Critical line for *H* superfluidity in strange quark matter?

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(Received 3 January 2000; published 8 August 2000)

Based upon the analogy to the electroweak phase diagram, I propose that in QCD there might be a critical line for a superfluid transition, in the plane of chemical potential and temperature. The order parameter has the quantum numbers of the *H* dibaryon, but the transition is driven by color superconductivity in strange quark matter.

PACS number(s): 21.65.+f, 12.38.Mh, 14.20.Pt

In QCD, there is a phase transition to a color superconducting phase at high quark density and low temperature [1–24]. At densities of interest for the collisions of heavy ions or quark stars, ''$2+1$'' flavors of quarks—up, down, and strange—enter.

The order of the phase transition to a color superconducting phase at zero temperature, as function of the quark chemical potential, was analyzed previously [6]. The zero temperature transition is simple because the effective theory is four dimensional over large distances [25]. For a second order transition, couplings can only flow into the origin, with mean field behavior corrected by logarithms. Most importantly, quark loops screen gluons, so that gluons do not contribute over long distances. For $2+1$ flavors, this analysis predicts a first order transition [7].

The transition at nonzero temperature is much more complicated. Over large distances, the effective theory is three dimensional; a second order transition typically flows toward a fixed point which lies in a regime of strong coupling. Also, while static electric fields are screened by quark loops, static magnetic fields are not. Thus the phase transition involves scalar fields coupled to gauge fields in three dimensions.

In this paper I consider the effective theories which are of relevance for the phase transition to color superconductivity for $2+1$ flavors of quarks [6]. This enables me to unify a large number of model dependent results in a simple manner. Because of an instanton induced term [4,14], I find one surprise. As suggested previously [1,3,4,6,19,23], in a chirally symmetric, color superconducting phase, an (approximate) spontaneous violation of parity can be large. The pattern, however, is unexpected: *if* instantons are important, then (approximate) parity violation is greater for the up-strange and the down-strange superconducting condensates than it is for the up-down condensate.

While the phase transitions of scalars coupled to gauge fields in three dimensions is a complicated problem, because of the possibility of generating a cosmological baryon asymmetry at the electroweak scale, much is known about such phase diagrams from numerical simulations on the lattice [27,28]. Using this information, I conjecture how the phase diagrams for the effective three-dimensional theories for color superconductivity might look. Following especially the phase diagram for adjoint scalars coupled to a SU(3) gauge field [28], I suggest that for $2+1$ flavors, there might be a *line* of second order phase transitions, in the plane of chemical potential and temperature. The transition is induced by color superconductivity for $2+1$ flavors, assuming color-flavor locking [4]. Even so, it is properly a superfluid transition, where the order parameter is an operator for the *H* dibaryon [5,6]. As in ordinary superfluidity, ''*H* superfluidity'' lies in the universality class of an $O(2)$ vector.

All of my arguments are qualitative and, on occasion, speculative. However, the phase diagram for the effective three-dimensional theory is directly testable by lattice simulations involving only scalars and gauge fields. The effective theory in three dimensions can be computed analytically using perturbation theory in QCD. While these (perturbative) coefficients are of course only valid at very high densities, by comparing to the lattice results, one could at least estimate if the critical line for *H* superfluidity might begin. If a critical line does occur, it is manifestly of experimental interest, as is a critical end point for the chiral phase transition [29].

I. EFFECTIVE THEORIES

In this section I first review the order parameters for color superconductivity with massless quarks [1,6,7,10], and then use them to construct effective Lagrangians in a standard fashion. I assume that if a condensate with (total) spin zero can form—as is true for two and three flavors—that it does, and dominates over condensates with higher spin.

A. General analysis

Massless quarks naturally decompose into eigenstates of chirality. In a Fermi sea, particles have zero energy near the Fermi surface, and dominate over antiparticles, which always have nonzero energy. Thus it is natural to introduce projectors for chirality and energy,

$$\mathcal{P}_{R,L}=\frac{1}{2}(1\pm\gamma_5),\quad \mathcal{P}^{\pm}=\frac{1}{2}(1\pm\gamma_0\gamma\cdot\hat{k}), \tag{1}$$

where $\vec{k}$ is the momentum of the quark, and $\vec{k}=k\hat{k}$, $\hat{k}^2=1$. There are then four types of quark fields, right and left handed, and particle and antiparticle.

Quarks transform under a local gauge group of $SU(3)_c$ color; the color indices of the fundamental representation are denoted by $i,j=1,2,3$. For N_f flavors of massless quarks,

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with flavor indices $a,b=1\ldots N_f$, classically there is also a global flavor symmetry of $\mathrm{SU}(N_f)_R\times\mathrm{SU}(N_f)_L\times\mathrm{U}_A(1)\times\mathrm{U}(1)$. A right handed particle is given by

$$q^+_{i,a_R}=\mathcal{P}^+\,\mathcal{P}_R\,q_{i,a}\,, \tag{2}$$

where $q_{i,a}$ is a quark field with color i, flavor a, and momentum $\vec{k}$.

There are two right handed superconducting condensates with total spin zero: between two right handed particles, or two right handed antiparticles,

$$\Phi^{\pm}_{i,j;a_R,b_R}=(q^{\pm}_{i,a_R})^T C q^{\pm}_{j,b_R}\,, \tag{3}$$

and similarly for the left handed condensates. q has momentum $\vec{k}$, q^T is the Dirac transpose of a quark with momentum $-\vec{k}$, and C is the charge conjugation matrix. Gaps for (total) spin 1 are constructed similarly [10].

Superconductivity is due to pairing of particles near the Fermi surface, so it is natural to expect that only the particle condensates Φ^+, matter, and that the antiparticle condensates Φ^- can be neglected. In an effective Lagrangian approach, this happens as follows. As is evident from Eq. (3), for every particle condensate there is a corresponding antiparticle condensate. Thus in an effective Lagrangian the two fields mix,

$$g^2\,\mathrm{tr}[(\Phi^-)^\dagger\Phi^+ + \mathrm{c.c.}]+m_-^2\,\mathrm{tr}(|\Phi^-|^2). \tag{4}$$

I assume that Φ^- does not condense on its own, so that it has a positive mass squared $m_-^2>0$. For free fields, Φ^+ and Φ^- do not mix, but they do at $\sim g^2$, since interactions invariably mix particles and antiparticles [24]. Here g is the QCD coupling constant, although perhaps the g^2 is only a g, due to a logarithmic enhancement from forward scattering [8–15,24]. Whatever the value of the mixing term, though, excluding isolated points in the phase diagram, there is no generic reason why it should vanish. With Eq. (4), when Φ^+ condenses, it becomes a term linear in Φ^-, so it also condenses, $\langle\Phi^-\rangle\sim g^2\langle\Phi^+\rangle$. But the critical behavior, where $\langle\Phi^+\rangle\to 0$, is dominated by Φ^+ alone. Thus I consider only the particle condensates and drop the "+" superscript, $\Phi=\Phi^+$.

In addition to those for color superconductivity, I also require the order parameters for chiral symmetry breaking. Chiral symmetry is broken by a condensate between an antiquark and a quark. From group theory, the product of a color antitriplet and a triplet is a singlet plus an octet, $\bar{\mathbf{3}}\times\mathbf{3}=\mathbf{1}+\mathbf{8}$. There are then two chiral order parameters: a color singlet

$$\psi_{a_L,b_R}=\overline{q}_{i,a_L}\,q_{i,b_R} \tag{5}$$

and a color adjoint field

$$\tilde{\psi}^C_{a_L,b_R}=\overline{q}_{i,a_L}t^C_{ij}q_{j,b_R}; \tag{6}$$

t^C_{ij} is the generator for $\mathrm{SU}(3)_c$, with the adjoint index $C=1\ldots 8$. In the vacuum,

$$\langle\overline{q}_{i,a_L}\,q_{j,b_R}\rangle=\psi_0\frac{\delta_{ij}}{3}\,\delta_{a_Lb_R}; \tag{7}$$

this leaves $\mathrm{SU}(3)_c$ unbroken, and breaks the flavor $\mathrm{SU}(N_f)_R\times\mathrm{SU}(N_f)_L\to\mathrm{SU}(N_f)$. The color singlet chiral field develops an expectation value $\langle\psi\rangle\sim\psi_0$, and the color adjoint chiral field does not, $\langle\tilde{\psi}\rangle=0$.

Using this inelegant notation, one can write down how the fields transform under the non-Abelian symmetries. What is simpler and more useful is how they transform under the Abelian flavor symmetries of $\mathrm{U}_A(1)\times\mathrm{U}(1)$. Suppressing the color and flavor indices, the quark fields transform as

$$q_R\to \mathrm{e}^{i(\theta+\theta_A)}\,q_R\,,\quad q_L\to \mathrm{e}^{i(\theta-\theta_A)}\,q_L\,, \tag{8}$$

so the condensate fields transform as

$$\Phi_R\to\mathrm{e}^{2i(\theta+\theta_A)}\Phi_R\,,\quad \Phi_L\to\mathrm{e}^{2i(\theta-\theta_A)}\Phi_L\,,$$
$$\psi\to\mathrm{e}^{2i\theta_A}\,\psi,\quad \tilde{\psi}\to\mathrm{e}^{2i\theta_A}\,\tilde{\psi}. \tag{9}$$

θ generates rotations for the $\mathrm{U}(1)$ symmetry of quark number, which is an exact symmetry of the Lagrangian. In contrast, θ_A generates a rotation for the $\mathrm{U}_A(1)$ symmetry of anomalous quark number; this is badly broken in the vacuum, but at high density or temperature, is very nearly a good symmetry of the Lagrangian [31]. Note that $\Phi_{R,L}$ transform nontrivially under both $\mathrm{U}_A(1)$ and $\mathrm{U}(1)$, while ψ and $\tilde{\psi}$ transform only under the anomalous $\mathrm{U}_A(1)$.

Color superconductivity involves quarks pairing with quarks, so Fermi statistics implies a nontrivial relation. For a spin zero condensate, Φ must be symmetric in the simultaneous exchange of color and flavor indices

$$\Phi^T_{R,L}=+\Phi_{R,L}\,. \tag{10}$$

(Condensates with spin one satisfy a more complicated relationship, but are essentially antisymmetric [10].) There is no such relationship for chiral symmetry breaking, which involves the condensation of quarks with antiquarks.

Group theory tells us that the product of two color triplets is an antitriplet plus a sextet, $\mathbf{3}\times\mathbf{3}=\bar{\mathbf{3}}_a+\mathbf{6}_s$; the subscripts denote antisymmetric and symmetric representations, respectively. By Eq. (10), the color antitriplet piece of Φ, which I denote ϕ, combines with an antisymmetric flavor representation, while the color sextet part χ combines with a symmetric flavor representation. Under singlet gluon exchange, the antitriplet channel is attractive, and the sextet repulsive.

Defering the precise definitions of ϕ and χ for now, the lowest order effective Lagrangian, including gauge interactions, but neglecting terms which are nonlinear in the condensate fields, is

$$\mathcal{L}^0=\mathcal{L}^0_\psi+\mathcal{L}^0_{\tilde{\psi}}+\mathcal{L}^0_\phi+\mathcal{L}^0_\chi+\mathcal{L}^m_\psi+\mathcal{L}_g\,. \tag{11}$$

For massless quarks, the effective Lagrangian is composed of four terms: for the color singlet chiral field

$$\mathcal{L}^0_\psi=\mathrm{tr}(|\partial_\alpha\psi|^2)+m^2_\psi\,\mathrm{tr}(\psi^\dagger\psi), \tag{12}$$

the color adjoint chiral field

$$\mathcal{L}^0_{\tilde{\psi}} = \mathrm{tr}(|D_\alpha \tilde{\psi}|^2) + m^2_{\tilde{\psi}} \mathrm{tr}(\tilde{\psi}^\dagger \tilde{\psi}), \tag{13}$$

the color antitriplet superconducting field

$$\mathcal{L}^0_{\phi} = \mathrm{tr}(|D_\alpha \phi|^2) + m^2_{\phi} \mathrm{tr}(\phi^\dagger \phi), \tag{14}$$

and the color sextet superconducting field

$$\mathcal{L}^0_{\chi} = \mathrm{tr}(|D_\alpha \chi|^2) + m^2_{\chi} \mathrm{tr}(\chi^\dagger \chi); \tag{15}$$

D_α is the covariant color derivative in the appropriate representation. For massive quarks, one also needs

$$\mathcal{L}^m_{\psi} = + \mathrm{tr}(\psi \mathcal{M}). \tag{16}$$

The diagonal elements of $\mathcal{M}$ are proportional to the current quark masses, $\mathcal{M}_{a_L b_R} \sim m_a \delta_{a_L b_R}$, with m_a is the current quark mass for flavor a. From current algebra and lattice simulations [30], the quark masses for up, down, and strange are $m_u \sim 4$, $m_d \sim 8$, and $m_s \sim 100$ MeV, respectively.

The Lagrangian for the color gauge field $\mathcal{L}_g$ is the usual action plus a term for hard dense loops [33]. I assume that the Debye mass for hard dense loops is always nonzero.

While these terms are all completely standard, given the multiplicity of fields, it helps to be explicit. I assume that the adjoint chiral field and the color sextet field always represent repulsive channels, with positive mass squared:

$$m^2_{\tilde{\psi}} > 0, \quad m^2_{\chi} > 0. \tag{17}$$

In contrast, one expects that at low densities and temperature, chiral symmetry is broken in the color singlet channel, $m^2_{\psi} < 0$; if all current quark masses vanish, $\mathcal{M} = 0$, the pattern is

$$\langle \psi_{a_L, b_R} \rangle = \psi_0 \delta_{a_L, b_R}, \tag{18}$$

as is consistent with Eq. (7). Chiral symmetry is restored at high density or temperature, $m^2_{\psi} > 0$. Ignoring the coupling to other fields, ψ still develops an expectation value from the mass term $\mathcal{M}$

$$\langle \psi_{a_L, b_R} \rangle = \psi' m_a \delta_{a_L, b_R}. \tag{19}$$

For color superconductivity, I assume that the color antitriplet channel is favored at high density [8–12,14,15,24], with $m^2_{\phi} < 0$, and disfavored at low density, with $m^2_{\phi} > 0$. How the chiral transition and color superconductivity are coupled is one of the principle questions to be addressed.

B. Two flavors

I start with the case of two flavors. For flavor SU(2), $\mathbf{2} \times \mathbf{2} = \mathbf{1}_a + \mathbf{3}_s$. The color antitriplet superconducting field is then a flavor singlet [1]:

$$\phi_{i,R} = \epsilon^{ijk} \epsilon^{a_R b_R} \Phi_{j,k;a_R, b_R}. \tag{20}$$

For two flavors I ignore the adjoint chiral field and the color sextet field for color superconductivity, since they always vanish: $\langle \tilde{\psi} \rangle = \langle \chi \rangle = 0$ at all densities. Under the Abelian flavor symmetries, $\phi_{i,R}$ transforms as Φ_R, Eq. (9), etc.

Many interaction terms need to be added to $\mathcal{L}^0$; those which violate $\mathrm{U}_A(1)$ are especially interesting. For N_f flavors of massless quarks, the zero modes of an instanton with topological charge Q generate an interaction between QN_f right-handed quarks and QN_f left-handed antiquarks [31]. From Eq. (9), the corresponding operators transform as $\exp(2iQN_f\theta_A)$ under $\mathrm{U}_A(1)$ rotations. In vacuum instanton effects are large, since they give the η' its mass; thus they must continue to be important in a hadronic phase, at small chemical potential. Conversely, semiclassical methods are valid at large chemical potential, and it is certain then that instantons are very dilute. At intermediate chemical potential, it is not clear how the density of instantons is correlated with chiral symmetry breaking and color superconductivity. I discuss what might happen if the density of instantons is large in a chirally symmetric, color superconducting phase, but this might not occur in QCD: the density of instantons might drop precipitously when chiral symmetry is restored.

For two flavors, single instantons generate a determinantal term for the chiral fields [31],

$$\mathcal{L}^I_{\psi} \sim - \det(\psi), \tag{21}$$

which is quadratic in the ψ's. The superscript I is used to denote that the term is induced by instantons. The overall minus sign in Eq. (21) is important [31]. At $\theta = 0$, the instanton term not only acts to make the η meson, which has spin parity $J^P = 0^-$, massive, but it also drives chiral symmetry breaking in the 0^+ channel.

Single instantons generate a similar term for the ϕ's [16],

$$\mathcal{L}^I_{\phi} \sim - (\phi^*_{i,L} \phi_{i,R} + \phi^*_{i,R} \phi_{i,L}). \tag{22}$$

As for $\mathcal{L}^I_{\psi}$, I write $\mathcal{L}^I_{\phi}$ with an overall minus sign, so that it acts to drive color superconductivity [16].

In addition to the terms induced by single instantons $\mathcal{L}^I_{\psi}$ and $\mathcal{L}^I_{\phi}$, there are also terms induced by instantons with topological charge 2. Operators induced by $Q = 2$ instantons include $(\mathcal{L}^I_{\psi})^2$, $\mathcal{L}^I_{\psi} \mathcal{L}^I_{\phi}$, and $(\mathcal{L}^I_{\phi})^2$. For two flavors these operators are a curiosity, but they arise naturally for three flavors.

At high densities, where $\mathrm{U}_A(1)$ is essentially restored, one can write terms which respect axial $\mathrm{U}_A(1)$ by squaring each term in Eqs. (21) and (22), such as

$$|\det(\psi)|^2, \quad |\phi^*_{i,L} \phi_{i,R}|^2. \tag{23}$$

There are also terms which couple ϕ to ψ, such as

$$\mathrm{tr}(\psi^\dagger \psi)(|\phi_R|^2 + |\phi_L|^2) \tag{24}$$

and

$$\det(\psi)^* \phi^*_{i,L} \phi_{i,R} + \mathrm{c.c.} \tag{25}$$

The effect of Eqs. (25) and (24) is to couple the transition for color superconductivity to that for chiral symmetry breaking. Model dependent analyses indicate that chiral symmetry breaking and color superconductivity can coexist for some range of densities [16,17,19]. However, all such models manifestly leave out confinement: while two quarks may like to bind together in a color superconducting condensate, in a phase with chiral symmetry breaking—and so confinement—this could well be overwhelmed by the tendency of three quarks to form a color singlet baryon; see, also, Ref. [26]. On this basis, I assume that the sign of the coupling constants in Eqs. (25) and (24) is positive, so that chiral symmetry breaking suppresses color superconductivity.

When color superconductivity occurs, and $\mathcal{L}^{I}_{\phi}$ is important, the preferred condensate is

$$\langle \phi_{i,(R,L)} \rangle = e^{i\theta_{R,L}} \phi_0 \, \delta_{i3}, \quad \theta_R = \theta_L ; \tag{26}$$

a global color rotation is done to align the condensate in the color-3 direction. This breaks $SU(3)_c \to SU(2)_c$, and leaves flavor unbroken; ϕ_0 is real. There are two types of correlations in these expectation values. First, the phases of $\phi_{i,R}$ and $\phi_{i,L}$ are equal, $\theta_R = \theta_L$. Parity switches right and left handed fields, so if both fields have the same phase, it implies that the condensate has spin parity $J^P = 0^+$. Secondly, with Eq. (26) the direction of the right and left handed condensates are the same in color space [22,23].

What happens at high densities, when instantons are very dilute? There is always some density of instantons about, and they generate a term such as $\mathcal{L}^{I}_{\phi}$, albeit with a small coefficient. In this limit, $U_A(1)$ symmetry is effectively restored, and θ_R and θ_L are not correlated, except over very large scales. This is the (approximate) spontaneous breaking of parity [1,3,4,6,19,23]. Phrased in another way, the η meson is very light: its mass is determined by Eq. (25), $m_\eta^2 \sim m_u m_d$ [6].

What about the coupling between the directions of $\phi_{i,R}$ and $\phi_{i,L}$ in color space? While the instanton term is no longer important, there are many other terms in the effective Lagrangian which couple the color direction of the two condensates. One example is Eq. (23); in weak coupling, this first appears at $\sim g^4$ [22,23], where g is the QCD coupling constant. Thus while the phase of the right and left handed condensates are not (strongly) correlated at high density, they are correlated in color; for a dynamical explanation, see Ref. [23].

C. Three flavors

This completes my discussion for two flavors. In QCD, the case of interest for dense quark matter is really that of three flavors. A chemical potential does not matter until it is greater than the mass of a particle, so there is no Fermi sea until the quark chemical potential μ is greater than one third of the nucleon mass $\mu > 313$ MeV (because of binding in nuclear matter, it is actually a little less). As μ is always at least three times the strange quark mass, any complete analysis must include three flavors. Of course this counting is only valid in a chirally symmetric phase; with chiral symmetry breaking, the constituent quark masses are large, ~ 313 MeV, and strange baryons are suppressed. In this region my caveats about confinement apply.

For three flavors, the color antitriplet superconducting field is a flavor antitriplet; for right handed particles, this is

$$\phi_{i,a_R} = \epsilon^{ijk} \epsilon^{a_R b_R c_R} \Phi_{j,k;b_R,c_R}. \tag{27}$$

I also introduce the color sextet, flavor sextet superconducting field by symmetrizing with respect to the color and flavor indices; for right handed particles,

$$\chi_{i,j;a_R,b_R} = \{[\Phi_{i,j;a_R,b_R} + (i \leftrightarrow j)] + (a_R \leftrightarrow b_R)\}. \tag{28}$$

My notation is somewhat confusing: for either color or flavor, the indices on ϕ are antitriplet, while those on χ are triplet.

There are several terms which are special to three flavors. For three flavors I keep track of all fields, including those which are not favored to condense: the color adjoint chiral field $\tilde{\psi}$ and the color sextet superconducting field χ. As will be seen, because of cubic operators they develop expectation values when color superconductivity occurs.

I first consider operators induced by single instantons. The simplest is a determinant for the chiral fields $\mathcal{L}^{I}_{\psi}$ in Eq. (21). This is just like that for two flavors, except now it is cubic in the component fields ψ_{a_L,b_R}. Analogously, there is also a determinantal operator for three color adjoint chiral fields

$$\mathcal{L}^{I}_{\tilde{\psi}} \sim \mathrm{tr}[\det(\tilde{\psi})] \sim d^{ABC} \epsilon^{a_R b_R c_R} \epsilon^{a_L b_L c_L} \tilde{\psi}^{A}_{a_R a_L} \tilde{\psi}^{B}_{b_R b_L} \tilde{\psi}^{C}_{c_R c_L} \tag{29}$$

[d^{ABC} is the symmetric structure constant for $SU(3)_c$], and between two color adjoint chiral fields and one color singlet chiral field

$$\mathcal{L}^{I}_{\psi\tilde{\psi}} \sim \epsilon^{a_R b_R c_R} \epsilon^{a_L b_L c_L} \psi_{a_R a_L} \tilde{\psi}^{C}_{b_R b_L} \tilde{\psi}^{C}_{c_R c_L}. \tag{30}$$

For the ϕ fields, in obvious analogy there are two cubic operators which are invariant under the non-Abelian symmetries, but transform under $U_A(1)$:

$$H_R = \det(\phi_{i,a_R}), \quad H_L = \det(\phi_{i,a_L}). \tag{31}$$

For later reference, I introduce

$$H_{\pm} = \frac{1}{2}(H_R \pm H_L). \tag{32}$$

$H_{\pm}$ has spin-parity $J^P = 0^{\pm}$, so H_+ has the quantum numbers of the H dibaryon [32]. However, unlike $\det(\psi)$ and $\det(\tilde{\psi})$, H_R and H_L cannot appear in an effective Lagrangian, because they transform not only under the anomalous $U_A(1)$, but also under the (good) $U(1)$ symmetry for quark number [5]. Using only the ϕ fields, one can construct terms which are invariant under all symmetries except $U_A(1)$:

$$\mathcal{L}_H^I \sim H_L^* H_R + H_R^* H_L . \tag{33}$$

This is similar to $\mathcal{L}_\phi^I$ for two flavors, but given the transformation properties of ϕ under $U_A(1)$, Eq. (9), $\mathcal{L}_H^I$ is not induced by single instantons, but by instantons with topological charge 2.

An operator induced by a single instanton is given by combining two ϕ's and one color singlet chiral field ψ [4,14]:

$$\mathcal{L}_{\phi\psi}^I \sim -(\phi_{i,a_L}^* \phi_{i,b_R} \psi_{a_L,b_R} + \text{c.c.}). \tag{34}$$

I assume the sign is negative, as it is for two flavors. Single instantons also induce a similar term between two ϕ's and one color adjoint chiral field $\tilde{\psi}$

$$\mathcal{L}_{\phi\tilde{\psi}}^I \sim \phi_{i,a_L}^* t_{i,j}^C \phi_{j,b_R} \tilde{\psi}_{a_L,b_R}^C + \text{c.c.} \tag{35}$$

I do not know the sign of Eq. (35), but it is unimportant.

As for two flavors, when chiral symmetry is broken in the color singlet channel, Eq. (18), $\mathcal{L}_{\phi\psi}^I$ helps to generate color superconductivity. I assume that quartic terms in the potential, such as

$$\text{tr}(\psi^\dagger \psi)\, \text{tr}(\phi_R^\dagger \phi_R + \phi_L^\dagger \phi_L), \tag{36}$$

where $\text{tr}(\phi_R^\dagger \phi_R) = \phi_{i,a_R}^* \phi_{i,a_R}$, etc., are sufficiently large and of positive sign, so that phases with chiral symmetry breaking and color superconductivity do not overlap.

This terminology is imprecise. Consider what happens when chiral symmetry is restored, $m_\psi^2 > 0$, so the expectation value of ψ is naively that of Eq. (19). The preferred condensate is color-flavor locked [4]:

$$\langle \phi_{i,a_{R,L}} \rangle = e^{i\theta_{R,L}} \phi_0\, \delta_{i,a_{R,L}}, \quad \theta_R = \theta_L ; \tag{37}$$

global color and flavor rotations are done to make the condensates diagonal. This pattern breaks $SU(3)_c \times SU(3)_L \times SU(3)_R \times U_A(1) \times U(1) \to SU(3)$. Because of $\mathcal{L}_{\phi\psi}^I$, the right and left handed condensates have the same phase, so the condensate has $J^P = 0^+$.

I remark that although the physics is very different, formally the pattern of symmetry breaking for color-flavor locking in Eq. (37) is identical to that for chiral symmetry breaking in (18) [20]. As for chiral symmetry breaking, Eq. (37) is not the only possible way in which color superconductivity could occur; for example, one might have $\langle \phi_{i,a_R} \rangle = \phi_0 \delta_{i,3} \delta_{i,a_R}$ [6]. It is easy to argue that this is disfavored [4,6]: such a condensate leaves at least two different colors and flavors ungapped, while with color-flavor locking, all colors and flavors of quarks are gapped.

Because of the instanton terms, however, when color superconductivity occurs, $\mathcal{L}_{\phi\psi}^I$ and $\mathcal{L}_{\phi\tilde{\psi}}^I$ become terms which are *linear* in ψ and $\tilde{\psi}$, respectively. Consequently, expectation values for ψ and $\tilde{\psi}$ are automatically generated when ϕ condenses. For the color singlet chiral field, this means that the expectation value of ψ never vanishes, even at high density in the chiral limit $\mathcal{M} = 0$. The color adjoint chiral field also develops an expectation value, as is seen in a three flavor instanton model [19].

Since the expectation value of ψ is always nonzero, there is no gauge invariant order parameter which distinguishes a phase with chiral symmetry breaking from one with color superconductivity, and at least formally, there is a continuity between strange hadronic matter and strange quark matter [5]. One might wonder if $\det(\tilde{\psi})$, Eq. (29), provides such an order parameter, but even though $\langle \tilde{\psi} \rangle = 0$ in the phase with chiral symmetry breaking, assuming the quark expectation value of Eq. (7), $\langle \det(\tilde{\psi}) \rangle \sim \psi_0^3 \neq 0$.

Even so, I argue that in QCD, phases with chiral symmetry breaking and color superconductivity appear to be rather different. In a hadronic phase, the relationship between chiral symmetry breaking and confinement helps us to understand the central mystery of nuclear physics: why the nuclear binding energy, ~ 16 MeV, is so small relative to *any* other scale in QCD [34]. The scale of hadronic superfluidity is smaller still, $\leqslant 3$ MeV [35]. If, as originally believed [1], the gaps for color superconductivity are also ~ 1 MeV, then continuity between hadronic and quark matter is automatic. From recent work with effective models, however, it appears that the color superconducting gaps are natural on a QCD scale, ~ 100 MeV [2,4,5,16–19], and so huge relative to the hadronic gaps. If true, I assume that this disparity in scales, by almost two orders of magnitude, is due to confinement.

Thus I distinguish between a phase driven by chiral symmetry breaking, where $m_\psi^2 < 0$ and $m_\phi^2 > 0$, from a phase driven by color superconductivity, with $m_\psi^2 > 0$ and $m_\phi^2 < 0$. At high density, the instanton terms $\mathcal{L}_{\phi\psi}^I$ and $\mathcal{L}_{\phi\tilde{\psi}}^I$ are very small, so the expectation values of ψ and $\tilde{\psi}$ are negligible.

At intermediate densities, the instanton term $\mathcal{L}_{\phi\psi}^I$ has several interesting effects. Remember that the ϕ field is antitriplet in the flavor indices. Thus the strange component, $\phi_{i,3}$ is an up-down condensate, while the up and down components $\phi_{i,1}$ and $\phi_{i,2}$, are condensates of down-strange and up-strange, respectively. Since $m_s \gg m_u, m_d$, $\mathcal{L}_{\phi\psi}^I$ is greatest for the up-down condensate $\sim -m_s \phi_{i,3}^* \phi_{i,3}$ and smallest for the down-strange and up-strange condensates $\sim -m_u \phi_{i,1}^* \phi_{i,1}$ and $\sim -m_d \phi_{i,2}^* \phi_{i,2}$. This is reasonable: because of the difference in the quark masses, it is easiest for color superconductivity to occur between up and down quarks, and hardest for it to form between up or down and strange quarks. With the overall minus sign, this is exactly what $\mathcal{L}_{\phi\psi}^I$ does.

As for $\mathcal{L}_\phi^I$ with two flavors, for three flavors $\mathcal{L}_{\phi\psi}^I$ correlates the overall phases of the right and left handed condensates. Because of the difference in the quark masses, though, it is most effective for the up-down condensate, and least effective for the up-strange and down-strange condensates, by a factor of $m_s/m_{u,d} \sim 20$. This implies that in a phase driven by color superconductivity, *if* instantons are important, then the (approximate) spontaneous violation of parity is *smallest* for the up-down condensate, and *greatest* for the up-strange and down-strange condensates. If instantons are not important, then all three condensates exhibit the same

(approximate) parity violation, and the η' is the lightest pseudo-Goldstone boson [21,22].

It is not clear how to observe the (approximate) spontaneous violation of parity in the up-strange and down-strange condensates. As an effect from a Fermi sea, this appears most directly in baryons; the pattern above suggests effects are large for Λ baryons, and negligible for any baryons which have two quarks of the same flavor. Any effect is obscured by the fact that even in vacuum, the decays of the Λ are not parity conserving, for reasons which are not well understood.

What about effects in a phase with (approximate) $U_A(1)$ symmetry? The color directions of the right and left handed fields are correlated through terms of quartic order in the potential, including

$$(\phi^*_{i,a_R}\phi_{i,b_L})(\phi^*_{j,b_L}\phi_{j,a_R}), \tag{38}$$

which is analogous to the quartic coupling for two flavors in Eq. (23). Mass dependence for the ϕ's enter through terms such as Eq. (36) and

$$|\phi^*_{i,a_L}\phi_{i,b_R}\psi_{a_L,b_R}|^2. \tag{39}$$

These terms are analogous to those using nonlinear effective Lagrangians [21,22].

For three flavors, condensation of the color antitriplet superconducting field ϕ also drives that of the color sextet superconducting field χ. Consider the operators [6,20]

$$\mathcal{L}_{\phi\chi}\sim H^*_R\phi_{i,a_R}\phi_{j,b_R}\chi_{i,j;a_R,b_R}+\text{c.c.} \tag{40}$$

and

$$\mathcal{L}^I_{\phi\chi}\sim H^*_L\phi_{i,a_R}\phi_{j,b_R}\chi_{i,j;a_R,b_R}+\text{c.c.} \tag{41}$$

Both operators are invariant under the U(1) of quark number; $\mathcal{L}_{\phi\chi}$ is invariant under $U_A(1)$, while $\mathcal{L}^I_{\phi\chi}$ is induced by instantons with topological charge 2. When ϕ condenses according to Eq. (37), $\mathcal{L}_{\phi\chi}$ and $\mathcal{L}^I_{\phi\chi}$ become terms *linear* in χ, and generate an expectation value for χ. The terms in Eqs. (40) and (41) explain why for three flavors the preferred condensate always contains some (small) piece in the repulsive, color sextet channel [4,14,15,18,19].

Since $\mathcal{L}_{\phi\chi}$ is invariant under $U_A(1)$, it is present even at high density. This is in contrast to the instanton induced operators $\mathcal{L}^I_{\phi\chi}$, $\mathcal{L}^I_{\phi\psi}$, and $\mathcal{L}^I_{\phi\tilde{\psi}}$, which are small at high density. There is no anomaly in perturbation theory, so in weak coupling one finds that because of $\mathcal{L}_{\phi\chi}$, χ condenses, but not ψ or $\tilde{\psi}$ [14,15].

I conclude this section by discussing the fields related to H superfluidity, H_+ and H_- [5,6]. Each is a complex valued scalar field with two real degrees of freedom. In a color superconducting phase, one mode of H_+ is the Goldstone boson for the spontaneous breaking of the U(1) symmetry of quark number, and so is massless. The other H_+ mode is massive except near a second order transition where $\langle H_+\rangle \to 0$, at which point both modes form a $O(2)$ multiplet. The H_- field is similar to H_+, except that both components obtain a mass from instantons, from $\mathcal{L}^I_H$ in Eq. (33).

Since H_+ only cares about the (spontaneous) breaking of the U(1) for quark number, it is *not* affected by nonzero, nondegenerate quark masses. All that matters is that all three colors and flavors of quarks become color superconducting. Alternately, one may consider separate fields for the three condensates, and construct the operator analogous to H_+; see Ref. [21] of Ref. [6].

It is possible to construct gauge invariant, superfluid order parameters for two flavors by using both the particle $\phi^+_{i,R}$ and antiparticle $\phi^-_{i,R}$ condensates

$$\epsilon^{ijk}\phi^+_{i,L}\phi^+_{j,R}\phi^-_{k,L}. \tag{42}$$

Similar to H_+ and H_-, this is invariant under all but the Abelian flavor symmetries. However, there is no reason to believe that this quantity is ever nonzero, since all three fields most likely lie in the same direction in color space. Thus H superfluidity is uniquely a consequence of color superconductivity through color-flavor locking for $2+1$ flavors.

II. PHASE DIAGRAMS

In this section I begin by proposing phase diagrams for the effective three-dimensional theories which describe the phase transition to color superconductivity at nonzero temperature. At a temperature T, a three dimensional effective theory is valid over distances $\gg 1/T$; for the phase transition to color superconductivity, three dimensional effective theory should be valid when the gap $\phi(T)\ll T$. I then conjecture how this might relate to the phase diagram in the plane of chemical potential and temperature.

I assume that the density of instantons is always large, so that the right and left handed condensates are equal. The more general case, allowing for different right and left handed condensates, is discussed later. The moral of the preceding section is that for three flavors, because of cubic terms involving two ϕ's and the other fields—either ψ, $\tilde{\psi}$, or χ—expectation values of these other fields are generated by color superconductivity. Even so, assuming that these other fields all have positive mass squared, then as the ϕ field becomes critical, $m^2_\phi\to 0$, all of the other fields remain noncritical. Thus we can safely neglect all fields except for the color antitriplet superconducting field.

For two flavors, I denote the condensate field as $\overline{\phi}_i \equiv \overline{\phi}_{i,R}=\overline{\phi}_{i,L}$; the effective Lagrangian is

$$\mathcal{L}_2=\frac{1}{2}\text{tr}(G^2_{\mu\nu})+|D_\mu\overline{\phi}|^2+m^2_\phi|\overline{\phi}|^2+\overline{\lambda}(|\overline{\phi}|^2)^2; \tag{43}$$

$G_{\mu\nu}$ is the field strength for the gauge field A_μ, and $D_\mu=\partial_\mu+igA_\mu$ is the covariant derivative for an antitriplet color field $\overline{\phi}_i$. I distinguish the condensate field $\overline{\phi}_i$ from ϕ_i in the previous section, due to an overall rescaling explained below, Eq. (47). The effective Lagrangian in Eq. (43) is valid near the transition temperature, and so is meant to be used in

three dimensions. I assume that all coupling constants are just the temperature T times those in four dimensions, which is approximately true.

The phase transition occurs as $m_\phi^2 \to 0$; in this case it is natural to introduce the ratio of the $\overline{\phi}$ coupling constant $\overline{\lambda}$ to that for the gauge field g

$$\lambda = \frac{\overline{\lambda}}{g^2}. \tag{44}$$

This ratio has a more physical interpretation. At zero temperature, if $m_\phi^2 < 0$, the expectation value of $\overline{\phi}$ is $\overline{\phi}_0 \sim (-m_\phi^2/\overline{\lambda})^{1/2}$. In the broken phase, the Higgs mass is the mass for the $\overline{\phi}$ field $\sim m_\phi$ while the gluon has a mass $m_A \sim g\overline{\phi}_0$. Thus λ is proportional to the Higgs mass divided by the gluon mass, squared $\lambda \sim m_H^2/m_A^2$.

Contrast this to the situation at zero temperature, where the theory is four dimensional. At zero temperature, instead of m_ϕ^2 varying with temperature, one only finds a transition if one varies m_ϕ^2 by hand. A phase transition is expected to occur as $m_\phi^2 \to 0$. Instead, because of the Coleman-Weinberg phenomenon, one finds that the (renormalized) correlation length never diverges [36–38]. This can be understood from the renormalization group. For $g=0$, the β function for $\overline{\lambda}$ has an infrared stable fixed point at the origin. When $g \neq 0$, however, $\overline{\lambda}$ cannot flow into the origin; instead, it flows from positive to negative values, which then generates a first order transition. Naively, one expects that λ is a free parameter, but because of dimensional transmutation, this is an illusion. By letting the coupling constants flow, one can always go from a regime with large λ to one with small λ. Physically, the square of the ratio of the masses for the Higgs boson to the gluon fields is not a free parameter, but is fixed, $\sim g^2$.

Return now to a theory at nonzero temperature, where the effective theory is three dimensional. Then λ is a free parameter: different values of λ correspond to different theories at zero temperature. We can then consider the phase diagram as a function of λ [39–44]. For small λ, fluctuations in the gauge field dominate, and a one loop analysis reliably indicates a first order phase transition. This is the ''type-I'' regime of ordinary superconductivity. As λ increases, one moves into the ''type-II'' regime, where fluctuations in the scalar field become important. An expansion from $4-\epsilon$ dimensions to three dimensions predicts that for large λ, that the transition is driven first order by fluctuations in the scalar fields. Consequently, the simplest hypothesis is an unbroken line of first order transitions for all λ.

This is not what lattice simulations find [27]. There is indeed a first order transition for small λ, but as λ increases, the strength of the first order transition decreases, until it ends at a critical point, at λ_c. The critical point occurs when the masses of the gauge and scalar fields are approximately equal. For $\lambda > \lambda_c$, there is no first order transition, only a smooth crossover between the two phases.

The existence of a critical end point is certainly possible. Since $\overline{\phi}$ is in the fundamental representation of the gauge group, there is no gauge invariant order parameter which distinguishes between the two phases [45]. The question, however, is why in the type-II regime does the first order transition at small ϵ turn into a smooth crossover at $\epsilon = 1$?

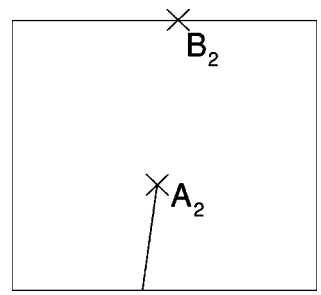

FIG. 1. Phase diagram for one $\overline{\phi}_i$ field coupled to a SU(3) gauge field. In Figs. 1 and 2, the y axis is λ, from 0 to ∞, and the x axis is $\sim m_\phi^2$.

One possibility is simply that the ϵ expansion breaks down at large $\epsilon \sim 1$. Nevertheless, I *assume* that the ϵ expansion is reliable in its prediction of a fluctuation induced transition when the theory only involves scalar fields. Numerous examples are known in condensed matter physics [40]; for careful analyses in models where the strength of the first order transition can be controlled, see Ref. [43].

Instead, I suggest that the ϵ expansion fails uniquely for theories of scalars coupled to gauge fields. Since the theory is three dimensional, when the vacuum expecation value of the scalar field vanishes, one inevitably enters a strongly coupled phase of the theory. In this phase, the proper way to think of the spectrum is in terms of gauge invariant excitations, such as glueballs and mesons formed from scalars [44]. A fluctuation induced first order transition occurs when the quartic couplings for the scalar run from positive to negative values. For this to happen, however, the couplings must flow. Perhaps the crossover regime is simply a manifestation of confinement in three dimensions: when $m_\phi^2 \to 0$, scalars become strongly bound into relatively heavy mesons. If the scalars are heavy, the scalar self couplings never run by much, even as $m_\phi^2 \to 0$. That is, confinement in three dimensions ''eats'' the running of the coupling constants. While only a qualitative explanation, it is reasonable that crossover begins when the (zero temperature) masses of the Higgs and gauge fields are approximately equal. How (three-dimensional) confinement can stop the running of (effective) coupling constants is analogous to how, in four dimensions, the strong coupling constant might freeze at low momenta [46].

The complete phase diagram can then be sketched. Consider the limit of large λ; this may not make sense in the continuum (because of triviality bounds), but is perfectly reasonable on the lattice. For infinite λ, the gauge fields decouple, and there is only a scalar field; the universality class is that of an U(3) vector, which is the same as an $O(6)$ vector. Thus there is a second order phase transition when $\lambda = \infty$. At large but finite value of λ, however, confinement presumably eats the running of the scalar coupling constants, so the theory exhibits crossover for $\lambda_c < \lambda < \infty$. This phase diagram, from Ref. [38], is illustrated again in Fig. 1: there

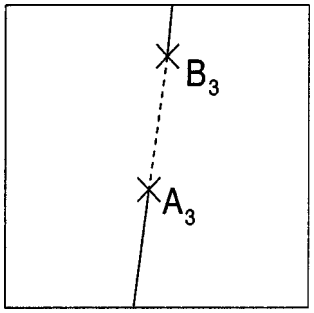

FIG. 2. Phase diagram for $\overline{\phi}_{i,a}$ coupled to a SU(3) gauge field.

are critical points at A_2, where $\lambda=\lambda_c$, and at B_2, where $\lambda=\infty$.

Consider now the model for three flavors, where the antitriplet condensate field is $\overline{\phi}_{i,a}$:

$$\mathcal{L}_3=\frac{1}{2}\mathrm{tr}(G_{\mu\nu}^2)+\mathrm{tr}(D_\mu\overline{\phi}^\dagger D_\mu\overline{\phi})+m_\phi^2\,\mathrm{tr}(\overline{\phi}^\dagger\overline{\phi})+\overline{\lambda}_1[\mathrm{tr}(\overline{\phi}^\dagger\overline{\phi})]^2+\overline{\lambda}_2\,\mathrm{tr}(\overline{\phi}^\dagger\overline{\phi})^2. \tag{45}$$

There are now two quartic coupling constants $\overline{\lambda}_1$ and $\overline{\lambda}_2$, so there are two λ parameters similar to that of Eq. (44); for simplicity I speak only of one, assuming that $\overline{\lambda}_2\neq 0$, so that there is not an accidental $O(18)$ symmetry. At $\lambda=\infty$, gauge fields can be neglected, and $\overline{\phi}_{i,a}$ is a $\mathrm{SU}(3)\times\mathrm{SU}(3)\times\mathrm{U}(1)$ vector field. In $4-\epsilon$ dimensions, this has a fluctuation induced first order transition [47]. Assuming this persists to three dimensions, there is a first order transition at $\lambda=\infty$; even with confinement at $\lambda<\infty$, the first order transition continues for some finite range of large λ. Assuming a crossover regime for intermediate λ, the phase diagram, illustrated in Fig. 2, has a critical end point for small λ, at a point A_3, and for large λ, at a point B_3. Both critical end points are in the Ising universality class in three dimensions, as are points A_2, B, A_3, and B_3 in Figs. 3 and 4.

Between A_3 and B_3, there is a gauge invariant order parameter, H_+, which distinguishes between the two phases. The expectation value of H_+ either goes to zero continuously or discontinuously. If the latter, there is a first order transition, and no crossover regime. Assuming there is a crossover regime for the non-Abelian fields, between the

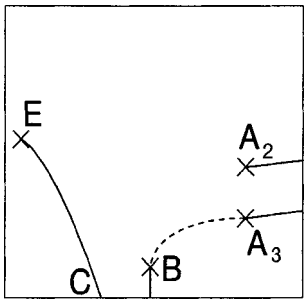

FIG. 3. Cartoon of the QCD phase diagram in the small gap limit. In Figs. 3 and 4, the y axis is temperature, and the x axis quark chemical potential.

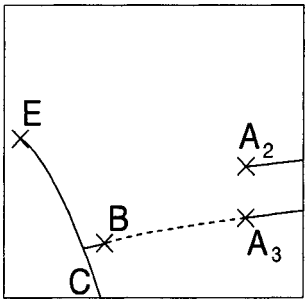

FIG. 4. Cartoon of the QCD phase diagram in the large gap limit.

critical end points A_3 and B_3 there must be a *line* of second order phase transitions, at which the H_+ field becomes critical. This is in the universality class of an $O(2)$ vector in three dimensions.

This critical line is analogous to that found by Kajantie *et al.* in their study of an adjoint scalar field coupled to a SU(3) gauge field [28]. (For the model of Ref. [28], there is only one quartic scalar coupling, and so the point B_3 lies at $\lambda=\infty$.) In the present case, one might wonder why the critical fluctuations for the H_+ are not eaten by confinement. The same comment applies to the adjoint model of Ref. [28], and is easy to dismiss. For color superconductivity, the fluctuations in H_+ are associated entirely with fluctuations in the U(1) symmetry of quark number. The non-Abelian $\mathrm{SU}(3)_c$ gluons cannot eat the running of the coupling constant for the U(1) of quark number because they cannot ''taste'' it; the H_+ field is neutral under any $\mathrm{SU}(3)_c$ transformation. Thus near the critical line for H superfluidity, the only critical modes are those for H_+.

These effective theories can be directly related to color superconductivity, although there is one surprise. As is known in ordinary superconductivity [1], the condensate field does not have canonical normalization; the effective Lagrangian for ϕ is

$$\mathcal{L}=\frac{\mu^2}{\phi_0^2}|D_\mu\phi|^2-\mu^2|\phi|^2+\frac{\mu^2}{\phi_0^2}(|\phi|^2)^2+\cdots. \tag{46}$$

The terms in Eq. (43) are only correct up to coefficients of order one, and I am sloppy about which quartic terms enter, because all I really care about is how the scales in the problem—the chemical potential μ and the value of the condensate at zero temperature ϕ_0—enter. What is interesting is that because particles have an energy $\sim\phi_0$ near the Fermi surface, the kinetic and quartic terms in the potential have factors of μ^2/ϕ_0^2. This can be understood as follows. The mass term $\sim|\phi|^2$ is not singular, with an overall mass dimension set by the chemical potential μ. Expanding the two point function of ϕ in momentum, the natural scale for the momenta to vary is over ϕ_0; thus the kinetic term is $\mu^2|\partial_\mu\phi|/\phi_0^2$. The gauge invariant generalization is $\mu^2|D_\mu\phi|/\phi_0^2$, which includes a quartic interaction between

two ϕ's and two A_μ's $\sim\mu^2/\phi_0^2$. Thus it is not surprising that the quartic interaction between four ϕ's also has an overall factor $\sim\mu^2/\phi_0^2$.

To relate this to the effective Lagrangians in Eqs. (43) and (45), it is necessary to rescale the fields and coupling constants, so that

$$\phi\sim\frac{\phi_0}{\mu}\overline{\phi},\quad \lambda\ \sim\left(\frac{\phi_0}{g\mu}\right)^2. \tag{47}$$

It is also clear from the form of the potential that terms of higher order, such as a six point term $(\mu/\phi_0)^4(|\phi|^2)^3 \rightarrow(\phi_0/\mu)^2(|\overline{\phi}|^2)^3$, are just as important as the quartic term. Indeed, all of the effective terms in the previous section should be multiplied by corresponding powers of μ/ϕ_0. None of the results change qualitatively, since they were a consequence of symmetry, and not of the assumption of limiting oneself to operators with the smallest mass dimension.

These powers of μ/ϕ_0 imply that, as in ordinary superconductivity, generally a Landau-Ginzburg approach is a terrible approximation. The one exception is near a point of phase transition, where $\langle\phi\rangle\rightarrow 0$. In this case, fluctuations are controlled by the term with the largest mass dimension; i.e., cubic and quartic terms.

Using the conjectured results for the phase diagram of the effective theories in three dimensions, we can then draw cartoons for the possible phase diagrams of color superconductivity in the μ-T plane. At large μ, whereby asymptotic freedom $g(\mu)$ is small, the gap is exponentially small in $1/g$ [6,8–15,23,24]:

$$\phi_0=512\pi^4\left(\frac{2}{g^2N_f}\right)^{5/2}\ \exp\left(-\frac{3\pi^2}{\sqrt{2}g}\right)\mu b_0'; \tag{48}$$

b_0' is a pure number, determined in Ref. [11]. Notice that the gap decreases as the number of (massless) flavors N_f increases. In mean field theory, the transition temperature is as in the theory of Bardeen, Cooper, and Schrieffer, $T_c \sim 0.567\phi_0$, and is of second order [10]. Thus at a large but fixed value of μ, as the temperature increases, there is a transition at which superconductivity for three flavors evaporates, and then a higher temperature at which that for two flavors evaporates. When fluctuations are included, both transitions turn first order. Since the condensate is small, the effective coupling $\lambda\sim(\phi_0/g\mu)^2$ is very small, and the theory is in a regime of extreme type I, with a tiny latent heat $\sim\lambda$.

I note that the coefficient of the kinetic term $|D_\mu\phi|^2$ in Eq. (46) has been computed in weak coupling perturbation theory in QCD [24]. What is remaining is to determine the coefficients of the terms which are quadratic and quartic in ϕ. Technically, this is easiest to do directly at the critical temperature. While of course such calculations are only valid at large chemical potential, by comparing to lattice results, they will give one an idea if it is reasonable that the theory goes from the type-I into the type-II regime.

As μ decreases, λ increases, so one moves up the phase diagrams of Figs. 1 and 2. The crucial question is how color superconductivity matches onto the chiral phase transition. I henceforth *assume* that the type-I regime ends before the chiral phase transition. In this case, there are critical end points for color superconductivity, at points A_2 and A_3, respectively, in Figs. 3 and 4; they correspond precisely to the same points in Figs. 1 and 2.

Now consider the opposite limit, working up for small chemical potential μ. I assume that at zero temperature there is a first order chiral transition, at a point C in Figs. 3 and 4. As the temperature increases, the chiral transition occurs at smaller μ, so the chiral transition bends back, extending to a point E, which is a critical end point for the chiral phase transition [29].

How do the phase transitions for color superconductivity and the chiral transition match onto each other? Based upon the discussion in the previous section, I assume that at zero temperature the chiral transition coincides with that for color superconductivity. There are then two cases: if the gap for color superconductivity at the point C is small, relative to m_s^2/μ. Increasing μ at $T=0$, one first enters a phase in which only up and down quarks superconduct [18], and then a phase in which all three flavors superconduct. This is illustrated in Fig. 3. Following [6], at zero temperature the transition from two flavor to three flavor superconductivity is of first order.

(A crucial assumption in Ref. [6] is that hard dense loops give the gluons a ''mass,'' so they decouple from the phase transition. One might question if this remains true in strong coupling; even with the hard dense loop mass, perhaps the four dimensional gluons eat the running of the coupling constants for the condensate field? With some effort, this can be analyzed on the lattice: it would be necessary to add dummy fields to generate hard dense loops for the gluons, and then couple the gluons to a condensate field.)

Alternately, the gap at the point C could be large, relative to m_s^2/μ. In this limit, one goes directly from a phase with (large) chiral symmetry breaking, to one with three flavor color superconductivity, Fig. 4. As discussed in the previous section, there need not be a true phase transition at the point C [5]; I assume there is, based on the disparity in scales for superfluidity between hadronic and quark matter.

The crucial assumption in Figs. 3 and 4 is that the color superconductor does not remain in the type-I regime, but enters the type-II, or crossover, regime. Thus the line for two flavor superconductivity terminates in a critical endpoint, A_2, with no true phase transition for μ's less than some value. For three flavor superconductivity, if one enters the crossover regime there must be a critical line for H superfluidity. Even if one enters the type-II regime, one may not reach the second critical endpoint, B_3 of Fig. 2. In Figs. 3 and 4 I assume not. In Fig. 3, the point B represents where the critical theory goes from being four-dimensional to a three-dimensional theory in the crossover regime. In Fig. 4, since a critical line probably does not attach directly to the line of first order chiral transitions, the point B represents the end of a first order line induced by the chiral transition.

There is a caveat to the phase diagrams of Figs. 3 and 4. Even in a confined phase, as long as strange quarks populate a Fermi sea, Λ baryons may well be superfluid, so that at

$T=0$, $\langle H_+\rangle \sim \langle \Lambda\Lambda\rangle \neq 0$. The phase transition for such ''hadronic'' H superfluidity is probably of second order for all μ and T. As discussed in Sec. II, though, hadronic superfluid gaps appear to be much smaller than color superconducting gaps. Since critical temperatures are proportional to the gap, in the μ-T plane the lines for hadronic H superfluidity lie very close to the zero temperature axis. Similarly, even between the two first order transitions at $T=0$ in Fig. 3, $\langle H_+\rangle \neq 0$. This is a phase in which only up and down, but not strange, quarks, superconduct. In this region, I also assume that $\langle H_+\rangle$ is small, on the order of that in the hadronic phase.

I stress that I assume that the theory goes from the type-I to the type-II regime as μ decreases. It is conceivable that the color superconducting transitions remain in the type-I regime for all μ. In this instance, the point A_2 would reach all the way to the chiral line, and A_3 and B would coincide, with no critical line for H superfluidity. Alternately, it is also possible that at small μ the theory goes so deep into the type-II regime that the point B in Figs. 3 and 4 coincides with the critical end point B_3 in Fig. 2.

In the above, I have assumed that because the effects of instantons are large, that left and right handed condensates are correlated. Of course in the full theory they oscillate independently, coupled through mass and anomaly terms. Effective theories which include independent left and right handed fields are constructed following the analysis of Sec. I. There is no signficant modification for two flavors, but there is for three. For three massless flavors, with no anomaly, in Fig. 2 there must be an unbroken line of first order transitions. This is because chiral symmetry is also broken by color-flavor locking, so we can treat $\phi^*_{i,a_l}\phi_{i,a_r}$ as a gauge invariant order parameter for chiral symmetry breaking. A standard renormalization group analysis then predicts that the transition is of first order [47,48].

My arguments are admittedly speculative, and meant only to suggest what the QCD phase diagram might look like. While at present the lattice cannot tell us about QCD with $\mu \neq 0$, it can study the effective theories of relevance to color superconductivity. Moreover, by using perturbation theory in QCD, one can work down from large μ to match onto the lattice results. How far this can be pushed at small μ is open to question, but is testable. The most interesting question is where the critical endpoint A_3 is, since that tells us if and when the critical line for H superfluidity begins.

ACKNOWLEDGMENTS

I thank P. Arnold, K. Enqvist, M. Laine, L. Yaffe, and especially G. Moore for discussions on the electroweak phase diagram; D. Son, for comments on how particle and antiparticle condensates mix; T. Schafer, for noting how the phase diagram changes in the chiral limit for three massless flavors; D. Rischke, for numerous discussions, especially for pointing out the importance of the color adjoint chiral field, and the effects of instantons with topological charge $Q>1$; lastly, G. G. De Roux Taylor, for the figures. This work was supported in part by DOE Grant No. DE-AC02-98CH10886.

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