Inequality for approximate halo nuclear cross sections

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It is shown that the ''folding model'' approximation to Glauber's theory of the high-energy scattering of composites of strongly absorbed particles always overestimates the reaction cross section.

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In collisions between nuclei at incident energies above 100 MeV per nucleon it is a good approximation to treat the internal degrees of freedom of the nuclei adiabatically (frozen) and to use straight line trajectories for the motion of the individual nucleons. The approach based on these two approximations is often referred to as Glauber theory [1] and has been widely used [2]. However, in practical applications of the theory further approximations are often made. In particular the ''folding model'' approximation, which ignores certain correlation effects in multiple scattering terms, is frequently used. This approximation has been shown [3–5] to have serious shortcomings when a loosely bound neutron halo nucleus is involved as one partner in the collision. To reproduce the experimental fragmentation cross section data, the improved calculations of [3,4] required significantly larger halo radii than those deduced using the ''folding model,'' because the improved calculations reduced the contribution of the valence halo particles to the calculated reaction cross sections. It is shown here that this is a very general result.

We use a generalization of the formalism of [4], pp. 1844–1846. We consider the scattering of a projectile P consisting of $n-1$ valence nucleons and a core C. The valence nucleons and the core interact with the target through potentials V_{NT} and V_{CT}, respectively. In [4] these potentials are themselves given explicitly in terms of the nucleon-nucleon interaction and the target and core densities, but this detail is not necessary here.

If we assume that the internal coordinates which describe the configuration of the valence particles relative to each other and to the core are frozen during the collision (adiabatic approximation), and that the valence particles and the core travel on straight line trajectories (eikonal approximation), then the projectile-target elastic S matrix is given by [4]

$$S_P(b)=\left\langle \Phi_0\left|\exp i\left[\chi_{CT}(b_C)+\sum_{j=1}^{n-1}\chi_{NT}(b_j)\right]\right|\Phi_0\right\rangle, \quad (1)$$

where

$$\chi_{NT}(b_j)=-\frac{1}{\hbar v}\int_{-\infty}^{\infty}dz_j V_{NT}(\sqrt{b_j^2+z_j^2}), \quad (2)$$

and a similar formula for $\chi_{CT}(b_C)$ in terms of V_{CT}. Here v is the incident projectile speed and $\vec{b}$, $\vec{b}_j$, and $\vec{b}_C$ are the impact parameters of the projectile center of mass and of its constituent bodies, that is, their coordinates transverse to the direction of the incident projectile momentum; similarly, z, z_j, and z_C are their coordinates along the direction of the incident momentum.

The state vector Φ_0 describes the ground state of the projectile in its rest system and is a function of the relative coordinates of the valence particles and the core together with the internal variables of the latter. In the models used in [4], Φ_0 is explicitly antisymmetric in the valence particles but not in the valence and core particles. It is basic to the idea of a halo nucleus that such an approximation is a good starting point.

We shall compare $S_P(b)$, Eq. (1), with the expression

$$\hat{S}_P(b)=\exp[i\hat{\chi}_P(b)], \quad (3)$$

where

$$\hat{\chi}_P(b)=\left\langle \Phi_0\left|\left[\chi_{CT}(b_C)+\sum_{j=1}^{n-1}\chi_{NT}(b_j)\right]\right|\Phi_0\right\rangle. \quad (4)$$

We will refer to the approximation in Eq. (4) as the ''folding model'' [5].

Because the term $\Sigma_{j=1}^{n-1}\chi_{NT}(b_j)$ is a one-body operator in the valence-particle coordinates, the second term on the right hand side of Eq. (4) can be expressed in terms of the one-body matter density associated with the valence particles. If in addition V_{NT} and V_{CT} are approximated in terms of an average nucleon-nucleon scattering amplitude and the target matter density, in the standard way associated with Glauber microscopic scattering theory, we find that $\hat{S}_P(b)$ reduces to the ''static density'' or ''optical limit'' expressions given in [4] in terms of the matter density of the projectile and the target. We shall not need these expressions here, however.

A consequence of the upward concavity of the exponential function is the inequality

$$\exp y \geqslant 1+y, \quad y \text{ real} \quad (5)$$

[[6], Eq. (4.2.30)]. From this follows the inequality of expectation values:

$$\langle \exp Y \rangle \geqslant \langle 1 + Y \rangle, \tag{6}$$

where Y is a Hermitian operator. If we set $Y = F - \langle F \rangle$, we obtain

$$\langle \exp F \rangle \geqslant \exp \langle F \rangle, \tag{7}$$

where F is any Hermitian operator. In the field of statistical mechanics this inequality is usually referred to as Peierls' inequality (or theorem) [7]; it seems to have been first given by Delbrück and Molière [8].

The inequality (7) applies to $S_P(b)$ and $\hat{S}_P(b)$, defined in the previous section, when the interaction with the target is purely absorptive (very nearly true for a collision at sufficiently high energy with a nuclear target). The potentials V_{NT} and V_{CT} are then purely imaginary and $\hat{S}_P(b)$ and $S_P(b)$ are purely real.

Taking

$$F = i \left[\chi_{CT}(b_C) + \sum_{j=1}^{n-1} \chi_{NT}(b_j) \right], \tag{8}$$

we obtain

$$S_P(b) \geqslant \hat{S}_P(b), \tag{9}$$

and hence that

$$|S_P(b)| \geqslant |\hat{S}_P(b)|. \tag{10}$$

Note that the result in Eq. (10) may not be valid when the interactions between the constituents of the projectile and the target have real parts. For instance, for purely real V_{NT} and V_{CT}, $|\hat{S}_P(b)| = 1$, so in this case the inequality in Eq. (10) is reversed (since $|S_P(b)| \leqslant 1$ always).

An immediate deduction from Eq. (10) is that for purely absorptive two-body interactions and a given projectile wave function the ''folding model'' will always underestimate the transmitted amplitude at each impact parameter and hence it will always overestimate the total reaction cross section

$$\sigma_R(P) = 2\pi \int_0^\infty db\, b\, [1 - |S_P(b)|^2]. \tag{11}$$

There will therefore be a tendency for projectile sizes to be underestimated. This was just what was found by direct calculation in [3–5]. A particularly clear example of the effect within the context of a simple model for Φ_0 can be found in [10], Secs. 2.1–2.4. The inequality (7) was also used implicitly in [9] in a discussion of the effects of the composite nature of hadrons on high-energy hadron-nucleus total cross sections.

The result of this paper is that, for purely imaginary $\chi_{CT}(b_C)$ and $\chi_{NT}(b_j)$,

$$\langle \Phi_0 | \exp(i\chi_P) | \Phi_0 \rangle \geqslant \exp \langle \Phi_0 | i\chi_P | \Phi_0 \rangle, \tag{12}$$

where

$$\chi_P = \chi_{CT}(b_C) + \sum_{j=1}^{n-1} \chi_{NT}(b_j). \tag{13}$$

The inequality arises because of the spatial extension of the probability density $|\Phi_0|^2$. When all the internal coordinates of the projectile are completely localized the *equality* will hold. Of course in the case of halo nuclei the halo particles are not strongly localized in space. The two sides of the inequality are therefore very different and the static density approximation will seriously overestimate the reaction cross section.

Higher-order terms in the expansion of the exponentials in Eq. (12) provide corrections for multiple scattering and interference effects which are particularly important if the projectile constituents lie in each other's ''shadow'' [1]. The ''folding model'' calculates these effects using scattering phases averaged over the ground state spatial probability distribution of the core and valence nucleons. This is not unreasonable for the contribution from a compact massive core, but for valence nucleons which can roam over a large volume it will produce inaccuracies. Our quantitative discussion here closely resembles the qualitative suggestion of [11] for the source of the improvements identified in [3,4].

Note that calculations using either the more complete Eq. (1) or (folding model) Eqs. (3) and (4), can both be regarded as based on an adiabatic treatment of the internal variables of the projectile. The implications of adiabaticity for the reaction cross sections of light nuclei were studied in [12,13]. There the probability distribution for the projectile coordinates and black disk reaction probabilities for its constituents were handled in a purely classical way. It was found that the adiabatic effect reduced the calculated reaction probability compared with geometrical estimates. Calculations based on Eq. (11) and either Eq. (1) or (3) include quantum mechanical effects which are not included in the work of Nishioka and Johnson [12,13]. The inequality proved here relates to a comparison between two reaction cross section estimates, both of which differ from the estimates of Nishioka and Johnson [12,13] and include additional physical effects.

The results of this work have important implications for the interpretation of high-energy nuclear reactions in terms of the structure of halo nuclei. Our results confirm and make rigorous the conclusions from the work of [3–5,14] that the folding model may be seriously misleading for nuclei with a halolike structure.

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[1] R. J. Glauber, in *Lectures in Theoretical Physics*, edited by W. E. Brittin (Interscience, New York, 1959), Vol. 1, p. 315.
[2] V. Franco and G. K. Varma, Phys. Rev. C **18**, 349 (1978).
[3] J. S. Al-Khalili and J. A. Tostevin, Phys. Rev. Lett. **76**, 3903 (1996).
[4] J. S. Al-Khalili, I. J. Thompson, and J. A. Tostevin, Phys. Rev. C **54**, 1843 (1996).
[5] H. Esbensen and G. F. Bertsch, Phys. Rev. C **59**, 3240 (1999).
[6] *Handbook of Mathematical Functions*, edited by M. Abramowitz and Irene A. Stegun (Dover, New York, 1965).
[7] R. Peierls, Phys. Rev. **54**, 918 (1938).
[8] M. Delbrück and G. Molière, Abh. Preuss. Akad. Wiss. 1 (1936).
[9] D. R. Harrington, Phys. Rev. C **52**, 926 (1995).
[10] J. A. Tostevin, R. C. Johnson, and J. S. Al-Khalili, Nucl. Phys. **A630**, 340c (1998).
[11] P. G. Hansen, Nature (London) **384**, 413 (1996).
[12] H. Nishioka and R. C. Johnson, Phys. Rev. C **22**, 2457 (1980).
[13] H. Nishioka and R. C. Johnson, J. Phys. G **8**, 39 (1982).
[14] R. C. Johnson, J. S. Al-Khalili, and J. A. Tostevin, Phys. Rev. Lett. **79**, 2771 (1997).