-box diagrams M_2 and M_3 are

$$M_2=(Ze^2)^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2-\lambda^2+i\epsilon}\frac{1}{(k+q)^2-\lambda^2+i\epsilon}\times\left[\bar{u}(p_3)\gamma_\nu\frac{1}{\not p_1-\not k-m+i\epsilon}\gamma_\mu u(p_1)\right]\times\left[\bar{u}(p_4)\Gamma^\nu[(k+q)^2]\frac{1}{\not p_2+\not k-M+i\epsilon}\Gamma^\mu(k^2)u(p_2)\right], \tag{3.20}$$

$$M_3=(Ze^2)^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2-\lambda^2+i\epsilon}\frac{1}{(k+q)^2-\lambda^2+i\epsilon}\times\left[\bar{u}(p_3)\gamma_\nu\frac{1}{\not p_1-\not k-m+i\epsilon}\gamma_\mu u(p_1)\right]\times\left[\bar{u}(p_4)\Gamma^\mu(k^2)\frac{1}{\not p_4-\not k-M+i\epsilon}\Gamma^\nu[(k+q)^2]u(p_2)\right]. \tag{3.21}$$

After rationalizing the propagators, the required integrals can, for form factors of the form (2.2), all be written in terms of four-point functions; in principle they can be evaluated using [3], Sec. 6, and [8], Appendix E. In the present work, however, we have chosen to evaluate these matrix elements in an approximate manner, but one which is less drastic than that employed in Ref. [1]. We note first that in M_2 and M_3 the integrands have two infrared divergent factors $[(k^2-\lambda^2+i\epsilon)((k-q)^2-\lambda^2+i\epsilon)]^{-1}$. The integrands are thus peaked when either of the two exchanged photons is soft, and become divergent when $k\to 0$ or when $k\to q$. We therefore evaluate the numerators in M_2 and M_3 at these two points but make no changes to the denominators. A simple calculation shows in fact that each of the numerators has the same value for $k=0$ as for $k=q$, viz., $4ip_1\cdot p_2q^2M_1$ in the case of M_2 and $4ip_3\cdot p_2q^2M_1$ in the case of M_3. We then take this factor outside of the integral and are left with a scalar four-point function to evaluate. The result has been given in [3], Sec. 6 and Appendix E (b) and is expressed simply in terms of logarithms

$$M_2 = -\frac{\alpha Z}{\pi}\frac{\epsilon_1}{|\mathbf{p}_1|}\ln\left(\frac{\epsilon_1+|\mathbf{p}_1|}{m}\right)\ln\left(\frac{-q^2}{\lambda^2}\right)M_1 \quad (3.22)$$

and

$$M_3 = \frac{\alpha Z}{\pi}\frac{\epsilon_3}{|\mathbf{p}_3|}\ln\left(\frac{\epsilon_3+|\mathbf{p}_3|}{m}\right)\ln\left(\frac{-q^2}{\lambda^2}\right)M_1 . \quad (3.23)$$

By contrast, in Ref. [1], in addition to the approximation just described, a soft-photon approximation is made in the infrared denominators: Specifically, when $k=0$ the factor $(k-q)^2-\lambda^2$ is set equal to $q^2-\lambda^2$ and when $k=q$ the factor $k^2-\lambda^2$ is set equal to $q^2-\lambda^2$, thus giving two terms and reducing the four-point function to three-point functions

$$M_2 = -\frac{\alpha Z}{2\pi}[K(p_2,-p_1)+K(p_4,-p_3)]M_1 \quad (3.24)$$

and

$$M_3 = \frac{\alpha Z}{2\pi}[K(p_2,p_3)+K(p_4,p_1)]M_1 \quad (3.25)$$

[see Ref. [1], Eqs. (II.9) and (II.11)]. The infrared divergent terms (those with a factor $\ln\lambda^2$) are, for M_2, the same in Eqs. (3.22) and (3.24), and, for M_3, the same in (3.23) and (3.25). However, Eqs. (3.24),(3.25) differ significantly from Eqs. (3.22),(3.23). These latter expressions are functions of the momentum transfer, q^2. The integrals $K(p_i,p_j)$, on the other hand, are functions only of the scalar invariants p_i^2, p_j^2 and $p_i\cdot p_j$. In Eqs. (3.24),(3.25), M_2 and M_3 therefore depend only on the initial and final electron energies, and not on the momentum transfer ($p_2\cdot p_1 = p_4\cdot p_3 = \epsilon_1 M$; $p_2\cdot p_3 = p_4\cdot p_1 = \epsilon_3 M$).

D. Contribution of proton form factor

Using our results obtained for M_n, we find up to order α

$$|\mathcal{M}|^2 = |M_1|^2\left\{\begin{array}{c} 1+\frac{\alpha}{\pi}\left[\frac{13}{6}\ln\left(\frac{-q^2}{m^2}\right)-\frac{28}{9}-K(p_1,p_3)+K(p_1,p_1)\right] \\ -\frac{2\alpha Z}{\pi}\ln\eta\ln\left(\frac{-q^2}{\lambda^2}\right) \\ +\frac{\alpha Z^2}{\pi}[-K(p_2,p_4)+K(p_2,p_2)] \end{array}\right\} + 2\,\mathrm{Re}\{M_1^\dagger \overline{M}_6^{(1)}\}. \quad (3.26)$$

This holds provided that $-q^2 \gg m^2$.

Finally, we consider the remaining contribution $2\,\mathrm{Re}\{M_1^\dagger\overline{M}_6^{(1)}\}$ in Eq. (3.26), coming from the inclusion of form factors for the proton and integration over the entire range of four-momenta of the virtual photon in the proton vertex correction. The term $M_6^{(1)}$ is obtained from the full proton vertex correction by subtracting the infrared divergent matrix element $M_6^{(0)}$ which is independent of the proton form factor. We may therefore introduce $G_1'(q^2)$ and $X_2'(q^2)$ to be the expressions $G_1(q^2)$ and $X_2(q^2)$ from which we have omitted the terms with factor $\phi_1(\lambda^2)$. As a result we get (apart from factors)

$$\overline{M}_6^{(1)} = \frac{\alpha Z^2}{2\pi}\langle p_3|\gamma^\mu|p_1\rangle\langle p_4|\tilde{\Gamma}_\mu|p_2\rangle \quad (3.27)$$

and

$$2\,\mathrm{Re}\{M_1^\dagger\overline{M}_6^{(1)}\} = \frac{\alpha Z^2}{\pi}(\langle p_3|\gamma^\nu|p_1\rangle\langle p_4|\Gamma_\nu|p_2\rangle)^\dagger(\langle p_3|\gamma^\mu|p_1\rangle \times\langle p_4|\tilde{\Gamma}_\mu|p_2\rangle), \quad (3.28)$$

where

$$\tilde{\Gamma}_\mu \equiv \tilde{F}_1(q^2)\gamma_\mu + \kappa\tilde{F}_2(q^2)\frac{i\sigma_{\mu\nu}q^\nu}{2M} \quad (3.29)$$

with

$$\tilde{F}_1(q^2) \equiv F(q^2)[(G_1'(q^2)-G_1'(0))+\kappa X_1(q^2)], \quad (3.30)$$

$$\kappa\tilde{F}_2(q^2) \equiv F(q^2)[G_2(q^2)+\kappa(X_2'(q^2\}-G_1'(0))]. \quad (3.31)$$

Equation (3.28) has the same form as

$$M_1^\dagger M_1 = (\langle p_3|\gamma^\nu|p_1\rangle\langle p_4|\Gamma_\nu|p_2\rangle)^\dagger(\langle p_3|\gamma^\mu|p_1\rangle\langle p_4|\Gamma_\mu|p_2\rangle) \quad (3.32)$$

with the exception of the replacement $\Gamma_\mu \to \tilde{\Gamma}_\mu$ in the right-hand term. Thus, in place of the Rosenbluth cross section, obtained from $\Sigma_{\mathrm{spins}} M_1^\dagger M_1$, we have

$$\sum_{\mathrm{spins}} 2\,\mathrm{Re}\{M_1^\dagger\overline{M}_6^{(1)}\} = \frac{\alpha^2\cos^2(\theta/2)}{4\epsilon_1^2\eta\sin^4(\theta/2)}\left(\frac{\alpha Z^2}{\pi}\right)\{\ \}, \quad (3.33)$$

where

$$\{\ \} = \left(F_1\tilde{F}_1 - \frac{\kappa^2 q^2}{4M^2} F_2\tilde{F}_2 \right) - \frac{q^2}{2M^2}(F_1+\kappa F_2)(\tilde{F}_1+\kappa\tilde{F}_2)\tan^2\frac{\theta}{2}. \quad (3.34)$$

The purely elastic cross section, including radiative corrections to order α, can thus be written as

$$\left(\frac{d\sigma_0}{d\Omega}\right)(1+\delta_{el}^{(0)}+\delta_{el}^{(1)}), \quad (3.35)$$

where

$$\delta_{el}^{(0)} = \frac{\alpha}{\pi}\left\{ -\left[\ln\left(\frac{-q^2}{m^2}\right)-1\right]\ln\left(\frac{m^2}{\lambda^2}\right) + \frac{13}{6}\ln\left(\frac{-q^2}{m^2}\right) - \frac{28}{9} - \frac{1}{2}\ln^2\left(\frac{-q^2}{m^2}\right) + \frac{\pi^2}{6}\right\} - \frac{2\alpha Z}{\pi}\ln\eta\ln\left(\frac{-q^2}{\lambda^2}\right) + \frac{\alpha Z^2}{\pi}\left\{ -\left(\frac{\epsilon_4}{|\mathbf{p}_4|}\ln x - 1\right)\ln\left(\frac{M^2}{\lambda^2}\right) + \frac{\epsilon_4}{|\mathbf{p}_4|}\left[-\ln x\ln\left(\frac{\rho^2}{M^2}\right) + \frac{1}{2}\ln^2 x + 2L\left(-\frac{1}{x}\right) + \frac{\pi^2}{6}\right]\right\}, \quad (3.36)$$

in which $L(z)$ is defined in Eq. (B9) and

$$\delta_{el}^{(1)} = \frac{\alpha Z^2}{\pi}\left\{ \frac{[F_1\tilde{F}_1 - (\kappa^2 q^2/4M^2)F_2\tilde{F}_2] - (q^2/2M^2)(F_1+\kappa F_2)(\tilde{F}_1+\kappa\tilde{F}_2)\tan^2(\theta/2)}{\{[F_1^2 - (\kappa^2 q^2/4M^2)F_2^2] - (q^2/2M^2)(F_1+\kappa F_2)^2\tan^2(\theta/2)]} \right\}. \quad (3.37)$$

IV. SOFT BREMSSTRAHLUNG CROSS SECTION

In this section we calculate the contribution of soft photon emission to the radiative correction; emission by both electron and proton are included. The relevant diagrams, with corresponding matrix elements, M_{b1} and M_{b2}, are shown in Fig. 2. These matrix elements are given by

$$M_{b1} = -iZe^3(2\pi)^4\delta^4(p_3+p_4+k-p_1-p_2) \times \frac{mM}{\sqrt{2\omega\epsilon_1\epsilon_3\epsilon_2\epsilon_4}}\bar{u}(p_3)\left[\not{\epsilon}\frac{1}{\not{p}_3+\not{k}-m+i\epsilon}\gamma_\mu + \gamma_\mu\frac{1}{\not{p}_1-\not{k}-m+i\epsilon}\not{\epsilon}\right] u(p_1)\bar{u}(p_4)\Gamma^\mu u(p_2) \times \frac{1}{(p_1-p_3-k)^2+i\epsilon}, \quad (4.1)$$

$$M_{b2} = iZ^2e^3(2\pi)^4\delta^4(p_3+p_4+k-p_1-p_2) \times \frac{mM}{\sqrt{2\omega\epsilon_1\epsilon_3\epsilon_2\epsilon_4}}\bar{u}(p_3)\gamma_\mu u(p_1) \times \bar{u}(p_4)\left[\not{\epsilon}\frac{1}{\not{p}_4+\not{k}-M+i\epsilon}\Gamma^\mu + \Gamma^\mu\frac{1}{\not{p}_2-\not{k}-M+i\epsilon}\not{\epsilon}\right] u(p_2)\frac{1}{(p_1-p_3)^2+i\epsilon}. \quad (4.2)$$

Making the soft photon approximation, we rationalize the denominators and drop terms of relative order k in the numerator and denominator (but not in the delta function). The cross section, summed over photon polarizations, is then

$$d\sigma_b = -\frac{Z^2e^6}{(2\pi)^9}\frac{m^2M^2}{\sqrt{(p_1\cdot p_2)^2-m^2M^2}} \times \sum_{\text{spins}}\int \frac{d^3p_3}{\epsilon_3}\frac{d^3p_4}{\epsilon_4}\frac{d^3k}{2\omega}(2\pi)^4 \times \delta^4(p_3+p_4+k-p_1-p_2)|\bar{u}(p_3)\gamma_\mu u(p_1)\bar{u}(p_4) \times \Gamma^\mu u(p_2)|^2\frac{1}{(q^2)^2}\left(\frac{p_3}{p_3\cdot k} - \frac{p_1}{p_1\cdot k} - Z\frac{p_4}{p_4\cdot k} + Z\frac{p_2}{p_2\cdot k}\right)^2. \quad (4.3)$$

The range of integration in the above expression is determined by the experimental conditions. We assume, as in Ref.

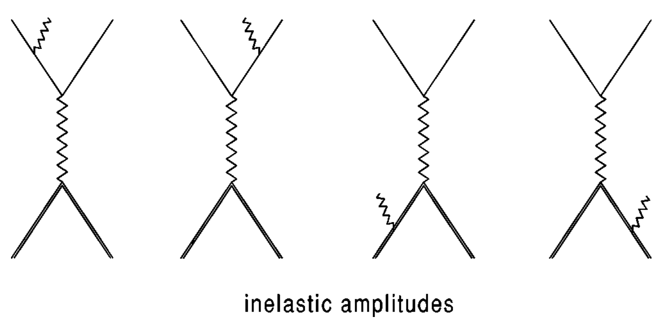

FIG. 2. Feynman diagrams for inelastic amplitudes.

[1], that the final proton and emitted photon are undetected; a different result would be obtained in the case of coincidence experiments. Integrating first over d^3p_4 we obtain

$$d\sigma_b=-\frac{Z^2e^6}{(2\pi)^5}\frac{m^2M^2}{\sqrt{(p_1\cdot p_2)^2-m^2M^2}}\sum_{\text{spins}}\int\frac{d^3p_3}{\epsilon_3}\int\frac{d^3k}{\omega}\times\delta\big((t-k)^2-M^2\big)\theta(\epsilon_4)|\overline{u}(p_3)\gamma_\mu u(p_1)\overline{u}(p_4)\times\Gamma^\mu u(p_2)|^2\frac{1}{(q^2)^2}\left(\frac{p_3}{p_3\cdot k}-\frac{p_1}{p_1\cdot k}-Z\frac{p_4}{p_4\cdot k}+Z\frac{p_2}{p_2\cdot k}\right)^2 \tag{4.4}$$

with $t\equiv p_1+p_2-p_3=p_4+k$. The succeeding integrations are simplified by transforming to the special frame S^0 (defined by $\mathbf{t}=0$), in which the argument of the delta function in Eq. (4.4) is independent of the direction of the emitted photon. The photon energy is then given solely by the final electron energy. We integrate next over the photon energy and angles in S^0. Assuming that the elastic cross section has no significant variation over the range of photon momenta, we may take $|\overline{u}(p_3)\gamma_\mu u(p_1)\overline{u}(p_4)\Gamma^\mu u(p_2)|^2/(q^2)^2$ outside of the integration, setting p_3 and p_4 equal to their values for purely elastic scattering. We may then transform back to the lab frame and integrate over the angles of the final electron, determined by the entrance slit and spectrometer, giving

$$d\sigma_b=-\frac{\alpha}{4\pi^2}d\sigma_0\int'\frac{d^3k}{\omega}\left(\frac{p_3}{p_3\cdot k}-\frac{p_1}{p_1\cdot k}-Z\frac{p_4}{p_4\cdot k}+Z\frac{p_2}{p_2\cdot k}\right)^2, \tag{4.5}$$

where $\omega=\sqrt{\mathbf{k}^2+\lambda^2}$. The integration over photon energy is restricted to $|\mathbf{k}|\le\Delta\epsilon$, where $\Delta\epsilon$ is the maximum momentum of the photon in the frame S^0, which is related to the final electron detector acceptance in the lab frame ΔE by $\Delta\epsilon=\eta\Delta E$ (see Appendix C for details). It is assumed that $\Delta\epsilon$ is less than any of the other energies. The relevant integrals are all of the form

$$L_{ij}=\int'\frac{d^3k}{\omega}\frac{1}{(p_i\cdot k)(p_j\cdot k)}, \tag{4.6}$$

in terms of which

$$d\sigma_b=-\frac{\alpha}{4\pi^2}d\sigma_0\left\{\begin{matrix} m^2L_{11}+m^2L_{33}-2p_1\cdot p_3L_{13}\\ +Z(-2p_1\cdot p_2L_{12}+2p_3\cdot p_2L_{32}+2p_1\cdot p_4L_{14}-2p_3\cdot p_4L_{34})\\ +Z^2(M^2L_{22}+M^2L_{44}-2p_2\cdot p_4L_{24})\end{matrix}\right\}. \tag{4.7}$$

These integrals have been evaluated by 't Hooft and Veltman [3], Sec. 7. We give here only their final result, rewritten using our metric; the essential steps in the derivation are given in their work. As shown in [3], Sec. 7, for the case in which the momenta p_i and p_j are all on the mass shell, the integrals L_{ij} can, provided p_i is not a multiple p_j, be written in the form

$$L_{ij}=\frac{2\pi}{\sqrt{(p_i\cdot p_j)^2-m_i^2m_j^2}}\{S_{ij}^{(1)}+S_{ij}^{(2)}\}, \tag{4.8}$$

where

$$S_{ij}^{(1)}=2\ln\left(\frac{p_i\cdot p_j+\sqrt{(p_i\cdot p_j)^2-m_i^2m_j^2}}{m_im_j}\right)\ln\left(\frac{2\Delta\epsilon}{\lambda}\right) \tag{4.9}$$

and

$$S_{ij}^{(2)}=\ln^2\left(\frac{\beta_i}{m_i\sqrt{t^2}}\right)-\ln^2\left(\frac{\beta_j}{m_j\sqrt{t^2}}\right)+L\left(1-\frac{\beta_i l\cdot t}{t^2\gamma_{ij}}\right)+L\left(1-\frac{m_i^2 l\cdot t}{\beta_i\gamma_{ij}}\right)-L\left(1-\frac{\beta_j l\cdot t}{\alpha t^2\gamma_{ij}}\right)-L\left(1-\frac{m_j^2 l\cdot t}{\alpha\beta_j\gamma_{ij}}\right), \tag{4.10}$$

in which $L(z)$ is defined in Eq. (B9) and

$$\alpha=\frac{p_i\cdot p_j+\sqrt{(p_i\cdot p_j)^2-m_i^2m_j^2}}{m_i^2},\quad l=\alpha p_i-p_j, \tag{4.11}$$

$$\beta_{i,j}\equiv p_{i,j}\cdot t+\sqrt{(p_{i,j}\cdot t)^2-m_{i,j}^2t^2},\quad \gamma_{ij}\equiv\sqrt{(p_i\cdot p_j)^2-m_i^2m_j^2}. \tag{4.12}$$

[Note that for $\Delta\epsilon\to0$, $S_{ij}^{(2)}$ remains finite; the only singularity is confined to the term $\ln(2\Delta\epsilon/\lambda)$, evident in $S_{ij}^{(1)}$.] The evaluation of Eq. (4.6) for $p_i=p_j$ is straightforward. The result written in terms of relativistic invariants is

$$L_{ii}=\frac{4\pi}{m_i^2}\left[\ln\left(\frac{2\Delta\epsilon}{\lambda}\right)-\frac{p_i\cdot t}{\sqrt{(p_i\cdot t)^2-m_i^2t^2}}\ln\left(\frac{\beta_i}{m_i\sqrt{t^2}}\right)\right]. \tag{4.13}$$

In Ref. [3], Sec. 7, the expression for $S_{ij}^{(2)}$ is evaluated in the frame S^0. Since we want finally to express the cross section in terms of lab frame energies and momenta, we have, in Eq. (4.10), written $S_{ij}^{(2)}$ in terms of relativistic invariants. The terms of leading order in $\ln\lambda$ are apparent in Eqs. (4.9) and (4.13). Substituting these in Eq. (4.7) gives the infrared divergent terms in $d\sigma_b$. They are cancelled exactly by the $\ln\lambda$ terms in the elastic cross section.

TABLE I. Contributions to the radiative correction δ for electron-proton scattering as given in this paper (MTj) and in Mo and Tsai [2] (MoTsai) for three initial electron energies and four-momentum transfers. Values in the rows marked Z^0, Z^1, and Z^2 refer to contributions from terms with these factors in Eq. (5.2).

	$\epsilon_1=4.4$ GeV $Q^2=6\,(\mathrm{GeV}/c)^2$		$\epsilon_1=12$ GeV $Q^2=16\,(\mathrm{GeV}/c)^2$		$\epsilon_1=21.5$ GeV $Q^2=31.3\,(\mathrm{GeV}/c)^2$	
	MTj	MoTsai	MTj	MoTsai	MTj	MoTsai
Z^0	−0.2187	−0.2171	−0.2330	−0.2322	−0.2323	−0.2317
Z^1	−0.0569	−0.0506	−0.0517	−0.0479	−0.0625	−0.0571
Z^2	−0.0242	−0.0232	−0.0359	−0.0347	−0.0452	−0.0440
$\delta_{el}^{(1)}$	+0.0068		+0.0116		+0.0185	
δ	−0.2930	−0.2908	−0.3090	−0.3149	−0.3214	−0.3328

In writing the explicit expressions for the terms in L_{ij}, we choose i and j such that L_{ij} simplifies readily for lab frame electron energies and momentum transfers which are very large compared to the electron rest mass. When $m_i\neq m_j$, this is achieved by choosing i and j such that $m_i=m$ and $m_j=M$. $d\sigma_b$ can now be calculated in the high-energy approximation as defined in Sec. II. Using the results as given in Appendix D, we obtain for the inelastic cross section

$$d\sigma_b=\frac{\alpha}{\pi}d\sigma_0\left\{\begin{aligned}&\left[\ln\left(\frac{-q^2}{m^2}\right)-1\right]\ln\left(\frac{(\eta m\Delta E)^2}{\epsilon_1\epsilon_3\lambda^2}\right)+\frac{1}{2}\ln^2\left(\frac{-q^2}{m^2}\right)-\frac{1}{2}\ln^2\eta+L\left(\cos^2\frac{1}{2}\theta\right)-\frac{1}{3}\pi^2\\&+2Z\left[\ln\eta\ln\left(\frac{(2\eta\Delta E)^2}{x\lambda^2}\right)+L\left(1-\frac{\eta}{x}\right)-L\left(1-\frac{1}{\eta x}\right)\right]\\&+Z^2\left[\left(\frac{\epsilon_4}{|\mathbf{p}_4|}\ln x-1\right)\ln\left(\frac{(2\eta\Delta E)^2}{\lambda^2}\right)-\frac{\epsilon_4}{|\mathbf{p}_4|}\left[\ln^2x-\ln x+L\left(1-\frac{1}{x^2}\right)\right]-1\right]\end{aligned}\right\}, \tag{4.14}$$

where η is the lab system recoil factor and $x=(\rho+\rho_1)/4M^2$, introduced in Sec. II.

V. RADIATIVE CORRECTIONS TO ELASTIC ELECTRON-PROTON SCATTERING

The results given in Eqs. (3.36),(3.37), and (4.14) may be added to give the radiative correction δ. The analytic expression is given below in Eqs. (5.1),(5.2). Numerical evaluation of the radiative correction for various values of the pertinent parameters (initial beam energy, momentum transfer, final electron detector resolution, and target nucleus) are given in Tables I–III. We note that the infrared ($\ln\lambda$) terms, which appear in both the purely elastic (3.36) and inelastic (4.14) contributions to the radiative correction, cancel exactly when added to give the cross section for elastic electron-proton scattering with radiative corrections to first order in α:

$$d\sigma=d\sigma_0(1+\delta), \tag{5.1}$$

where

TABLE II. Contributions to the radiative correction δ as given in this paper (MTj) and in Mo and Tsai [2] (MoTsai) for several nuclei, with $\epsilon_1=4.4$ GeV, $Q^2=6$ $(\mathrm{GeV}/c)^2$; other symbols as in Table I.

	^{2}H		^{4}He		^{12}C		^{40}Ca	
	MTj	MoTsai	MTj	MoTsai	MTj	MoTsai	MTj	MoTsai
Z^0	−0.2476	−0.2467	−0.2535	−0.2532	−0.2615	−0.2609	−0.2632	−0.2625
Z^1	−0.0187	−0.0183	−0.0066	−0.0077	−0.0151	−0.0168	−0.0147	−0.0173
Z^2	−0.0094	−0.0088	−0.0077	−0.0071	−0.0156	−0.0145	−0.0188	−0.0178
$\delta_{el}^{(1)}$	+0.0010		+0.0002		+0.0001		+0.0001	
δ	−0.2747	−0.2739	−0.2677	−0.2680	−0.2920	−0.2922	−0.2966	−0.2975

TABLE III. Contributions to the radiative correction δ as given in this paper (MTj) and in Mo and Tsai [2] (MoTsai) for several nuclei, with $\epsilon_1 = 21.5$ GeV, $Q^2 = 31.3\,(\mathrm{GeV}/c)^2$; other symbols as in Table I.

	^{2}H		^{4}He		^{12}C		^{40}Ca	
	MTj	MoTsai	MTj	MoTsai	MTj	MoTsai	MTj	MoTsai
Z^0	−0.2707	−0.2704	−0.2817	−0.2814	−0.2876	−0.2875	−0.2896	−0.2894
Z^1	−0.0182	−0.0189	−0.0149	−0.0166	−0.0134	−0.0164	−0.0123	−0.0173
Z^2	−0.0231	−0.0221	−0.0379	−0.0352	−0.0583	−0.0535	−0.0761	−0.0708
$\delta_{el}^{(1)}$	+0.0045		+0.0026		+0.0006		+0.0001	
δ	−0.3076	−0.3114	−0.3319	−0.3333	−0.3587	−0.3573	−0.3786	−0.3775

$$\delta = \frac{\alpha}{\pi}\left[\frac{13}{6}\ln\left(\frac{-q^2}{m^2}\right) - \frac{28}{9} - \left[\ln\left(\frac{-q^2}{m^2}\right) - 1\right]\ln\left(\frac{4\epsilon_1\epsilon_3}{(2\eta\Delta E)^2}\right) - \frac{1}{2}\ln^2\eta + L\left(\cos^2\frac{1}{2}\theta\right) - \frac{\pi^2}{6}\right]$$
$$+\frac{2\alpha Z}{\pi}\left[-\ln\eta\,\ln\left(\frac{-q^2x}{(2\eta\Delta E)^2}\right) + L\left(1-\frac{\eta}{x}\right) - L\left(1-\frac{1}{\eta x}\right)\right]$$
$$+\frac{\alpha Z^2}{\pi}\left[\frac{\epsilon_4}{|\mathbf{p}_4|}\left(-\frac{1}{2}\ln^2 x - \ln x\,\ln\left(\frac{\rho^2}{M^2}\right) + \ln x\right) - \left(\frac{\epsilon_4}{|\mathbf{p}_4|}\ln x - 1\right)\ln\left(\frac{M^2}{(2\eta\Delta E)^2}\right) + 1 + \frac{\epsilon_4}{|\mathbf{p}_4|}\left(-L\left(1-\frac{1}{x^2}\right) + 2L\left(-\frac{1}{x}\right) + \frac{\pi^2}{6}\right)\right] + \delta_{el}^{(1)}. \tag{5.2}$$

Here, $\delta_{el}^{(1)}$ is the contribution coming from the inclusion of form factors for the proton and integration over the entire range of four-momenta of the virtual photon in the proton vertex correction; it is thus not included in the analysis given in Refs. [1,2], denoted here as the soft photon approximation. Moreover, $\delta_{el}^{(1)}$ has no infrared divergent terms; these are all included in the soft photon approximation.

In Tables I–III we compare the values of the radiative correction, δ, calculated in this paper (denoted by MTj) with those given by Mo and Tsai in Ref. [2] for various kinematics. The initial beam energies and momentum transfers have been chosen to correspond to experiments proposed or already performed at Jefferson Lab [14] and SLAC [15]. The final electron detector acceptance ΔE has been taken throughout to be one percent of the final electron energy ϵ_3. In the form factors [see Eq. (2.2)], the parameter Λ has been chosen to be 700 MeV/c throughout. The contribution of the terms in Eq. (5.2) are grouped according to the power of Z which appears there as a factor. The numerical value of each of these groups of terms is given in the rows denoted by Z^0, Z^1, Z^2. Values given in the column MTj in the row Z^2 do not include the contribution of the proton form factor (which are contained in $\delta_{el}^{(1)}$); they are given for comparison with the values in Ref. [2]. In the range of energies and momentum transfers considered here, the correction $\delta_{el}^{(1)}$, due to the finite size of the nucleon (and integration over the entire range of four-momenta of the virtual photon in the proton vertex correction), is found in general to be much smaller than the other contributions with factor Z^2, labeled explicitly in Eq. (5.2) and in Tables I–III. The values given in these tables include only the contribution of electron-positron pairs in the vacuum polarization; the contribution of muon and tau pairs is given by Eqs. (3.17),(3.18).

The curves in Figs. 3 and 4 illustrate the two aspects of the present work: (1) the contribution of nucleonic size effects to the radiative correction, and (2) the improvement of the mathematical treatment of the integrations given in the

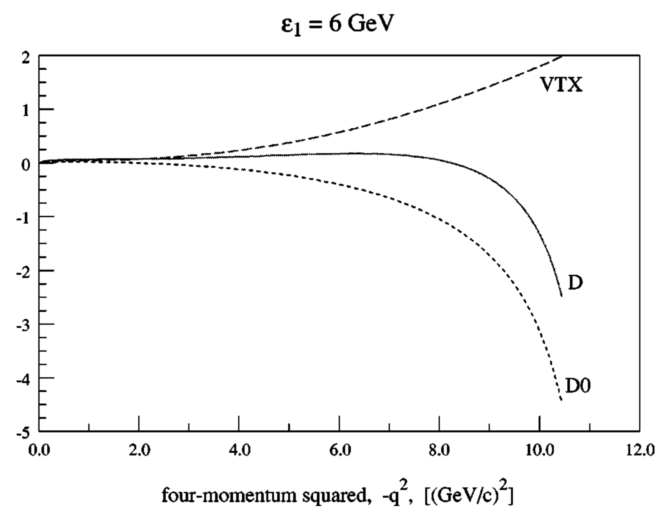

FIG. 3. The curves show the contribution of nucleonic size effects (VTX, dashed curve), mathematical refinement (D0, dotted curve), and the resulting difference (D, solid curve) between the radiative correction given by Mo and Tsai [2] and that given in this paper δ_{MTj} as a function of four-momentum transfer for an initial electron energy of 6 GeV. (VTX, D0, and D are defined in Sec. V.)

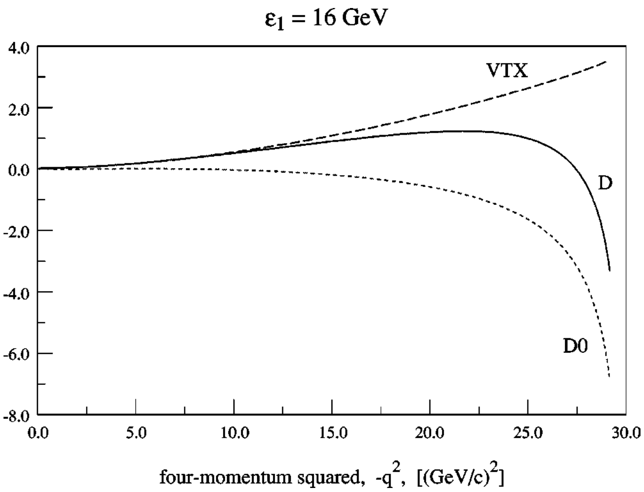

FIG. 4. As in Fig. 3, but with an initial electron energy of 16 GeV.

work of Mo and Tsai [1,2]. The nucleonic size effects are all contained in the term $\delta_{el}^{(1)}$ [Eq. (3.37)]; its contribution relative to the overall radiative correction factor $(1+\delta_{\rm MTj})$ is given by the dashed curves marked VTX, where VTX $=100\times\delta_{el}^{(1)}/(1+\delta_{\rm MTj})$. The dotted curve shows D0 $=100\times(\delta_{\rm MTj}^{(0)}-\delta_{\rm Tsai})/(1+\delta_{\rm MTj})$, which is that part of the difference between the radiative correction given by Mo and Tsai [2] and the one given in this paper due solely to the improvement of the mathematical treatment of the integrations. Here, $\delta_{\rm MTj}^{(0)}$ is the radiative correction given in Eq. (5.2), *excluding* the term $\delta_{el}^{(1)}$. It will be noticed that VTX is always positive, and that for most of the range of allowed momentum transfers $D0$ is negative. Thus their sum, which is the difference between the radiative correction $\delta_{\rm MTj}$ given in this paper in Eq. (5.2), and $\delta_{\rm Tsai}$, given in Ref. [2], is rather small except for the region corresponding to large scattering angles. This sum is given by the solid curves marked D, where D=D0+VTX=$100\times(\delta_{\rm MTj}-\delta_{\rm Tsai})/(1+\delta_{\rm MTj})$.

VI. CONCLUSION

We have calculated the radiative correction to elastic electron-proton scattering to lowest order in α using a hadronic model which includes the finite size of the nucleon. The contribution from the emission of real soft photons by the electron and the proton is calculated exactly. The contributions of the box and crossed-box (two-photon exchange) diagrams are calculated in a soft photon approximation which is less drastic than that employed in Ref. [1]. A number of observations may be made from the values given in Tables I–III. First, the contributions of the electron vertex correction, vacuum polarization, and real soft photon emission by the electron [the terms in Eq. (5.2) with factor α/π] dominate the radiative correction δ. Since our expression for these terms differs from that given by Mo and Tsai [2] solely in that they have omitted the term $(\alpha/\pi)[L(\cos^2\frac{1}{2}\theta)-\pi^2/6]$ in Eq. (5.2), we find values for δ which differ from theirs by at most 2% for the initial energies and momentum transfers considered here. Further, we note that, except for the proton, and at the higher energies considered here, the contribution of $\delta_{el}^{(1)}$ is negligible. However, for the two highest energies, $\delta_{el}^{(1)}$ is between 2% and 3% of the factor $(1+\delta)$ by which the uncorrected cross section must be multiplied, and hence should be considered in precision measurements for electron-proton scattering at energies above 8 GeV. As an empirical guide, we find that $\delta_{el}^{(1)}=0.02(1+\delta)$ for initial energies and scattering angles satisfying $\epsilon_1\sin\theta\approx 8$ for beam energies between 8 and 16 GeV. Finally, we note that a considerable simplification of the expression in Eq. (5.2) occurs if, in addition to the last two terms multiplying α/π, we neglect the last two terms multiplying $2\alpha Z/\pi$ as well as the last three terms multiplying $\alpha Z^2/\pi$, each of these sets of terms being always less than $\pi^2/6$ in magnitude. From this study we see that at the energies and momentum transfers considered here, the nucleonic finite size effects are rather small but are expected to become more important at higher energies. The corrections due to the improvement of the high-energy behavior of the radiative corrections as described in this paper are not negligible and need to be taken into account at the energies and momentum transfers we have considered.

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APPENDIX A: PROTON VERTEX CORRECTION

As noted in Sec. III A, the terms $ggg,gsg,\ldots,$ can be expressed in terms of the integrals given in Eq. (3.1). There, the integrals I_0, J_0, and K_0 are scalars and hence are functions of the scalars p_2^2, p_4^2, and $p_2\cdot p_4$ (and, of course, λ^2 and Λ^2). Since we have on-shell particles in the initial and final states ($p_2^2=p_4^2=M^2$), these integrals are functions of M^2 and q^2. The integrals I_μ and J_μ are vectors and hence in principal can be written in the form

$$I_\mu=ap_{2\mu}+bp_{4\mu} \tag{A1}$$

with a similar equation for J_μ, where a and b are functions of M^2 and q^2. However, the calculation is simplified greatly if we express I_μ in terms of the four vectors $\rho=p_4+p_2$ (which is symmetric in p_4 and p_2) and $q=p_4-p_2$ (which is antisymmetric in p_4 and p_2), i.e., $I_\mu=A\rho_\mu+Bq_\mu$. Here A and B are functions of M^2 and q^2 and hence are symmetric in p_4 and p_2. Further, since the integrands for the vectors I_μ and J_μ are symmetric in p_4 and p_2, it follows that $B=0$. We thus have

$$I_\mu=A\rho_\mu \tag{A2}$$

and a similar equation for J_μ. These same considerations of symmetry allow for the simplification of the tensors $I_{\mu\nu}$ and $J_{\mu\nu}$, which are also symmetric functions of p_4 and p_2. We can therefore write

$$I_{\mu\nu}=a_1\rho_\mu\rho_\nu+a_2q_\mu q_\nu+a_3g_{\mu\nu} \tag{A3}$$

and a similar equation for $J_{\mu\nu}$. That the terms $\rho_\mu q_\nu$ and $q_\mu \rho_\nu$ are absent follows directly by multiplying $I_{\mu\nu}$ successively by $\rho^\mu q^\nu$ and $q^\mu \rho^\nu$, using $\rho\cdot q=0$ and the fact that $I_{\mu\nu}\rho^\mu q^\nu$ and $I_{\mu\nu}q^\mu\rho^\nu$ are antisymmetric in p_2 and p_4. Multiplying $I_\mu\,(J_\mu)$ by ρ^μ, and $I_{\mu\nu}\,(J_{\mu\nu})$ successively by $\rho^\mu\rho^\nu$, $q^\mu q^\nu$, and $g^{\mu\nu}$, the coefficients in the expressions for I_μ, J_μ, $I_{\mu\nu}$ and $J_{\mu\nu}$ may be expressed in terms of their moments, defined by

$$g_1=\frac{1}{\rho^2}I_\mu\rho^\mu,\quad h_1=\frac{1}{\rho^2}J_\mu\rho^\mu, \tag{A4}$$

$$g_{11}=\frac{1}{\rho^4}I_{\mu\nu}\rho^\mu\rho^\nu,\quad h_{11}=\frac{1}{\rho^4}J_{\mu\nu}\rho^\mu\rho^\nu, \tag{A5}$$

$$g_{22}=\frac{1}{\rho^4}I_{\mu\nu}q^\mu q^\nu,\quad h_{22}=\frac{1}{\rho^4}J_{\mu\nu}q^\mu q^\nu. \tag{A6}$$

Conversely, from Eqs. (A2) and (A3) the integrals in Eq. (3.3) may be written in terms of the moments. The terms $ggg,\ldots,sss$ can then be expressed in terms of these moments. Substituting Eqs. (2.1) and (2.2) in the expression for the proton vertex correction, Eq. (3.2), the integrals are all of the form given in Eq. (3.3), which are in turn expressed in terms of the moments. In so doing we find

$$ggg=-ie^2F(q^2)\left\{\begin{array}{c}\left[\begin{array}{c}2(2M^2-q^2)g_0-4(2M^2-q^2)g_1\\-2(8M^2+q^2)g_{11}-2(q^4/\rho^2)g_{22}+8(M^2/\rho^2)h_0\end{array}\right]\gamma_\mu\\+[-8M^2g_1+24M^2g_{11}+8M^2(q^2/\rho^2)g_{22}-8(M^2/\rho^2)h_0]\dfrac{i\sigma_{\mu\nu}q^\nu}{2M}\end{array}\right\}, \tag{A7}$$

$$gsg=-ie^2\kappa F(q^2)\left\{\begin{array}{c}[-2q^2g_1]\gamma_\mu\\+[2(2M^2-q^2)g_0-4(2M^2-q^2)g_1]\dfrac{i\sigma_{\mu\nu}q^\nu}{2M}\end{array}\right\}, \tag{A8}$$

$$ggs+sgg=-2ie^2\kappa F(q^2)\left\{\begin{array}{c}[-12M^2g_{11}-(q^4/\rho^2)g_{22}+2(1+2M^2/\rho^2)h_0-3h_1]\gamma_\mu\\+\left[\begin{array}{c}(16M^2-q^2)g_{11}+(q^2/\rho^2)(8M^2-q^2)g_{22}\\-4(1+M^2/\rho^2)h_0+3h_1\end{array}\right]\dfrac{i\sigma_{\mu\nu}q^\nu}{2M}\end{array}\right\}, \tag{A9}$$

$$gss+ssg=-2ie^2\kappa^2F(q^2)\left\{\begin{array}{c}(q^2/4M^2)\left[\begin{array}{c}-(8M^2+q^2)g_{11}-(q^4/\rho^2)g_{22}\\+(-2+4M^2/\rho^2)h_0+h_1\end{array}\right]\gamma_\mu\\+[3q^2g_{11}+4M^2(q^2/\rho^2)g_{22}-(q^2/\rho^2)h_0-h_1]\dfrac{i\sigma_{\mu\nu}q^\nu}{2M}\end{array}\right\}, \tag{A10}$$

$$sgs=-ie^2\kappa^2F(q^2)\left\{\begin{array}{c}\left[\begin{array}{c}-(8M^2+q^2)g_{11}-(q^4/\rho^2)g_{22}+(2+q^2/\rho^2)h_0\\-(8M^2+q^2)h_{11}/4M^2-(q^4/\rho^2)h_{22}/4M^2+(-1+q^2/\rho^2)k_0/4M^2\end{array}\right]\gamma_\mu\\+\left[\begin{array}{c}12M^2g_{11}+4M^2(q^2/\rho^2)g_{22}-2(1+2M^2/\rho^2)h_0\\+3h_{11}+4(q^2/\rho^2)h_{22}-k_0/\rho^2\end{array}\right]\dfrac{i\sigma_{\mu\nu}q^\nu}{2M}\end{array}\right\}, \tag{A11}$$

$$sss=-ie^2\kappa^3F(q^2)\left\{\begin{array}{c}(q^2/4M^2)\left[\begin{array}{c}-12M^2g_{11}-4M^2(q^2/\rho^2)g_{22}+2(1+2M^2/\rho^2)h_0\\-4h_1+3h_{11}+(q^2/\rho^2)h_{22}-k_0/\rho^2\end{array}\right]\gamma_\mu\\+\left[\begin{array}{c}2(2M^2+q^2)g_{11}+4M^2(q^2/\rho^2)g_{22}\\-(4M^2/\rho^2+q^2/2M^2)h_0+(q^2/M^2)h_1\\-2(2M^2+q^2)h_{11}/4M^2+2(q^2/\rho^2)(2M^2-q^2)h_{22}/4M^2\\+(q^2/\rho^2)k_0/4M^2\end{array}\right]\dfrac{i\sigma_{\mu\nu}q^\nu}{2M}\end{array}\right\}. \tag{A12}$$

For convenience of notation, we have defined

$$g_0 = I_0, \quad h_0 = J_0, \quad k_0 = K_0. \tag{A13}$$

These expressions do not depend on the particular form of the form factors; we have assumed only that $F_1 = F_2 = F$. However, for form factors of the form given in Eq. (2.2), the moments may all be expressed more simply in terms of the functions $C(\Lambda^2)$:

$$\{C_0(\Lambda^2); C_\mu(\Lambda^2); C_{\mu\nu}(\Lambda^2)\} = \int d^4k \{1; k_\mu; k_{\mu\nu}\}/D(\Lambda^2). \tag{A14}$$

For the choice of form factors (2.2) we readily find

$$\{I\} = N'_m (\Lambda^2)^m T^{m-1} \left\{ \frac{C(\Lambda^2) - C(\lambda^2)}{\Lambda^2 - \lambda^2} \right\}, \tag{A15}$$

with $m = 2n$, $T \equiv \partial/\partial(\Lambda^2)$ and where $N'_m = (-1)^m/(m-1)!(2\pi)^4$. In Eq. (A15) I and C denote any one of $I_0, I_\mu, I_{\mu\nu}$ and $C_0, C_\mu, C_{\mu\nu}$, respectively. We see from Eq. (A15) that terms in $C(\Lambda^2)$ which are independent of Λ^2 do not appear in the expression for I. In particular, this applies to $C_{\mu\nu}(\Lambda^2)$, which may be evaluated using either dimensional regularization or a convergence factor. The infinities in $C_{\mu\nu}(\Lambda^2)$ are indeed independent of Λ^2, thus giving a finite result for $I_{\mu\nu}$ as it should. In similar fashion, we have

$$\{J\} = N'_m (\Lambda^2)^m T^{m-1} \left\{ \frac{\Lambda^2 C(\Lambda^2) - \lambda^2 C(\lambda^2)}{\Lambda^2 - \lambda^2} \right\} \tag{A16}$$

in which J and C denote any one of $J_0, J_\mu, J_{\mu\nu}$ and $C_0, C_\mu, C_{\mu\nu}$, respectively. We see from Eq. (A16) that any terms in $C(\Lambda^2)$ which are independent of Λ^2 do not appear in the expression for J provided $m > 1$. Finally, for K_0 we have

$$K_0 = N'_m (\Lambda^2)^m T^{m-1} \left\{ \frac{\Lambda^4 C_0(\Lambda^2) - \lambda^4 C_0(\lambda^2)}{\Lambda^2 - \lambda^2} \right\}. \tag{A17}$$

We note that, apart from trivial factors, the integrals in Eq. (A14) are the three-point functions defined in Ref. [3], Eq. (5.1), and Ref. [8], Eq. (E.1); C_0 has been evaluated in terms of Spence functions in Ref. [3]. The details of the algebra in Refs. [3,8] being rather lengthy, we choose instead to evaluate the integrals in Eq. (A14) using Feynman parameters, writing

$$\frac{1}{D(\Lambda^2)} = 2\int_0^1 \int_0^1 \frac{x\,dx\,dy}{[k^2 - 2xk\cdot p_y - \Lambda^2(1-x) + i\epsilon]^3}, \tag{A18}$$

where $p_y = p_2 y + p_4(1-y) = \rho/2 + q(1-2y)/2$. Using Eq. (A18) and neglecting terms which are independent of Λ, we may express C_0, C_μ, and $C_{\mu\nu}$ in terms of the functions

$$\phi_k(\Lambda^2) \equiv \int_0^1 \int_0^1 \frac{x^k\,dx\,dy}{p_y^2 x^2 + \Lambda^2(1-x)}. \tag{A19}$$

We obtain

$$C_0 = -i\pi^2 \phi_1(\Lambda^2), \quad C_\mu = -i\pi^2 \frac{1}{2}\rho_\mu \phi_2(\Lambda^2), \tag{A20}$$

$$\begin{aligned} C_{\mu\nu} = -i\pi^2 \Bigg\{ &\frac{1}{4}\rho_\mu\rho_\nu\phi_3(\Lambda^2) - \frac{1}{4}\frac{\rho^2}{q^2}q_\mu q_\nu \phi_3(\Lambda^2) \\ &- \frac{q_\mu q_\nu}{q^2}\Lambda^2[\phi_1(\Lambda^2) - \phi_2(\Lambda^2)] \\ &+ \frac{1}{2}g_{\mu\nu}\Lambda^2\left[\phi_1(\Lambda^2) - \frac{1}{2}\phi_2(\Lambda^2)\right]\Bigg\}. \end{aligned} \tag{A21}$$

We now describe the procedure for calculating the functions ϕ_k. As shown in Appendix B, ϕ_k obey a three-term inhomogeneous recursion, which can be used to calculate ϕ_k for $k > 1$:

$$\begin{aligned} &(k+1)\rho^2\phi_{k+2}(\Lambda^2) - 2(2k+1)\Lambda^2\phi_{k+1}(\Lambda^2) \\ &\quad + 4k\Lambda^2\phi_k(\Lambda^2) \\ &= \frac{2\rho}{\rho_1}\ln\left(\frac{\rho+\rho_1}{\rho-\rho_1}\right) + 2\Lambda^2[\phi^{(0)}_{k+1}(\Lambda^2) - 2\phi^{(0)}_k(\Lambda^2)] \end{aligned} \tag{A22}$$

with $\rho_1^2 = -q^2$ and where

$$\phi^{(0)}_k(\Lambda^2) \equiv \phi_k(\Lambda^2)|_{q^2=0} = \int_0^1 \frac{x^k\,dx}{M^2x^2 + (1-x)\Lambda^2}. \tag{A23}$$

The functions $\phi^{(0)}_k(\Lambda^2)$ may in turn be calculated from the recursion

$$M^2\phi^{(0)}_{k+2}(\Lambda^2) - \Lambda^2\phi^{(0)}_{k+1}(\Lambda^2) + \Lambda^2\phi^{(0)}_k(\Lambda^2) = \frac{1}{k+1}. \tag{A24}$$

To implement the recursions (A22) and (A24) we need

$$\phi^{(0)}_0(\Lambda^2) = \frac{1}{\Lambda\Lambda_1}\ln\left(\frac{\Lambda+\Lambda_1}{\Lambda-\Lambda_1}\right),$$

$$\phi^{(0)}_1(\Lambda^2) = \frac{1}{2M^2}\left[\ln\frac{M^2}{\Lambda^2} + \frac{\Lambda}{\Lambda_1}\ln\left(\frac{\Lambda+\Lambda_1}{\Lambda-\Lambda_1}\right)\right] \tag{A25}$$

which follow from Eq. (A23), and $\phi_1(\Lambda^2)$, which can be expressed in terms of dilogarithms (see Appendix B)

$$\phi_1(\Lambda^2)=\frac{1}{\rho\rho_1}\left\{L\left(1-\frac{1}{xy}\right)-L\left(1-\frac{x}{y}\right)-2\ln(x)\ln\left(1+\frac{1}{y}\right)\right\},\tag{A26}$$

where

$$x=\frac{\rho+\rho_1}{\rho-\rho_1}=\frac{(\rho+\rho_1)^2}{4M^2},\quad y=\frac{\Lambda+\Lambda_1}{\Lambda-\Lambda_1}=\frac{(\Lambda+\Lambda_1)^2}{4M^2},\tag{A27}$$

and $\Lambda_1^2=\Lambda^2-4M^2$.

In view of Eqs. (A15)–(A17) we also want to take the limit $\lambda\to 0$. Neglecting all terms which vanish in this limit, we find

$$\phi_1(\lambda^2)\underset{\vec{\lambda}\to 0}{\to}\frac{1}{\rho\rho_1}\left\{-2L\left(-\frac{1}{x}\right)-\frac{\pi^2}{6}-\frac{1}{2}\ln^2 x+\ln x\,\ln\left(\frac{\rho^2}{\lambda^2}\right)\right\},\tag{A28}$$

$$\phi_1^{(0)}(\lambda^2)\underset{\vec{\lambda}\to 0}{\to}\frac{1}{M^2}\ln\left(\frac{M}{\lambda}\right),\tag{A29}$$

and for $k>1$

$$\phi_k(0)=\frac{2}{(k-1)\rho\rho_1}\ln x,\quad \phi_k^{(0)}(0)=\frac{1}{(k-1)M^2}.\tag{A30}$$

Consequently, g_0 is the only moment which is infrared divergent. We have

$$g_0=-N_1\phi_1(\lambda^2)+N_m(\Lambda^2)^m T^{m-1}\left\{\frac{1}{\Lambda^2}\phi_1(\Lambda^2)\right\},\tag{A31}$$

where $N_m=-i\pi^2N_m'$.

Using the above results, the terms $ggg,\ldots,sss$ can now be expressed in terms of the functions ϕ_k. We get

$$ggg=-ie^2F(q^2)\left\{\begin{array}{c}-2(2M^2-q^2)N_1\phi_1(\lambda^2)\\ +2(2M^2-q^2)\left[S^{m-1}\left\{\frac{1}{\Lambda^2}\phi_1(\Lambda^2)\right\}-S^{m-1}\left\{\frac{1}{\Lambda^2}[\phi_2(\Lambda^2)-\phi_2(0)]\right\}\right]\\ +S^{m-1}\{\phi_2(\Lambda^2)\}-4M^2S^{m-1}\left\{\frac{1}{\Lambda^2}[\phi_3(\Lambda^2)-\phi_3(0)]\right\}\end{array}\right\}\gamma_\mu$$
$$-ie^2F(q^2)\left\{-4M^2S^{m-1}\left\{\frac{1}{\Lambda^2}[\phi_2(\Lambda^2)-\phi_2(0)]\right\}+4M^2S^{m-1}\left\{\frac{1}{\Lambda^2}[\phi_3(\Lambda^2)-\phi_3(0)]\right\}\right\}\frac{i\sigma_{\mu\nu}q^\nu}{2M},\tag{A32}$$

$$gsg=-ie^2\kappa F(q^2)\left\{-q^2S^{m-1}\left\{\frac{1}{\Lambda^2}[\phi_2(\Lambda^2)-\phi_2(0)]\right\}\right\}\gamma_\mu$$
$$-ie^2\kappa F(q^2)\left\{2(2M^2-q^2)\left[\begin{array}{c}-N_1\phi_1(\lambda^2)+S^{m-1}\left\{\frac{1}{\Lambda^2}\phi_1(\Lambda^2)\right\}\\ -S^{m-1}\left\{\frac{1}{\Lambda^2}[\phi_2(\Lambda^2)-\phi_2(0)]\right\}\end{array}\right]\right\}\frac{i\sigma_{\mu\nu}q^\nu}{2M},\tag{A33}$$

$$ggs+sgg=-2ie^2\kappa F(q^2)\left\{\begin{array}{c}\left(\frac{q^2}{4}-3M^2\right)S^{m-1}\left\{\frac{1}{\Lambda^2}[\phi_3(\Lambda^2)-\phi_3(0)]\right\}\\ +\frac{3}{2}S^{m-1}\left\{\phi_1(\Lambda^2)-\frac{1}{2}\phi_2(\Lambda^2)\right\}\end{array}\right\}\gamma_\mu$$
$$-2ie^2\kappa F(q^2)\left\{\begin{array}{c}2M^2S^{m-1}\left\{\frac{1}{\Lambda^2}[\phi_3(\Lambda^2)-\phi_3(0)]\right\}\\ -4S^{m-1}\left\{\phi_1(\Lambda^2)-\frac{1}{2}\phi_2(\Lambda^2)\right\}\end{array}\right\}\frac{i\sigma_{\mu\nu}q^\nu}{2M},\tag{A34}$$

$$gss+ssg=-2ie^2\kappa^2F(q^2)\frac{q^2}{4M^2}\left\{\begin{array}{c}-2M^2S^{m-1}\left\{\frac{1}{\Lambda^2}[\phi_3(\Lambda^2)-\phi_3(0)]\right\}\\ -2S^{m-1}\left\{\phi_1(\Lambda^2)-\frac{1}{2}\phi_2(\Lambda^2)\right\}\end{array}\right\}\gamma_\mu$$

$$-2ie^2\kappa^2F(q^2)\left\{\begin{array}{c}\left(\frac{3}{4}q^2-M^2\right)S^{m-1}\left\{\frac{1}{\Lambda^2}[\phi_3(\Lambda^2)-\phi_3(0)]\right\}\\ -\frac{1}{2}S^{m-1}\left\{\phi_1(\Lambda^2)-\frac{1}{2}\phi_2(\Lambda^2)\right\}\end{array}\right\}\frac{i\sigma_{\mu\nu}q^\nu}{2M}, \quad \text{(A35)}$$

$$sgs=-ie^2\kappa^2F(q^2)\left\{\begin{array}{c}-2M^2S^{m-1}\left\{\frac{1}{\Lambda^2}[\phi_3(\Lambda^2)-\phi_3(0)]\right\}-\frac{1}{2}S^{m-1}\{\phi_3(\Lambda^2)\}\\ +S^{m-1}\{\phi_1(\Lambda^2)\}+\frac{1}{2}S^{m-1}\{\phi_2(\Lambda^2)\}\\ -\frac{1}{2M^2}S^{m-1}\left\{\Lambda^2\left[\phi_1(\Lambda^2)-\frac{1}{4}\phi_2(\Lambda^2)\right]\right\}\end{array}\right\}\gamma_\mu$$

$$-ie^2\kappa^2F(q^2)\left\{\begin{array}{c}2M^2S^{m-1}\left\{\frac{1}{\Lambda^2}[\phi_3(\Lambda^2)-\phi_3(0)]\right\}\\ -2S^{m-1}\{\phi_1(\Lambda^2)\}+\frac{1}{2}S^{m-1}\{\phi_3(\Lambda^2)\}\end{array}\right\}\frac{i\sigma_{\mu\nu}q^\nu}{2M}, \quad \text{(A36)}$$

$$sss=-ie^2\kappa^3F(q^2)\frac{q^2}{4M^2}\left\{\begin{array}{c}-2M^2S^{m-1}\left\{\frac{1}{\Lambda^2}[\phi_3(\Lambda^2)-\phi_3(0)]\right\}+\frac{1}{2}S^{m-1}\{\phi_3(\Lambda^2)\}\\ +2S^{m-1}\{\phi_1(\Lambda^2)-\phi_2(\Lambda^2)\}\end{array}\right\}\gamma_\mu$$

$$-ie^2\kappa^3F(q^2)\left\{\begin{array}{c}\frac{1}{2}q^2S^{m-1}\left\{\frac{1}{\Lambda^2}[\phi_3(\Lambda^2)-\phi_3(0)]\right\}-\left(1+\frac{q^2}{2M^2}\right)S^{m-1}\{\phi_1(\Lambda^2)\}\\ -\frac{1}{2}S^{m-1}\{\phi_3(\Lambda^2)\}\\ +\frac{1}{2}\left(1+\frac{q^2}{M^2}\right)S^{m-1}\{\phi_2(\Lambda^2)\}-\frac{1}{4M^2}S^{m-1}\{\Lambda^2[\phi_1(\Lambda^2)-\phi_2(\Lambda^2)]\}\end{array}\right\}\frac{i\sigma_{\mu\nu}q^\nu}{2M}, \quad \text{(A37)}$$

where $S^{m-1}=N_m(\Lambda^2)^mT^{m-1}$.

It should be noted that the terms with $\phi_1(\lambda^2)$ appear only in ggg and gsg. They constitute the well-known infrared divergence. They are, apart from the hard-photon proton interaction (2.1) independent of the proton form factor (in this case independent of Λ and M). This is to be expected, since this term is cancelled by a similar infrared divergent term coming from the cross section for the emission of a real soft photon, which is given by the elastic cross section multiplied by a factor independent of the proton form factor.

APPENDIX B: THE FUNCTIONS $\phi_k(\Lambda^2)$

In this Appendix we derive the three-term recurrence relation for the function $\phi_k(\Lambda^2)$ given in Eq. (A22) as well as the expression for the function $\phi_1(\Lambda^2)$, defined in Eq. (A19) and given in terms of Spence functions in Eq. (A26). Integrating first over y in Eq. (A19) we have

$$\phi_k(\Lambda^2)=\frac{1}{\rho_1}\int_0^1\frac{x^{k-1}}{R}\ln\left\{\frac{R+x\rho_1}{R-x\rho_1}\right\}dx, \quad \text{(B1)}$$

where $R^2=\rho^2x^2+4(1-x)\Lambda^2$, $\rho_1^2=-q^2>0$, and $x=(\rho+\rho_1)^2/4M^2$. Noting that

$$\frac{d}{dx}\{x^kR\}=x^{k-1}\{kR^2+\rho^2x^2-2x\Lambda^2\}R^{-1}, \quad \text{(B2)}$$

we obtain

$$(k+1)\rho^2\phi_{k+2}-2(2k+1)\Lambda^2\phi_{k+1}+4k\Lambda^2\phi_k = \frac{2}{\rho_1}\int_0^1 \ln\left\{\frac{R+x\rho_1}{R-x\rho_1}\right\} d(x^k R). \tag{B3}$$

Integration by parts then gives

$$(k+1)\rho^2\phi_{k+2}-2(2k+1)\Lambda^2\phi_{k+1}+4k\Lambda^2\phi_k = \frac{2\rho}{\rho_1}\ln\left(\frac{\rho+\rho_1}{\rho-\rho_1}\right)-2\Lambda^2\int_0^1 \frac{x^k(2-x)}{M^2x^2+(1-x)\Lambda^2} \tag{B4}$$

from which the recursion (A22) follows at once, using Eq. (A23).

From Eq. (B4) it is clear that ϕ_2 and ϕ_3 follow once we have evaluated ϕ_1. Setting $k=1$ in Eq. (A19) and integrating first over x gives

$$\int_0^1 \frac{x\,dx}{p_y^2x^2+\Lambda^2(1-x)} = \frac{1}{2p_y^2}\left[\ln\left(\frac{p_y^2}{\Lambda^2}\right)+\frac{\Lambda}{\Delta}\ln\left(\frac{\Lambda+\Delta}{\Lambda-\Delta}\right)\right], \tag{B5}$$

where $\Delta^2=\Lambda^2-4p_y^2$. We next make the change of variable $y=(1+\omega)/2$, which gives $\Delta^2=\rho_1^2\omega^2+\Lambda^2-\rho^2$, and then make the further change of variable $\Delta=\rho_1\omega+s$, from which

$$\omega=\frac{\Lambda^2-\rho^2-s^2}{2\rho_1 s}, \quad \Delta=\frac{\Lambda^2-\rho^2+s^2}{2s}. \tag{B6}$$

Integrating Eq. (A19) over y gives

$$\phi_1(\Lambda^2)=\frac{2}{\rho_1}\int_{s_-}^{s_+}\frac{ds}{\rho^2-(\Lambda-s)^2}\ln\left[\frac{(\Lambda+s)^2-\rho^2}{4s\Lambda}\right] - \frac{2}{\rho_1}\int_{s_-}^{s_+}\frac{ds}{(\Lambda+s)^2-\rho^2}\ln\left[\frac{\rho^2-(\Lambda-s)^2}{4s\Lambda}\right], \tag{B7}$$

where $s_\pm=\Lambda_1\pm\rho_1$, $\Lambda_1^2=\Lambda^2-4M^2$. Factoring the expressions which appear as factors to the logarithms as well as in their arguments, ϕ_1 can be further reduced to partial fractions. Performing explicitly some of the occurring integrals we obtain

$$\phi_1(\Lambda^2)=\frac{1}{\rho\rho_1}\left\{\ln\left(\frac{\alpha_-}{\alpha_+}\right)\ln\left(\frac{4\sigma_+\sigma_-}{(1-\alpha_+^2)(1-\alpha_-^2)}\right) -2\ln\left(\frac{\alpha_-}{\alpha_+}\right)\ln(2\Lambda)+L(1-\alpha_+^2)-L(1-\alpha_-^2)\right\}, \tag{B8}$$

where $\sigma_\pm=\Lambda\pm\rho$ and $\alpha_\pm=(\rho\mp\rho_1)/(\Lambda+\Lambda_1)$. In Eq. (B8) L is the dilogarithm (Spence) function, defined as

$$L(z)=-\int_0^z \frac{\ln(1-t)}{t}dt. \tag{B9}$$

APPENDIX C: FINAL ELECTRON DETECTOR ACCEPTANCE

In this Appendix, we express $\Delta\epsilon$, the maximum momentum of the photon in the frame S^0, in terms of the final electron detector acceptance in the lab frame ΔE. In S^0 ($\mathbf{p}_4+\mathbf{k}=0$), if $|\mathbf{k}|=\Delta\epsilon\ll M$, we have, from $(p_1+p_2-p_3)^2=(p_4+k)^2$, neglecting terms of order $(\Delta\epsilon/M)^2$ and $(m/M)^2$,

$$p_2\cdot(p_1-p_3)-p_1\cdot p_3=M\Delta\epsilon. \tag{C1}$$

Writing this in terms of lab frame energies, we have, for high energies

$$M(\epsilon_1-\epsilon_3)-\epsilon_1\epsilon_3(1-\cos\theta)=M\Delta\epsilon. \tag{C2}$$

For elastic scattering in the lab frame, we have

$$M(\epsilon_1-\epsilon_3^{el})-\epsilon_1\epsilon_3^{el}(1-\cos\theta)=0. \tag{C3}$$

Subtracting gives

$$\Delta E\left(1+\frac{\epsilon_1}{M}(1-\cos\theta)\right)=\Delta\epsilon, \tag{C4}$$

where

$$\Delta E=\epsilon_3^{el}-\epsilon_3. \tag{C5}$$

Thus, in terms of lab frame quantities we have

$$\Delta\epsilon=\eta\Delta E. \tag{C6}$$

APPENDIX D: HIGH-ENERGY APPROXIMATION FOR $S_{ij}^{(2)}$

In this Appendix we give the high-energy approximation of the terms $S_{ij}^{(2)}$ defined in Eq. (4.10), in which we note in particular that for $i=1$ or 3 we have $l\cdot t=(\alpha p_i-p_j)\cdot t\approx\alpha p_i\cdot t$. Using transformations of the Spence functions [3], p. 389 (B.3),

$$L(z)=-L\left(\frac{1}{z}\right)-\frac{1}{6}\pi^2-\frac{1}{2}\ln^2(-z), \tag{D1}$$

$$L(z)=-L\left(\frac{z}{z-1}\right)-\frac{1}{2}\ln^2(1-z), \tag{D2}$$

the terms in $S_{ij}^{(2)}$ simplify considerably. We then obtain

$$S_{12}^{(2)} = -\ln^2\left(\frac{2\epsilon_3}{m}\right) - \ln^2 x + \frac{1}{2}\ln^2\left(\frac{x}{\eta}\right) - \frac{1}{6}\pi^2 - L\left(1 - \frac{1}{x\eta}\right) + L\left(1 - \frac{\eta}{x}\right), \tag{D3}$$

$$S_{32}^{(2)} = -\ln^2\left(\frac{2\epsilon_1}{m}\right) - \ln^2(x) + \frac{1}{2}\ln^2(x\eta) - \frac{1}{6}\pi^2 + L\left(1 - \frac{1}{x\eta}\right) - L\left(1 - \frac{\eta}{x}\right), \tag{D4}$$

$$S_{14}^{(2)} = -\ln^2\left(\frac{2\epsilon_3}{m}\right) - \frac{1}{6}\pi^2, \tag{D5}$$

$$S_{34}^{(2)} = -\ln^2\left(\frac{2\epsilon_1}{m}\right) - \frac{1}{6}\pi^2, \tag{D6}$$

$$S_{13}^{(2)} = -\ln^2\left(\frac{2\epsilon_1}{m}\right) - \ln^2\left(\frac{2\epsilon_3}{m}\right) - \frac{1}{3}\pi^2 + \frac{1}{2}\ln^2\left(\cos^2\frac{1}{2}\theta\right) + L\left(\cos^2\frac{1}{2}\theta\right), \tag{D7}$$

$$S_{24}^{(2)} = \frac{1}{2}\ln^2(x) + \frac{1}{2}L\left(1 - \frac{1}{x^2}\right). \tag{D8}$$

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