PHYSICAL REVIEW C, VOLUME 62, 034003

Field transformations and simple models illustrating the impossibility of measuring off-shell effects

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(Received 30 September 1999; published 11 August 2000)

In the context of simple models utilizing field transformations in Lagrangian field theories we discuss the impossibility of measuring off-shell effects in nucleon-nucleon bremsstrahlung, Compton scattering, and related processes. Our aim is to illustrate the general results using simple and familiar models. To that end we introduce a simple phenomenological Lagrangian describing nucleon-nucleon bremsstrahlung and perform an appropriate change of variables leading to different off-shell behavior in the nucleon-nucleon amplitude as well as the photon-nucleon vertex. As a result we obtain a class of equivalent Lagrangians, generating identical *S*-matrix elements, of which the original Lagrangian is but one representative. We make use of this property in order to show that what appears as an off-shell effect in an *S*-matrix element for one Lagrangian may originate in a contact term from an equivalent Lagrangian. By explicit calculation we demonstrate for the case of nucleon-nucleon bremsstrahlung as well as nucleon Compton scattering the equivalence of observables from which we conclude that off-shell effects cannot in any unambiguous way be extracted from an *S*-matrix element. Finally, we also discuss some implications of introducing off-shell effects on a phenomenological basis, resulting from the requirement that the description of one process be consistent with that of other processes described by the same Lagrangian.

PACS number(s): 13.40.Gp, 13.75.Cs, 13.60.Fz, 25.40.−h

I. INTRODUCTION

There has been a long history, within the context of calculations in medium-energy physics, of attempts to find effects of off-shell contributions in a particular process. Perhaps the prime example of this is nucleon-nucleon bremsstrahlung which has long been considered the best way to get information about the off-shell nucleon-nucleon amplitude. In an abstract mathematical sense, such an off-shell amplitude can be calculated from any potential, say by solving the Lippman-Schwinger equation, and the hope has always been that one might be able to distinguish between potentials which are equivalent on shell via a measurement of their off-shell behavior.

For nucleon-nucleon bremsstrahlung there have been a large number of calculations, mostly in nonrelativistic potential models which have this aim [1–5]. The usual procedure is to calculate an off-shell nucleon-nucleon amplitude as a separate building block and attach photons to the external legs of this amplitude. The photon-nucleon vertex may also have components involving off-shell nucleons. Usually the so-called double-scattering contribution, involving two strong scatterings with the photon attached in between, is also included. Less important corrections such as some relativistic effects, Coulomb corrections, and in some cases specific exchange-current diagrams are also included.

These state-of-the-art calculations are then compared with experiment. Most early experiments explored kinematics which were not far enough off shell to show anything, but the more recent ones, in particular the TRIUMF experiment [6] seemingly showed the need for off-shell effects at least within the context of current theories. A number of new experiments have been started in the past few years all designed at least in part to be more sensitive to kinematics in which the nucleons are as far off shell as possible [7–11]. These kinematics correspond to higher photon energies and in general to smaller opening angles between the two outgoing protons. Thus for example some of the new experiments have been designed to capture essentially all of the forward-going protons and thus get opening angles of only a few degrees.

Another case of interest in medium-energy physics where off-shell effects supposedly enter and have been considered is given by the electromagnetic interaction of a bound nucleon. Traditionally, electron-nucleus scattering experiments have been interpreted in terms of the free nucleon current operator in combination with some recipe to restore gauge invariance. It has only been recently that the influence of off-shell effects in the electromagnetic vertex on an interpretation of (e,e') and $(e,e'p)$ data has been investigated [12–15]. Off-shell effects at the electromagnetic vertex have also been considered for nucleon-nucleon bremsstrahlung and other processes starting with the early paper of Nyman [16] and continuing with the more recent works of Refs. [17–20].

Further processes which might be considered as a source of off-shell information are two-step processes involving the nucleon such as pion photo- and electroproduction [21] or real and virtual Compton scattering on the nucleon [22,23]. Similarly to the bremsstrahlung case the intermediate nucleon (or pion) in the pole diagrams is off shell and one might think of exploring the sensitivity of observables to the way the corresponding vertices behave off shell.

In all of these situations it is common to make some sort of model which generates off-shell effects in a vertex when that vertex is considered in isolation. In the nucleon-nucleon bremsstrahlung case almost any potential can generate a nucleon-nucleon *T*-matrix which can be extended off shell in some way determined by the potential. For the photon-nucleon vertex, which would appear in both bremsstrahlung and in Compton scattering, one can generate off-shell behavior from a simple phenomenological Lagrangian, or equivalently parametrize its off-shell structure by some sort of form factor in the off-shell variable. In Refs. [12–15,24,25] the off-shell effects were generated in terms of more or less sophisticated meson-loop calculations. In many other processes off-shell form factors are included. For example modern nucleon-nucleon potentials put in form factors at the meson-nucleon vertices, and one question which is at times heatedly discussed is the appropriate range of such form factors.

Thus within the context of standard calculations of medium-energy processes the concept of some sort of off-shell effect at the vertices is rather pervasive.

In contrast to this situation, within the context of field theory it has long been known that there is a certain ambiguity in the evaluation of off-shell effects. One can for example make field transformations, that is changes of representation of the fields involved in the Lagrangian, which do not change any measured quantity, but which can in fact change the off-shell behavior of a vertex building block of the process [26–30]. One also has the rather simple observation that, in an amplitude arising from a Feynman diagram, off-shell behavior at a vertex is cancelled by a similar factor coming from the intermediate propagator, resulting in an amplitude which could have been generated by a contact interaction, without reference to off-shell processes. This was already observed by Gell-Mann and Goldberger in their derivation of the Compton scattering low-energy theorem [31].

It is only relatively recently that these field-theory concepts have begun to be applied to gain an understanding of the role of, and the ambiguities in, the off-shell effects which are normally included in medium-energy processes. For example, in the context of chiral perturbation theory (ChPT), the off-shell electromagnetic vertex of the nucleon and pion were calculated in Refs. [24,32], respectively, by including in the Lagrangian certain terms proportional to the lowest-order equation of motion. For the pion it was shown explicitly [33] that this off-shell form factor did not contribute to pion Compton scattering. Likewise, in a similar model for spin zero bremsstrahlung it was shown [34] that off-shell effects arising from such equation-of-motion terms in the Lagrangian again could be transformed away, or alternatively replaced by contact terms which did not generate off-shell contributions at either strong or electromagnetic vertices. Thus in this context off-shell effects were shown to be unmeasurable, in contrast to the standard expectation for bremsstrahlung. In Refs. [35–37] the freedom of choosing appropriate field variables has been used to express the most general effective Lagrangian describing low-energy, i.e., below pion-production threshold, (virtual) Compton scattering in a canonical form such that any off-shell dependence has been removed from the electromagnetic vertex. Such a technique is the basis of modern methods of deriving the model-independent low-energy limit of (virtual) Compton scattering amplitudes. Off-shell form factors have been calculated via dispersion relations as well [16,38,39] and here it was shown [40] that the ambiguity in such effects corresponding to a freedom of choice of field representation was reflected in an ambiguity in the number of subtraction constants required in the dispersion relation. Furthermore, it was also demonstrated in Ref. [40] at the one-loop level in perturbation theory that not only real, but also the ''absorptive'' imaginary parts reveal such a representation-dependent asymptotic behavior.

In the present paper we continue and extend in several ways this discussion. Our aim is to illustrate the general results, namely the unmeasurability of off-shell effects, using simple and familiar models, which are closely related to the kind of phenomenological models being used in explicit calculations and which do not depend on specific expansions such as ChPT. In the following section, Sec. II, we thus take another look at the nonlinear σ model for the spin zero case and discuss the essential features of Ref. [34] in a simplified way. We then extend the approach to look at a simple spin 1/2 model which is much more closely allied to the types of phenomenological models which have been used than that of previous discussions. In particular, it does not involve ChPT, and so does not depend on any of the formalism there or in particular on being able to make and truncate an expansion in some small parameter. It involves spin 1/2 particles, in particular nucleons, as well as photons and thus should remove any lingering uncertainty that the results of the previous works somehow depended on the simplicity of a spin zero process. It is more ''realistic'' in the sense that it corresponds fairly closely to some phenomenological models being used to examine off-shell effects. However it is still sufficiently simple that we can focus on the principles rather than the details.

In Sec. III we will discuss the model, which is a somewhat simplified model Lagrangian for nucleons and photons, and calculate the leading contributions to nucleon-nucleon bremsstrahlung and Compton scattering on the nucleon. Section IV is devoted to a general discussion of field transformations and changes of representation and the way they lead to equivalent Lagrangians. In Sec. V we apply a specific field transformation to our starting Lagrangian to generate a new Lagrangian which is closely allied to commonly used phenomenological Lagrangians and which generates off-shell effects at both strong and electromagnetic vertices. Section VI is devoted to a calculation of bremsstrahlung and Sec. VII to a calculation of Compton scattering with the new Lagrangian. In both cases we can see explicitly how such off-shell effects are really not physically measurable quantities. The last section is then devoted to a summary and some conclusions.

II. SIMPLE EXAMPLE

As a first example let us consider the case of $\pi\pi$ bremsstrahlung which allows us to introduce the main concepts while avoiding complications due to spin. To be specific, we discuss the reaction $\pi^+ + \pi^0 \to \pi^+ + \pi^0 + \gamma$ in the framework of the nonlinear σ model. The present treatment simplifies results already discussed in Refs. [33,34] in the sense that it will not make use of higher orders in the momentum expansion of chiral perturbation theory (ChPT). In other words, a discussion only in terms of tree-level diagrams originating from the nonlinear σ model turns out to be sufficient.

The nonlinear σ model Lagrangian, describing pion interactions at low energies, is given by

$$\mathcal{L}=\frac{F^2}{4}\mathrm{Tr}[D_\mu U(D^\mu U)^\dagger]+\frac{F^2m_\pi^2}{4}\mathrm{Tr}(U+U^\dagger), \tag{1}$$

where $F=92.4$ MeV denotes the pion-decay constant, $m_\pi=135$ MeV is the pion mass, and the pion fields are contained in the SU(2) matrix U. The interaction with the electromagnetic field is generated through the covariant derivative $D_\mu U=\partial_\mu U+(i/2)eA_\mu[\tau_3,U]$, where $e^2/4\pi\approx 1/137$ and $e>0$. At this point we still have a choice how to represent the matrix U in terms of pion field variables. We will make use of two different parametrizations of U,

$$U(x)=\frac{1}{F}[\sigma(x)+i\vec{\tau}\cdot\vec{\pi}(x)],\quad \sigma(x)=\sqrt{F^2-\vec{\pi}^2(x)}, \tag{2}$$

$$U(x)=\exp\left[i\frac{\vec{\tau}\cdot\vec{\phi}(x)}{F}\right], \tag{3}$$

where in both cases the three Hermitian fields π_i and ϕ_i describe pion fields transforming as isovectors. The connection between the two different choices can be interpreted as a change of variables, leaving the free-field part of the Lagrangian unchanged [27,28],

$$\frac{\vec{\pi}}{F}=\hat{\phi}\sin\left(\frac{\phi}{F}\right)=\frac{\vec{\phi}}{F}\left(1-\frac{1}{6}\frac{\vec{\phi}^2}{F^2}+\cdots\right). \tag{4}$$

Let us now consider the tree-level invariant amplitude for $\pi^+(p_1)+\pi^0(p_2)\to\pi^+(p_3)+\pi^0(p_4)$. For that purpose we need to insert the expressions for U of Eqs. (2) and (3) into Eq. (1) and collect those terms containing four pion fields[1]:

$$\mathcal{L}_1^{4\pi}=\frac{1}{2F^2}\partial_\mu\vec{\pi}\cdot\vec{\pi}\partial^\mu\vec{\pi}\cdot\vec{\pi}-\frac{m_\pi^2}{8F^2}(\vec{\pi}^2)^2, \tag{5}$$

$$\mathcal{L}_2^{4\phi}=\frac{1}{6F^2}(\partial_\mu\vec{\phi}\cdot\vec{\phi}\partial^\mu\vec{\phi}\cdot\vec{\phi}-\vec{\phi}^2\partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi})+\frac{m_\pi^2}{24F^2}(\vec{\phi}^2)^2. \tag{6}$$

Observe that the two interaction Lagrangians depend differently on the respective pion fields. Expressing the physical pion fields in terms of the Cartesian components as $\pi^\pm=(1/\sqrt{2})(\pi_1\mp i\pi_2)$ and $\pi^0=\pi_3$ (and similarly for ϕ_i), it is straightforward to obtain the corresponding Feynman rules for $\pi^+(p_1)+\pi^0(p_2)\to\pi^+(p_3)+\pi^0(p_4)$:

$$\mathcal{M}_1^{\pi\pi}=\frac{i}{F^2}T_0(p_1,p_3), \tag{7}$$

$$\mathcal{M}_2^{\pi\pi}=\frac{i}{F^2}\left[T_0(p_1,p_3)-\frac{1}{3}(\Lambda_1+\Lambda_2+\Lambda_3+\Lambda_4)\right], \tag{8}$$

where $T_0(p_1,p_3)=(p_3-p_1)^2-m_\pi^2$ and $\Lambda_i=p_i^2-m_\pi^2$. If initial and final pions are on the mass shell, i.e., $\Lambda_i=0$, the result for the scattering amplitudes is the same which is a consequence of the equivalence theorem [26–29].[2] In fact, since our starting point is the nonlinear σ model, the on-shell result corresponds to the current-algebra prediction for low-energy $\pi\pi$ scattering [41]. On the other hand, if one of the momenta of the external lines is off mass shell, the amplitudes of Eqs. (7) and (8) differ.

According to the standard argument in nucleon-nucleon bremsstrahlung one would now try to discriminate between different on-shell equivalent $\pi\pi$ amplitudes through an investigation of the reaction $\pi^+(p_1)+\pi^0(p_2)\to\pi^+(p_3)+\pi^0(p_4)+\gamma(k)$. We will now critically examine this claim in the framework of the above model. To that end we include the electromagnetic field as in Eq. (1) and calculate the relevant tree-level diagrams.[3]

Inclusion of the electromagnetic interaction in combination with the first parametrization of Eq. (2) only generates electromagnetic interaction terms containing two oppositely charged pion fields with either one or two electromagnetic fields, i.e., the interaction is the same as for a point pion in scalar QED,

$$\begin{aligned}&\frac{F^2}{4}\mathrm{Tr}[D_\mu U(D^\mu U)^\dagger]\\&\quad=\frac{1}{2}(\partial_\mu\sigma\,\partial^\mu\sigma+\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi})\\&\qquad-ieA_\mu(\pi^-\partial^\mu\pi^+-\pi^+\partial^\mu\pi^-)+e^2A^2\pi^+\pi^-,\end{aligned} \tag{9}$$

where $\sigma=\sqrt{F^2-\vec{\pi}^2}$. This is due to the fact that in Eq. (2) the pion field appears only linearly in combination with the Pauli matrices such that the commutator with τ_3 in the covariant derivative also results only in a linear term. As a consequence, at tree level only the two diagrams, where the photon is radiated off the initial and final charged pions, contribute to the bremsstrahlung amplitude[4]:

[1]The Lagrangian of Eq. (1) only generates interaction terms containing an even number of pion fields, i.e., it is even under the substitution $U\to U^\dagger$ corresponding, respectively, to $\vec{\pi}\to-\vec{\pi}$ and $\vec{\phi}\to-\vec{\phi}$.

[2]For a general proof of the equivalence of S-matrix elements evaluated at tree level (phenomenological approximation), see Sec. 2 of Ref. [29].

[3]We have checked that first inserting the parametrizations of U into the nonlinear σ model without electromagnetic interaction and then performing minimal substitutions $\partial_\mu\pi^\pm\to(\partial_\mu\pm ieA_\mu)\pi^\pm$ (and similarly for $\phi^\pm$) generates the same result.

[4]For notational simplicity, we will omit the complex conjugation of the polarization vectors of final-state photons.

$$\mathcal{M}_1^{\pi\pi\gamma}=\frac{i}{F^2}T_0(p_1-k,p_3)\frac{i}{(p_1-k)^2-m_\pi^2}(-2iep_1\cdot\epsilon)$$
$$-2iep_3\cdot\epsilon\,\frac{i}{(p_3+k)^2-m_\pi^2}\,\frac{i}{F^2}\,T_0(p_1,p_3+k)$$
$$=e\left(\frac{p_3\cdot\epsilon}{p_3\cdot k}-\frac{p_1\cdot\epsilon}{p_1\cdot k}\right)\frac{i}{F^2}[T_0(p_1,p_3)-2(p_1-p_3)\cdot k]. \tag{10}$$

It is now natural to ask how the different off-shell behavior of the $\pi\pi$ amplitude of Eq. (8) enters into the calculation of the bremsstrahlung amplitude. Observe, in this context, that inserting the parametrization of Eq. (3) into Eq. (1) generates electromagnetic interactions involving $2n$ pion fields, where n is a positive integer. The additional interaction term relevant to the bremsstrahlung process reads

$$\mathcal{L}_2^{\phi^+\phi^-\phi^0\phi^0 A}=\frac{ieA_\mu}{3F^2}(\partial^\mu\phi^+\phi^- - \phi^+\partial^\mu\phi^-)\phi^0\phi^0. \tag{11}$$

Hence the total tree-level amplitude now contains a four-pion-one-photon contact diagram, in addition to the two diagrams involving radiation from the charged external legs which are the same as in $\mathcal{M}_1^{\pi\pi\gamma}$,

$$\mathcal{M}_2^{\pi\pi\gamma}=\frac{i}{F^2}\left\{T_0(p_1-k,p_3)-\frac{1}{3}[(p_1-k)^2-m_\pi^2]\right\}$$
$$\times\frac{i}{(p_1-k)^2-m_\pi^2}(-2iep_1\cdot\epsilon)$$
$$-2iep_3\cdot\epsilon\,\frac{i}{(p_3+k)^2-m_\pi^2}\,\frac{i}{F^2}\left\{T_0(p_1,p_3+k)\right.$$
$$\left.-\frac{1}{3}[(p_3+k)^2-m_\pi^2]\right\}+\frac{2ie}{3F^2}(p_1+p_3)\cdot\epsilon. \tag{12}$$

Combining the contribution due to the off-shell behavior in the $\pi\pi$ amplitude with the contact-term contribution, we find a precise cancellation of off-shell effects and contact interactions such that the final result is the same for both parametrizations, i.e., $\mathcal{M}_1^{\pi\pi\gamma}=\mathcal{M}_2^{\pi\pi\gamma}$. This is once again a manifestation of the equivalence theorem of field theory. What is even more important in the present context is the observation that the two mechanisms, i.e., contact term vs off-shell effects, are indistinguishable since they lead to the same measurable amplitude.

It should be noted that none of the above arguments relies on chiral perturbation theory or is a consequence of chiral symmetry. Also, even though gauge invariance poses restrictions on the result for the bremsstrahlung amplitude it is *not* the primary reason for the equivalence of the two results. This will become more evident in the next section, when we include spin in the problem and consider separately gauge-invariant terms.

III. SIMPLE MODEL FOR THE SPIN ONE-HALF CASE

We now include spin and consider a very simple model which can describe the interactions of nucleons and photons and which we can use to describe nucleon-nucleon bremsstrahlung and Compton scattering on the nucleon. Such a model is described by the Lagrangian

$$\mathcal{L}_0=\bar{\Psi}(i\not{D}-m)\Psi-\frac{e\kappa}{4m}\bar{\Psi}\sigma_{\mu\nu}F^{\mu\nu}\Psi+\bar{\Phi}(i\not{\partial}-m)\Phi$$
$$+g\bar{\Psi}\Psi\bar{\Phi}\Phi. \tag{13}$$

Here D is the covariant derivative $D_\mu\Psi=(\partial_\mu+ieA_\mu)\Psi$, e and κ are the proton charge and anomalous magnetic moment respectively, A_μ is the photon field and $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ is the electromagnetic field strength tensor. The fields Ψ correspond to protons and Φ to neutrons. We separate out the neutrons and neglect the electromagnetic coupling to the neutron magnetic moment purely to simplify the calculation of nucleon-nucleon bremsstrahlung, i.e., so that we need consider radiation from only two legs instead of four and so that we do not need to worry about antisymmetrization for identical particles. This simplifies the algebra, and reduces the number of diagrams to be considered from eight to two, but makes no substantive change in what one can learn from the model.

For the electromagnetic interaction of real photons with protons the above Lagrangian is exactly what would normally be used.[5] It is gauge invariant, since it involves only covariant derivatives and field strength tensors. It leads in momentum space to the standard photon nucleon vertex[6]

$$-ie\bar{u}(p_f)\left(\not{\epsilon}-\frac{i\kappa}{2m}\sigma_{\mu\nu}\epsilon^\mu k^\nu\right)u(p_i), \tag{14}$$

corresponding to the momenta $p_i=p_f+k$, i.e., for outgoing photons, and the photon polarization vector ϵ.

For the strong interaction corresponding to a nucleon-nucleon vertex this Lagrangian gives

$$ig\bar{u}_p(p_3)u_p(p_1)\bar{u}_n(p_4)u_n(p_2), \tag{15}$$

where p_1 and p_3 correspond to the protons and $p_1+p_2=p_3+p_4$. This sort of interaction could be generated by the exchange of a heavy scalar meson. It clearly grossly over-

[5]For a more general phenomenological gauge-invariant Lagrangian capable of describing also the interaction with virtual photons, see Eq. (3.1) of Ref. [36].

[6]For notational convenience we will include positive-energy spinors in our expression for vertices with the understanding that they have to be replaced by appropriate Feynman propagators where necessary.

simplifies the strong interaction, but is sufficient, as we will see, to illustrate all of the principles we wish to consider.

One should note that this $\mathcal{L}_0$, at least to lowest order, does not generate any off-shell effects at either of the strong or electromagnetic vertices.

Using these vertices we can calculate the Born amplitudes for both nucleon-nucleon bremsstrahlung and Compton scattering on the nucleon.

We find for nucleon-nucleon bremsstrahlung

$$\begin{aligned}\mathcal{M}_0^{NN\gamma}=&ieg\bar{u}_n(p_4)u_n(p_2)\bar{u}_p(p_3)\\&\times\left[\left(\not\epsilon-\frac{i\kappa}{2m}\sigma_{\mu\nu}\epsilon^\mu k^\nu\right)\right.\\&\times\frac{\not p_3+\not k+m}{(p_3+k)^2-m^2}+\frac{\not p_1-\not k+m}{(p_1-k)^2-m^2}\\&\left.\times\left(\not\epsilon-\frac{i\kappa}{2m}\sigma_{\mu\nu}\epsilon^\mu k^\nu\right)\right]u_p(p_1).\end{aligned}\tag{16}$$

This amplitude corresponds to the usual choice for nucleon-nucleon bremsstrahlung for the electromagnetic parts, but has a much simplified interaction for the strong part. Extension to the most general nucleon-nucleon interaction could be done along the lines of Ref. [42], but such extension adds nothing to the argument here.

Similarly the Born amplitude for Compton scattering on the nucleon, with $p_i+k_1=p_f+k_2$, is

$$\begin{aligned}\mathcal{M}_0^{CS}=&-ie^2\bar{u}(p_f)\left[\left(\not\epsilon_2-\frac{i\kappa}{2m}\sigma_{\mu\nu}\epsilon_2^\mu k_2^\nu\right)\right.\\&\times\frac{\not p_f+\not k_2+m}{(p_f+k_2)^2-m^2}\left(\not\epsilon_1+\frac{i\kappa}{2m}\sigma_{\mu\nu}\epsilon_1^\mu k_1^\nu\right)\\&+\left(\not\epsilon_1+\frac{i\kappa}{2m}\sigma_{\mu\nu}\epsilon_1^\mu k_1^\nu\right)\frac{\not p_f-\not k_1+m}{(p_f-k_1)^2-m^2}\\&\left.\times\left(\not\epsilon_2-\frac{i\kappa}{2m}\sigma_{\mu\nu}\epsilon_2^\mu k_2^\nu\right)\right]u(p_i).\end{aligned}\tag{17}$$

This amplitude for Compton scattering is exactly what is normally used for the Born part of the amplitude [35,43].

IV. EFFECT OF FIELD TRANSFORMATIONS

We now want to consider the effects of a field transformation on the fields which are contained in the Lagrangian $\mathcal{L}_0$. Such a transformation amounts to a change of representation for the fields. It will generate some new terms in the Lagrangian so that $\mathcal{L}_0\to\mathcal{L}_0+\Delta\mathcal{L}$. We know from general principles that this change of representation will not affect any physically measurable results [26–29]. This means that the physical amplitudes generated from $\mathcal{L}_0$ and from $\mathcal{L}_0+\Delta\mathcal{L}$ will be exactly the same, or alternatively that the sum of all terms containing a contribution obtained from $\Delta\mathcal{L}$ will add up to zero.

The general form for $\Delta\mathcal{L}$ can be obtained by making the substitution $\Psi\to\Psi+\delta\Psi$, where $\delta\Psi$ is not necessarily an infinitesimal transformation. In principle, a transformation on the neutron is also possible but for simplicity we discard this possibility since it does not add anything new to our argument. The resulting Lagrangian becomes

$$\mathcal{L}_0(\Psi+\delta\Psi)=\mathcal{L}_0(\Psi)+\Delta\mathcal{L}(\Psi)\tag{18}$$

with

$$\begin{aligned}\Delta\mathcal{L}=&\bar{\Psi}\left[(i\not D-m)-\frac{e\kappa}{4m}\sigma_{\mu\nu}F^{\mu\nu}+g\bar{\Phi}\Phi\right]\delta\Psi\\&+\delta\bar{\Psi}\left[(i\not D-m)-\frac{e\kappa}{4m}\sigma_{\mu\nu}F^{\mu\nu}+g\bar{\Phi}\Phi\right]\Psi\\&+\delta\bar{\Psi}\left[(i\not D-m)-\frac{e\kappa}{4m}\sigma_{\mu\nu}F^{\mu\nu}+g\bar{\Phi}\Phi\right]\delta\Psi.\end{aligned}\tag{19}$$

Observe that the first term is, up to a total derivative, proportional to the equation of motion for $\bar{\Psi}$ as obtained from $\mathcal{L}_0$. The second term is proportional to the equation of motion for Ψ. The last term of Eq. (19) however is of second order in $\delta\Psi$ and thus this situation is somewhat more general than that discussed in Ref. [34]. In that case the formalism of ChPT ensured that this last term was of higher order in the so-called momentum expansion and thus could be dropped. Here we have no such expansion criterion and so this term must be kept.

Now suppose that we take as our Lagrangian $\mathcal{L}_0+\Delta\mathcal{L}$. In simple cases, e.g., the ChPT example discussed in Ref. [34], the second order term in Eq. (19) can be dropped and $\Delta\mathcal{L}$ is simply proportional to an equation of motion. In more general situations the second order part would have to be kept [44]. Pieces of $\Delta\mathcal{L}$ generate off-shell contributions to the vertex functions and in general other pieces generate contact terms. However the full contribution of $\Delta\mathcal{L}$ vanishes and thus there are no contributions to measurable amplitudes from the complete set of terms from $\Delta\mathcal{L}$. Another way of saying this is that since $\Delta\mathcal{L}$ originated as a field transformation on $\mathcal{L}_0$ it can be completely transformed away, thus eliminating any dependence on both the off-shell pieces and other pieces involving contact terms.

To develop an approach, which is more closely allied to phenomenological calculations we divide $\Delta\mathcal{L}$ into two pieces, $\Delta\mathcal{L}=\Delta\mathcal{L}_1+\Delta\mathcal{L}_2$ where $\Delta\mathcal{L}_1$ contains those pieces which generate off-shell contributions to the various vertices generated by the Lagrangian, plus perhaps a few contact terms necessary for gauge invariance, and where $\Delta\mathcal{L}_2$ contains the remaining terms which generate pure contact type contributions to the amplitudes for the processes being considered. Now consider $\mathcal{L}_0+\Delta\mathcal{L}_1$ as a phenomenological Lagrangian. It is a different Lagrangian than $\mathcal{L}_0$ or $\mathcal{L}_0+\Delta\mathcal{L}$, and may have different physical consequences. It will generate off-shell contributions at the vertices and in general the amplitude calculated for a physical process will depend on the coefficients of this part of the Lagrangian, i.e., on coefficients which also multiply the off-shell contributions to the vertices. This is analogous to the procedure adopted in most calculations purporting to determine sensitivity of amplitudes to off-shell effects, where however $\Delta\mathcal{L}_1$ would be generated in a purely ad hoc fashion from phenomenological considerations.

In our case however, by the general result, the total contribution of $\Delta\mathcal{L}$ must be zero. This means that the Lagrangian $\mathcal{L}_0-\Delta\mathcal{L}_2$ will give exactly the same measurable amplitudes as $\mathcal{L}_0+\Delta\mathcal{L}_1$, depending on the same parameters of the Lagrangian, and thus could be considered as an alternative equivalent phenomenological Lagrangian. However in this case the Lagrangian generates only contact terms and does not give any off-shell contributions to the various vertices. Note that the specific way $\Delta\mathcal{L}$ is split into $\Delta\mathcal{L}_1$ and $\Delta\mathcal{L}_2$ will depend on the reaction in question.

Thus we have two Lagrangians, one which gives off-shell contributions to the vertex functions which are the building blocks for the full amplitude and one which does not. Both however give exactly the same measurable physical amplitudes and thus exactly the same dependence of these measured quantities on the parameters of the Lagrangian. One must thus conclude that the concept of off-shell contributions to a physical process is just not a meaningful concept. One can measure coefficients in a particular choice of phenomenological Lagrangian by comparing with data, but those coefficients cannot uniquely be associated with the strength off-shell contributions at the vertices in any meaningful way.

In the next sections we will see explicitly how these principles appear in our simple model. In particular we will apply a field transformation to the model to generate $\Delta\mathcal{L}$. We will then extract a $\Delta\mathcal{L}_1$ and $\Delta\mathcal{L}_2$ and see explicitly in the model the ambiguity described above.

V. EVALUATION OF THE TRANSFORMED LAGRANGIAN

Consider in this section a specific transformation or change of representation of the fields, which has been chosen to generate off-shell contributions at both strong and electromagnetic vertices in our simple model and to generate a Lagrangian corresponding to a phenomenological Lagrangian similar to one which has been used in investigations of off-shell effects. Thus take

$$\Psi\rightarrow\Psi+\tilde{a}g\bar{\Phi}\Phi\Psi+\tilde{b}e\sigma_{\mu\nu}F^{\mu\nu}\Psi. \tag{20}$$

Here $\tilde{a}$ and $\tilde{b}$ are real constants which determine the overall strength of the transformations.

This transformation generates $\Delta\mathcal{L}=\Delta\mathcal{L}_1+\Delta\mathcal{L}_2$ with

$$\begin{aligned}\Delta\mathcal{L}_1=&\tilde{a}g[\bar{\Psi}(i\overleftarrow{\not\partial}-m-e\not{A})\Psi\bar{\Phi}\Phi+\bar{\Phi}\Phi\bar{\Psi}(i\vec{\not\partial}-m-e\not{A})\Psi]\\&+\tilde{b}e[\bar{\Psi}(i\overleftarrow{\not\partial}-m-e\not{A})\sigma_{\mu\nu}F^{\mu\nu}\Psi\\&+\bar{\Psi}\sigma_{\mu\nu}F^{\mu\nu}(i\vec{\not\partial}-m-e\not{A})\Psi]\end{aligned} \tag{21}$$

and

$$\begin{aligned}\Delta\mathcal{L}_2=&-eg\left(\frac{\tilde{a}\kappa}{2m}-2\tilde{b}\right)\bar{\Phi}\Phi\bar{\Psi}\sigma_{\mu\nu}F^{\mu\nu}\Psi\\&-\frac{e^2\tilde{b}\kappa}{2m}\bar{\Psi}\sigma_{\mu\nu}F^{\mu\nu}\sigma_{\alpha\beta}F^{\alpha\beta}\Psi\\&+\tilde{a}\tilde{b}eg[\bar{\Psi}\sigma_{\mu\nu}F^{\mu\nu}(i\vec{\not\partial}-m-e\not{A})\Psi\bar{\Phi}\Phi\\&+\bar{\Phi}\Phi\bar{\Psi}(i\overleftarrow{\not\partial}-m-e\not{A})\sigma_{\mu\nu}F^{\mu\nu}\Psi]\\&+e^2\tilde{b}^2\bar{\Psi}\sigma_{\mu\nu}F^{\mu\nu}(i\not\partial-m-e\not{A})\sigma_{\alpha\beta}F^{\alpha\beta}\Psi.\end{aligned} \tag{22}$$

Here we have defined $\bar{\Psi}i\overleftarrow{\not\partial}\equiv-i(\partial_\mu\bar{\Psi})\gamma^\mu$. Both of these contributions to $\Delta\mathcal{L}$ have been expressed in terms of the covariant derivative, which means including some terms proportional to $\not{A}$ in $\Delta\mathcal{L}_1$ rather than in $\Delta\mathcal{L}_2$ so that both pieces will be manifestly gauge invariant. Some terms not contributing to either nucleon-nucleon bremsstrahlung or Compton scattering have also been dropped. One would have to keep those terms if one wanted to show, for more complicated processes than those considered here, the equivalence of the *S*-matrix elements obtained from the original Lagrangian and the transformed Lagrangian.

Consider first the Lagrangian $\mathcal{L}_0+\Delta\mathcal{L}_1$. This generates some new contributions to the vertices, in addition to those coming from $\mathcal{L}_0$ given in Eqs. (14) and (15) above. For the strong vertex we find from the first term of $\Delta\mathcal{L}_1$

$$i\tilde{a}g\bar{u}_p(p_3)[(\not{p}_3-m)+(\not{p}_1-m)]u_p(p_1)\bar{u}_n(p_4)u_n(p_2). \tag{23}$$

Clearly this represents an off-shell contribution to the strong vertex, of strength determined by the parameter $\tilde{a}$, analogous to what one would calculate in a potential model for the off-shell nucleon-nucleon vertex. It vanishes when the momenta p_1 and p_3 are on shell.

At the electromagnetic vertex we get from the second term in $\Delta\mathcal{L}_1$

$$-2ie\tilde{b}\bar{u}(p_f)[(\not{p}_f-m)i\sigma_{\mu\nu}\epsilon^\mu k^\nu+i\sigma_{\mu\nu}\epsilon^\mu k^\nu(\not{p}_i-m)]u(p_i). \tag{24}$$

Again this corresponds to an off-shell contribution, this time to the magnetic part of the photon-nucleon vertex. The overall strength is determined by the parameter $\tilde{b}$.

Finally there are the contact terms coming from the use of the covariant derivative as necessary for gauge invariance. The term

$$-2i\tilde{a}eg\bar{u}_p(p_3)\not{\epsilon}u_p(p_1)\bar{u}_n(p_4)u_n(p_2) \tag{25}$$

corresponds to a photon-four-nucleon vertex and will contribute to nucleon-nucleon bremsstrahlung. The term

$$+2ie^2\tilde{b}\overline{u}(p_f)(\not\epsilon_1 i\sigma_{\mu\nu}\epsilon_2^\mu k_2^\nu - i\sigma_{\mu\nu}\epsilon_1^\mu k_1^\nu \not\epsilon_2 + i\sigma_{\mu\nu}\epsilon_2^\mu k_2^\nu \not\epsilon_1 - \not\epsilon_2 i\sigma_{\mu\nu}\epsilon_1^\mu k_1^\nu)u(p_i) \tag{26}$$

gives a two-photon contact vertex which will contribute to Compton scattering.

This Lagrangian, $\mathcal{L}_0+\Delta\mathcal{L}_1$, corresponds very closely to some which have been used to investigate off-shell effects in nucleon-nucleon bremsstrahlung. The strong part is simplified, but produces off-shell effects in the nucleon-nucleon vertex amplitude analogous to those one might obtain from a potential model. The electromagnetic part is of the general form given by Bincer [38] and used by a variety of authors [12,17–20,24] to investigate the supposed sensitivity of nucleon-nucleon bremsstrahlung to off-shell effects. For example in the notation of Nyman [16], $F_2^-(m^2)\to\kappa+8m^2\tilde{b}$.[7]

The alternative Lagrangian $\mathcal{L}_0-\Delta\mathcal{L}_2$ generates just contact terms. The vertices generated are a contribution to the one-photon-four-nucleon amplitude from the first and third term in $\Delta\mathcal{L}_2$

$$\begin{aligned}&+2ieg\overline{u}_p(p_3)i\sigma_{\mu\nu}\epsilon^\mu k^\nu\left(\frac{\tilde{a}\kappa}{2m}-2\tilde{b}\right)u_p(p_1)\overline{u}_n(p_4)u_n(p_2)\\&-2ieg\tilde{a}\tilde{b}\overline{u}_p(p_3)[(\not p_1-\not k-m)i\sigma_{\mu\nu}\epsilon^\mu k^\nu\\&+i\sigma_{\mu\nu}\epsilon^\mu k^\nu(\not p_3+\not k-m)]u_p(p_1)\overline{u}_n(p_4)u_n(p_2)\end{aligned} \tag{27}$$

and a contribution to the two-photon-two-nucleon amplitude from the second and fourth terms

$$\begin{aligned}&+2i\frac{e^2\kappa}{m}\tilde{b}\overline{u}(p_f)(i\sigma_{\mu\nu}\epsilon_1^\mu k_1^\nu i\sigma_{\alpha\beta}\epsilon_2^\alpha k_2^\beta\\&+i\sigma_{\alpha\beta}\epsilon_2^\alpha k_2^\beta i\sigma_{\mu\nu}\epsilon_1^\mu k_1^\nu)u_p(p_i)-4ie^2\tilde{b}^2\overline{u}(p_f)\\&\times[i\sigma_{\mu\nu}\epsilon_1^\mu k_1^\nu(\not p_i-\not k_2-m)i\sigma_{\alpha\beta}\epsilon_2^\alpha k_2^\beta\\&+i\sigma_{\alpha\beta}\epsilon_2^\alpha k_2^\beta(\not p_i+\not k_1-m)i\sigma_{\mu\nu}\epsilon_1^\mu k_1^\nu]u(p_i).\end{aligned} \tag{28}$$

Several terms originating in the covariant derivatives have been dropped, as they contribute to neither bremsstrahlung nor Compton scattering.

From the general result that physical amplitudes must be independent of a field transformation we know that the amplitudes from $\mathcal{L}_0$ and $\mathcal{L}_0+\Delta\mathcal{L}$ must be the same. Thus, to the order we are considering, these two Lagrangians, $\mathcal{L}_0+\Delta\mathcal{L}_1$ and $\mathcal{L}_0-\Delta\mathcal{L}_2$, will give exactly the same physically measurable amplitudes, yet the first generates off-shell contributions to the vertices and the second does not. To see this explicitly we must calculate the amplitudes for these processes in detail, which we will do in the next two sections.

[7]Because we work at tree level a distinction between irreducible and reducible vertex [12] is not necessary.

VI. EXPLICIT EVALUATION OF NUCLEON-NUCLEON BREMSSTRAHLUNG

Consider now the nucleon-nucleon bremsstrahlung process which we will evaluate using each of the two Lagrangians, $\mathcal{L}_0+\Delta\mathcal{L}_1$ and $\mathcal{L}_0-\Delta\mathcal{L}_2$. We consider only tree-level diagrams and so must include radiation from each of the proton legs, with off-shell contributions at both strong and electromagnetic vertices together with the contact terms appropriate for each Lagrangian. As noted earlier, we treat the Φ fields as neutrons and neglect radiation from their magnetic moment. Hence there are only two diagrams with radiation from external legs.

Consider first the Lagrangian $\mathcal{L}_0+\Delta\mathcal{L}_1$. The contribution from $\mathcal{L}_0$ has already been given in Eq. (16). The amplitude coming from the parts of $\Delta\mathcal{L}_1$ corresponding to off-shell contributions at strong or electromagnetic vertices, that is the contribution from $\mathcal{L}_0$ at one vertex, $\Delta\mathcal{L}_1$ at the other, with a propagator in between, is

$$ieg\overline{u}_n(p_4)u_n(p_2)\overline{u}_p(p_3)\times\left[2\tilde{a}\not\epsilon-\left(\frac{\tilde{a}\kappa}{m}-4\tilde{b}\right)i\sigma_{\mu\nu}\epsilon^\mu k^\nu\right]u_p(p_1). \tag{29}$$

The amplitude with off-shell contributions from both strong and electromagnetic vertices simultaneously is

$$2ieg\tilde{a}\tilde{b}\overline{u}_n(p_4)u_n(p_2)\overline{u}_p(p_3)[i\sigma_{\mu\nu}\epsilon^\mu k^\nu(\not p_3+\not k-m)+(\not p_1-\not k-m)i\sigma_{\mu\nu}\epsilon^\mu k^\nu]u_p(p_1). \tag{30}$$

Finally, the contribution of the contact term originating in the use of covariant instead of regular derivatives is

$$-2ieg\tilde{a}\overline{u}_n(p_4)u_n(p_2)\overline{u}_p(p_3)\not\epsilon u_p(p_1). \tag{31}$$

The full amplitude is the sum of Eqs. (16), (29), (30), and (31). Note that the contact term of Eq. (31) actually cancels the similar term coming from the off-shell contributions at the strong vertex in Eq. (29) and that the net result coming from $\Delta\mathcal{L}_1$ consists of a number of magnetic type terms proportional to $\sigma_{\mu\nu}\epsilon^\mu k^\nu$.

From one viewpoint this Lagrangian, $\mathcal{L}_0+\Delta\mathcal{L}_1$, can be considered simply as a purely phenomenological Lagrangian. It leads to a bremsstrahlung amplitude, which can be compared with experiment so as to extract values of the phenomenological parameters $\tilde{a},\tilde{b}$. Such an approach is perfectly acceptable as long as one is completely clear that the Lagrangian is just phenomenological. Its usefulness will depend on how close the model Lagrangian reproduces the real physical situation. Difficulties arise however when one makes the traditional, though as we shall see incorrect, claim that the values of $\tilde{a},\tilde{b}$ so obtained correspond to some measure of off-shell effects.

To see how this claim arises and why it is incorrect let us first see how our simple model is closely analogous to the traditional approaches. Thus for example in standard nonrelativistic potential model approaches to nucleon-nucleon bremsstrahlung, one would first calculate in the abstract an off-shell nucleon-nucleon amplitude corresponding to a potential. This would give a result analogous to the off-shell amplitude of Eq. (23) calculated in our model. Different on-shell-equivalent potentials could still give different amplitudes, corresponding to different values of $\tilde{a}$.

Similarly at the electromagnetic vertex one traditionally parametrizes the off-shell behavior in some way, or uses some model to obtain something analogous to Eq. (24). Various authors have used dispersion relations, simple pion-loop models, chiral perturbation theory, or purely phenomenological considerations. In all cases the abstract off-shell electromagnetic vertex is governed by a strength parameter similar to our $\tilde{b}$.

Thus in these traditional approaches one argues that in the abstract the off-shell contributions to the strong and electromagnetic interaction vertices are proportional to $\tilde{a}$ and $\tilde{b}$, respectively. These are parameters of the Lagrangian, which appear in the amplitude, and can thus be determined from a comparison of the amplitude with measured quantities. Therefore, one traditionally concludes the values of these parameters measure in some physical way off-shell behavior in the strong and electromagnetic interactions.

It is this last part of the argument which is incorrect. In actual fact the values of $\tilde{a}$ and $\tilde{b}$ tell us nothing, in any unambiguous way, about off-shell behavior. To see this in detail let us calculate the bremsstrahlung amplitude using the alternative Lagrangian $\mathcal{L}_0-\Delta\mathcal{L}_2$. We obtain from $-\Delta\mathcal{L}_2$ two contact terms. From the first term of Eq. (27)

$$-ieg\left(\frac{\tilde{a}\kappa}{m}-4\tilde{b}\right)\overline{u}_n(p_4)u_n(p_2)\overline{u}_p(p_3)i\sigma_{\mu\nu}\epsilon^\mu k^\nu u_p(p_1) \tag{32}$$

and from the second

$$2ieg\tilde{a}\tilde{b}\overline{u}_n(p_4)u_n(p_2)\overline{u}_p(p_3)[i\sigma_{\mu\nu}\epsilon^\mu k^\nu(\not{p}_3+\not{k}-m) + (\not{p}_1-\not{k}-m)i\sigma_{\mu\nu}\epsilon^\mu k^\nu]u_p(p_1). \tag{33}$$

Clearly the full amplitude from $\mathcal{L}_0-\Delta\mathcal{L}_2$, Eqs. (16), (32), and (33) is exactly the same as that from $\mathcal{L}_0+\Delta\mathcal{L}_1$, Eqs. (16), (29), (30), and (31), as it must be, by virtue of the general results from field transformation arguments. Thus a comparison with data using this Lagrangian will produce exactly the same values of $\tilde{a},\tilde{b}$ as with the original Lagrangian. However $\mathcal{L}_0-\Delta\mathcal{L}_2$ produces no off-shell effects at either strong or electromagnetic vertices, so there cannot be any meaningful connection between the values of $\tilde{a},\tilde{b}$ and off-shell effects.

More generally consider a combination of these two Lagrangians, $\mathcal{L}(\eta)=\mathcal{L}_0+(1-\eta)\Delta\mathcal{L}_1-\eta\Delta\mathcal{L}_2$ where η is an arbitrary real parameter. Since $\Delta\mathcal{L}_1$ and $-\Delta\mathcal{L}_2$ produce the same amplitudes, the result will be independent of η. Thus $\mathcal{L}(\eta)$ for arbitrary η will all lead to the same bremsstrahlung amplitude, that given by Eqs. (16), (29), (30) and (31), and hence to exactly the same values of $\tilde{a},\tilde{b}$ if this amplitude is compared to data. However the off-shell strong and electromagnetic vertices calculated with $\mathcal{L}(\eta)$ will have strengths $(1-\eta)\tilde{a}$ and $(1-\eta)\tilde{b}$, respectively. Thus a single set of $\tilde{a},\tilde{b}$ corresponds to arbitrary values of the off-shell vertices. Thus it should be very clear that such off-shell behavior is not a physically measurable quantity. It makes no sense to equate sensitivity to $\tilde{a},\tilde{b}$ with sensitivity to off-shell behavior, or to claim to be able to measure off-shell behavior by measuring parameters appearing in the amplitude, as has traditionally been done in discussions of off-shell behavior.

In retrospect this result perhaps should not be surprising. The concept of an off-shell amplitude is in some sense a mathematical concept, which applies only to a piece of a diagram. An off-shell particle is not physical and one can never measure it directly. Such amplitudes have meaning only when put into a larger diagram which has appropriate interactions to put the particle back on shell. Thus one should perhaps expect a large measure of ambiguity in describing such an intermediate, unphysical state.

VII. EXPLICIT EVALUATION OF COMPTON SCATTERING

The Lagrangian which has just been used to evaluate nucleon-nucleon bremsstrahlung also leads to an amplitude for Compton scattering. Only the electromagnetic part is required, so the simplification of the strong interaction used for nucleon-nucleon bremsstrahlung is not necessary. In this section we evaluate that amplitude explicitly and will find a situation exactly similar to that of nucleon-nucleon bremsstrahlung. Namely, since the total contribution from the part of the Lagrangian generated by the field transformation, $\Delta\mathcal{L}_1+\Delta\mathcal{L}_2$, vanishes there are two alternate Lagrangians (actually an infinite set of linear combinations) which give the same Compton amplitude, $\mathcal{L}_0+\Delta\mathcal{L}_1$ which produces off-shell contributions in the electromagnetic vertices and $\mathcal{L}_0-\Delta\mathcal{L}_2$ which produces only contact terms.

Consider first the Compton amplitude originating in the Lagrangian $\mathcal{L}_0+\Delta\mathcal{L}_1$. This Lagrangian is essentially identical to that used in phenomenological calculations where a phenomenological off-shell part is introduced at the electromagnetic gamma-nucleon-nucleon vertex. The contribution from $\mathcal{L}_0$ has been given in Eq. (17). Just as for nucleon-nucleon bremsstrahlung the contribution from $\Delta\mathcal{L}_1$ leads to an amplitude with an off-shell contribution at one or the other of the electromagnetic vertices of the form

$$\begin{aligned}&+2i\tilde{b}e^2\overline{u}(p_f)\left[\left(\not{\epsilon}_2-\frac{i\kappa}{2m}\sigma_{\mu\nu}\epsilon_2^\mu k_2^\nu\right)i\sigma_{\alpha\beta}\epsilon_1^\alpha k_1^\beta\right.\\&\left.\quad-i\sigma_{\alpha\beta}\epsilon_2^\alpha k_2^\beta\left(\not{\epsilon}_1+\frac{i\kappa}{2m}\sigma_{\mu\nu}\epsilon_1^\mu k_1^\nu\right)\right]u(p_i)\\&-2i\tilde{b}e^2\overline{u}(p_f)\left[\left(\not{\epsilon}_1+\frac{i\kappa}{2m}\sigma_{\mu\nu}\epsilon_1^\mu k_1^\nu\right)i\sigma_{\alpha\beta}\epsilon_2^\alpha k_2^\beta\right.\\&\left.\quad-i\sigma_{\alpha\beta}\epsilon_1^\alpha k_1^\beta\left(\not{\epsilon}_2-\frac{i\kappa}{2m}\sigma_{\mu\nu}\epsilon_2^\mu k_2^\nu\right)\right]u(p_i)\end{aligned} \tag{34}$$

and a contribution with an off-shell part of the vertex at both electromagnetic vertices given by

$$+4i\tilde{b}^2e^2\overline{u}(p_f)[i\sigma_{\mu\nu}\epsilon_2^\mu k_2^\nu(\not p_f+\not k_2-m)i\sigma_{\alpha\beta}\epsilon_1^\alpha k_1^\beta]u(p_i)$$
$$+4i\tilde{b}^2e^2\overline{u}(p_f)[i\sigma_{\mu\nu}\epsilon_1^\mu k_1^\nu(\not p_f-\not k_1-m)i\sigma_{\alpha\beta}\epsilon_2^\alpha k_2^\beta]u(p_i). \tag{35}$$

There is also a two-photon contact term coming from the use of the covariant derivative in the last part of $\Delta\mathcal{L}_1$ which can be written as

$$+2ie^2\tilde{b}\overline{u}(p_f)(\not\epsilon_1 i\sigma_{\mu\nu}\epsilon_2^\mu k_2^\nu-i\sigma_{\mu\nu}\epsilon_1^\mu k_1^\nu\not\epsilon_2$$
$$+i\sigma_{\mu\nu}\epsilon_2^\mu k_2^\nu\not\epsilon_1-\not\epsilon_2 i\sigma_{\mu\nu}\epsilon_1^\mu k_1^\nu)u(p_i). \tag{36}$$

Note that, just as in nucleon-nucleon bremsstrahlung, this contact term cancels the $\not\epsilon$ terms from the off-shell contribution of Eq. (34) above, leaving purely magnetic contributions.

The full Compton amplitude originating in $\mathcal{L}_0+\Delta\mathcal{L}_1$ is then given by the sum of Eqs. (17), (34), (35), and (36). This corresponds exactly to the Lagrangian which has been used in phenomenological calculations. Naively in such an approach, one notes the appearance of the parameter $\tilde{b}$ and also the fact that it appears in the off-shell electromagnetic vertex and thus one might, as in nucleon-nucleon bremsstrahlung, claim to be able to ''determine'' the parameter $\tilde{b}$ and thus the off-shell vertex by a measurement of Compton scattering.

However, just as in the nucleon-nucleon bremsstrahlung case discussed above, this has to be wrong. We have the equivalent Lagrangian $\mathcal{L}_0-\Delta\mathcal{L}_2$ which involves only contact terms, but gives the same measurable result. We can see this specifically. The contributions from $-\Delta\mathcal{L}_2$ to the Compton amplitude are just those of Eq. (28), i.e.,

$$-2i\frac{e^2\kappa}{m}\tilde{b}\overline{u}(p_f)(i\sigma_{\mu\nu}\epsilon_1^\mu k_1^\nu i\sigma_{\alpha\beta}\epsilon_2^\alpha k_2^\beta$$
$$+i\sigma_{\alpha\beta}\epsilon_2^\alpha k_2^\beta i\sigma_{\mu\nu}\epsilon_1^\mu k_1^\nu)u_p(p_i)+4ie^2\tilde{b}^2\overline{u}(p_f)$$
$$\times[i\sigma_{\mu\nu}\epsilon_1^\mu k_1^\nu(\not p_i-\not k_2-m)i\sigma_{\alpha\beta}\epsilon_2^\alpha k_2^\beta$$
$$+i\sigma_{\alpha\beta}\epsilon_2^\alpha k_2^\beta(\not p_i+\not k_1-m)i\sigma_{\mu\nu}\epsilon_1^\mu k_1^\nu]u(p_i). \tag{37}$$

Clearly this is the same amplitude as generated by $\Delta\mathcal{L}_1$. Now the argument is exactly the same as given at the end of the previous section. The same measurable amplitude is produced by a Lagrangian which generates an off-shell component at the photon-nucleon-nucleon vertex as by the one which has only contact terms, or in fact by any linear combination of the two. Thus the constant $\tilde{b}$ appearing in the Lagrangian is not in any way a ''measure'' of off-shell behavior.

There is another interesting observation one can make using this simple model, though one somewhat peripheral to the main line of argument. Consider the Lagrangian $\mathcal{L}_0+\Delta\mathcal{L}_1$ which corresponds to a standard phenomenological Lagrangian used to describe the photon-nucleon-nucleon vertex. Sometimes the argument is given that even if the coefficient $\tilde{b}$ does not represent an off-shell effect it can be used to parametrize the unknown features of the interaction. That is certainly possible. However such parametrizations have implications for a variety of processes described by the Lagrangian and do not always lead to results consistent with the data.

For example with this Lagrangian we get the full amplitude for Compton scattering from the sum of Eqs. (17), (34), (35), and (36) and can extract from that amplitude an expression for the proton electromagnetic polarizabilities, for which there is some experimental data. To do this we make a two-component reduction of this amplitude in the lab frame using Coulomb gauge which leads to

$$\mathcal{M}^{CS}=i\chi_f^\dagger\Bigg[-\frac{e^2}{m}\vec{\epsilon}_1\cdot\vec{\epsilon}_2+\cdots \tag{38}$$
$$+\omega_1\omega_2\vec{\epsilon}_1\cdot\vec{\epsilon}_2\left(-\frac{4e^2}{m}(\kappa\tilde{b}+4m^2\tilde{b}^2)\right)$$
$$+\omega_1\omega_2\vec{\epsilon}_2\times\hat{k}_2\cdot\vec{\epsilon}_1\times\hat{k}_1\left(\frac{4e^2\kappa\tilde{b}}{m}\right)\Bigg]\chi, \tag{39}$$

where the χ's are two-component nucleon spinors and where we have kept only the leading Thomson term and those terms contributing to the polarizabilities [for details see, e.g., Eqs. (4) and (6) of Ref. [35]].

From this we can extract the electromagnetic polarizabilities [35] as

$$\overline{\alpha}=-\frac{e^2}{4\pi}\frac{4}{m}(\kappa\tilde{b}+4m^2\tilde{b}^2), \tag{40}$$

$$\overline{\beta}=\frac{e^2}{4\pi}\frac{4\kappa\tilde{b}}{m}, \tag{41}$$

where the factor of 4π has been inserted to put the polarizabilities into conventional units, since in our notation $e^2/4\pi\approx 1/137$.

Now using the experimental value for the proton polarizability, $\overline{\alpha}=(12.1\pm0.8\pm0.5)\times10^{-4}$ fm^3 from Ref. [45] we try to solve for $\tilde{b}$. We find

$$\tilde{b}=\frac{\kappa}{8m^2}\left(-1\pm\sqrt{1-\frac{16\pi m^3\overline{\alpha}}{e^2\kappa^2}}\right). \tag{42}$$

Since $16\pi m^3\overline{\alpha}/e^2\kappa^2=22\gg1$ there is no real solution possible for $\tilde{b}$. This means that simply introducing a phenomenological term in the Lagrangian which produces an off-shell photon-nucleon-nucleon vertex, as has been done in many calculations can lead to results which are inconsistent with data for related reactions. Clearly even at the phenomenological level one must be consistent and check the prediction of such phenomenological terms with other processes generated by the same Lagrangian.

VIII. SUMMARY AND CONCLUSIONS

In this paper we have used the concept of field transformations and several simple models to illustrate the impossibility of measuring off-shell effects in nucleon-nucleon bremsstrahlung, Compton scattering, and by implication other medium-energy processes. In the first example, the nonlinear σ model Lagrangian, together with two standard representations of the pion field, was used for pion-pion bremsstrahlung. We showed by specific example that these two representations gave different off-shell scattering amplitudes, but exactly the same measurable amplitude for the bremsstrahlung. Thus the measurable quantity clearly cannot distinguish different off-shell behavior at the vertices making up the full process.

For the rest of the paper we looked at a spin 1/2 model Lagrangian. Again as a result of a field transformation we could generate a modified Lagrangian which produced an off-shell contribution to both strong and electromagnetic vertices. This Lagrangian was very closely related to the kinds of phenomenological Lagrangians which have been used to investigate off-shell effects at these vertices. Again we showed by specific example that changes in the Lagrangian which produce different off-shell vertices lead to exactly the same measurable amplitude. In effect we can interpret pieces of the measurable amplitude as off-shell effects at the vertices or as contact terms or as any combination of the two. Thus the concept of ''off-shell vertex,'' while perfectly well defined as a mathematical abstraction, does not translate into a physically measurable quantity. In short, off-shell effects are not measurable. The model used here properly included spin 1/2, did not depend on ChPT, or on a momentum expansion, and did not depend on gauge invariance, which should put to rest any possible concerns that the original considerations of Refs. [33,34] depended somehow on the simplicity of the model considered there.

Finally we should comment that it is certainly possible to construct a microscopic model of a process which will generate an off-shell form factor at the vertices, and perhaps some contact terms. This is in fact standard procedure. It is a legitimate approach and we can speak of the off-shell form factor in this model, just as we speak of the off-shell amplitude generated from a potential by solving the Lippman-Schwinger equation. The generation of such a off-shell form factor is unique and a function only of the properties of the model. However one cannot measure this off-shell form factor and thus cannot determine the correctness or incorrectness of the model based on its prediction for the off-shell form factor.

ACKNOWLEDGMENTS

This work was supported in part by a grant from the Natural Sciences and Engineering Research Council of Canada and by the Deutsche Forschungsgemeinschaft (SFB 443).

[1] R. L. Workman and H. W. Fearing, Phys. Rev. C **34**, 780 (1986).
[2] H. W. Fearing, Nucl. Phys. **A463**, 95c (1987).
[3] See, e.g., V. R. Brown, P. L. Anthony, and J. Franklin, Phys. Rev. C **44**, 1296 (1991); V. Herrmann and K. Nakayama, *ibid.* **46**, 2199 (1992); A. Katsogiannis and K. Amos, *ibid.* **47**, 1376 (1993); M. Jetter and H. V. von Geramb, *ibid.* **49**, 1832 (1994); G. H. Martinus, O. Scholten, and J. A. Tjon, Phys. Lett. B **402**, 7 (1997).
[4] M. Jetter and H. W. Fearing, Phys. Rev. C **51**, 1666 (1995).
[5] J. A. Eden and M. F. Gari, Phys. Rev. C **53**, 1102 (1996).
[6] K. Michaelian *et al.*, Phys. Rev. D **41**, 2689 (1990); P. Kitching *et al.*, Phys. Rev. Lett. **57**, 2363 (1986); Nucl. Phys. **A463**, 87c (1987).
[7] B. von Przewoski *et al.*, Phys. Rev. C **45**, 2001 (1992).
[8] K. Yasuda *et al.*, Phys. Rev. Lett. **82**, 4775 (1999).
[9] R. Bilger *et al.*, Phys. Lett. B **429**, 195 (1998).
[10] J. Zlomanczuk, A. Johansson, and the WASA-PROMICE Collaboration, Nucl. Phys. **A631**, 622c (1998).
[11] H. Huisman, Ph.D. thesis, University of Groningen, 1999; (unpublished).
[12] H. W. L. Naus and J. H. Koch, Phys. Rev. C **36**, 2459 (1987).
[13] H. W. L. Naus, S. J. Pollock, J. H. Koch, and U. Oelfke, Nucl. Phys. **A509**, 717 (1990).
[14] P. C. Tiemeijer and J. A. Tjon, Phys. Rev. C **42**, 599 (1990).
[15] X. Song, J. P. Chen, and J. S. McCarthy, Z. Phys. A **341**, 275 (1992).
[16] E. M. Nyman, Nucl. Phys. **A154**, 97 (1970); **A160**, 517 (1971).
[17] V. Herrmann, K. Nakayama, O. Scholten, and H. Arellano, Nucl. Phys. **A582**, 568 (1995).
[18] H. C. Dönges, M. Schäfer, and U. Mosel, Phys. Rev. C **51**, 950 (1995).
[19] S. Kondratyuk, G. Martinus, and O. Scholten, Phys. Lett. B **418**, 20 (1998).
[20] Y. Li, M. K. Liou, and W. M. Schreiber, Phys. Rev. C **57**, 507 (1998).
[21] J. W. Bos, S. Scherer, and J. H. Koch, Nucl. Phys. **A547**, 488 (1992).
[22] A. Yu. Korchin, O. Scholten, and F. de Jong, Phys. Lett. B **402**, 1 (1997).
[23] S. I. Nagorny and A. E. L. Dieperink, Eur. Phys. J. A **5**, 417 (1999).
[24] J. W. Bos and J. H. Koch, Nucl. Phys. **A563**, 539 (1993).
[25] S. Kondratyuk and O. Scholten, nucl-th/9906044.
[26] R. Haag, Phys. Rev. **112**, 669 (1958).
[27] J. S. R. Chisholm, Nucl. Phys. **26**, 469 (1961).
[28] S. Kamefuchi, L. O'Raifeartaigh, and A. Salam, Nucl. Phys. **28**, 529 (1961).
[29] S. Coleman, J. Wess, and B. Zumino, Phys. Rev. **177**, 2239 (1969).
[30] B. K. Jennings, Phys. Lett. B **196**, 307 (1987).
[31] M. Gell-Mann and M. L. Goldberger, Phys. Rev. **96**, 1433 (1954).
[32] T. E. Rudy, H. W. Fearing, and S. Scherer, Phys. Rev. C **50**, 447 (1994).
[33] S. Scherer and H. W. Fearing, Phys. Rev. C **51**, 359 (1995).

[34] H. W. Fearing, Phys. Rev. Lett. **81**, 758 (1998).
[35] A. I. L'vov, Int. J. Mod. Phys. A **8**, 5267 (1993).
[36] S. Scherer, A. Yu. Korchin, and J. H. Koch, Phys. Rev. C **54**, 904 (1996).
[37] H. W. Fearing and S. Scherer, Few-Body Syst. **23**, 111 (1998).
[38] A. M. Bincer, Phys. Rev. **118**, 855 (1960).
[39] J. W. Bos, Ph.D. thesis, University of Amsterdam, 1993.
[40] R. M. Davidson and G. I. Poulis, Phys. Rev. D **54**, 2228 (1996).
[41] S. Weinberg, Phys. Rev. Lett. **18**, 188 (1967).
[42] H. W. Fearing, Phys. Rev. C **6**, 1136 (1972).
[43] J. L. Powell, Phys. Rev. **75**, 32 (1949).
[44] S. Scherer and H. W. Fearing, Phys. Rev. D **52**, 6445 (1995).
[45] B. E. MacGibbon *et al.*, Phys. Rev. C **52**, 2097 (1995).