$\Delta I=1$ staggering in octupole bands of light actinides: ''Beat'' patterns

Dennis Bonatsos,[1] C. Daskaloyannis,[2] S. B. Drenska,[3] N. Karoussos,[1] N. Minkov,[3] P. P. Raychev,[3] and R. P. Roussev[3]
[1]*Institute of Nuclear Physics, N.C.S.R. ''Demokritos,'' GR-15310 Aghia Paraskevi, Attiki, Greece*
[2]*Department of Physics, Aristotle University of Thessaloniki, GR-54006 Thessaloniki, Greece*
[3]*Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, 72 Tzarigrad Road, BG-1784 Sofia, Bulgaria*
(Received 18 October 1999; published 23 June 2000)

The $\Delta I=1$ staggering (odd-even staggering) in octupole bands of light actinides is found to exhibit a ''beat'' behavior as a function of the angular momentum I, forcing us to revise the traditional belief that this staggering decreases gradually to zero and then remains at this zero value. Various algebraic models [spf-interacting boson model (spdf-IBM), vector boson model, nuclear vibron model] are shown to predict in their su(3) limits constant staggering for this case, being thus unable to describe the ''beat'' behavior. An explanation of the ''beat'' behavior is given in terms of two Dunham expansions [expansions in terms of powers of $I(I+1)$] with slightly different sets of coefficients for the ground-state band and the negative parity band, the difference in the values of the coefficients being attributed to Coriolis couplings to other negative parity bands. Similar ''beat'' patterns have already been seen in rotational bands of some diatomic molecules, like AgH.

PACS number(s): 21.10.Re, 21.60.Fw, 21.60.Ev

I. INTRODUCTION

Rotational nuclear spectra have long been attributed to quadrupole deformations [1], corresponding to nuclear shapes produced by the revolution of an ellipsis around its maximum or minimum axis and rotating around an axis perpendicular to their axis of symmetry. In addition, it has been suggested that octupole deformation occurs in certain regions, most notably in the light actinides [2] and in the $A\approx150$ mass region [3,4], corresponding to pearlike nuclear shapes [5–8]. In even nuclei exhibiting octupole deformation the ground-state band, which contains energy levels with $I^\pi=0^+,2^+,4^+,6^+,\ldots$, is accompanied by a negative parity band containing energy levels with $I^\pi=1^-,3^-,5^-,7^-,\ldots$. After the first few values of angular momentum I the two bands become interwoven, forming a single octupole band with levels characterized by $I^\pi=0^+,1^-,2^+,3^-,4^+,5^-,\ldots$ [2–4]. (It should be noted, however, that in the light actinides alternative interpretations of these bands in terms of alpha clustering have been proposed [9,10].)

It has been observed [11] that in octupole bands the levels with odd I and negative parity ($I^\pi=1^-,3^-,5^-,\ldots$) are displaced relatively to the levels with even I and positive parity ($I^\pi=0^+,2^+,4^+,\ldots$), i.e., the odd levels do not lie at the energies predicted by an $E(I)=AI(I+1)$ fit to the energy levels, but all of them lie systematically above or all of them lie systematically below the predicted energies. This is an example of *odd-even staggering* or $\Delta I=1$ *staggering*, the latter term due to the fact that each energy level with angular momentum I is displaced relatively to its neighbors with angular momenta $I\pm1$.

A similar $\Delta I=1$ staggering effect (i.e., a relative displacement of the levels with odd I with respect to the levels of even I) is known to occur in rotational γ bands of even nuclei [12], the difference being that in γ bands all levels possess positive parity.

The $\Delta I=1$ staggering effect is different from the $\Delta I=2$ *staggering* effect recently observed [13–15] in superdeformed nuclear bands [16–18], since the $\Delta I=2$ staggering effect refers to the systematic displacement of the levels with $I=2,6,10,14,\ldots$ relative to the levels with $I=0,4,8,12,\ldots$, i.e., in this case the level with angular momentum I is displaced relative to its neighbors with angular momenta $I\pm2$.

On the other hand, rotational spectra of diatomic molecules [19] are known to show great similarities to nuclear rotational spectra, having in addition the advantage that observed rotational bands in several diatomic molecules are much longer than the usual rotational nuclear bands. In fact both $\Delta I=1$ [20] and $\Delta I=2$ staggering effects [21,22] have been recently observed in rotational spectra of several diatomic molecules. $\Delta I=2$ staggering has been attributed [22] to the presence of one or more bandcrossings [23,24], while $\Delta I=1$ staggering remains an open problem.

It should be noted that all these effects are much larger than the relevant experimental errors, with the notable exception of the $\Delta I=2$ staggering effect in superdeformed nuclear bands [13–15], for which only one case [the (a) band of ^{149}Gd [14]] is known to show an effect outside the limits of the experimental errors.

The dependence of the amplitude of the staggering effect on the angular momentum I presents much interest. The situation up to now is as follows:

(1) Algebraic models of nuclear structure appropriate for the description of octupole bands, like the spf-interacting boson model (spf-IBM) with u(11) symmetry [25], the spdf-IBM with u(16) symmetry [25,26], and the vector boson model (VBM) with u(6) symmetry [27–29], predict in their su(3) limits $\Delta I=1$ staggering of constant amplitude, i.e., all the odd levels are raised (or lowered) by the same amount of energy with respect to the even levels. In other words, $\Delta I=1$ staggering takes alternatively positive and negative values of equal absolute value as I increases.

(2) Algebraic models of nuclear structure suitable for the description of alpha clustering effects, like the nuclear vibron model (NVM) with u(6)⊗u(4) symmetry [9], also predict in

the su(3) limit $\Delta I=1$ staggering of constant amplitude.

(3) Older experimental work [2–4] on octupole nuclear bands suggests that $\Delta I=1$ staggering starts from large values and its amplitude decreases with increasing I. These findings are in agreement with the interpretation that an octupole band is gradually formed as angular momentum increases [5,6].

(4) Recent work on experimental data for diatomic molecules shows that in some rotational bands $\Delta I=1$ staggering of constant amplitude seems to appear [20], while in other bands a variety of shapes, reminiscent of beats, are exhibited [20].

Motivated by these recent findings, we make in the present work a systematic study in the light actinide region of all octupole bands for which at least 12 energy levels are known [30–36], taking advantage of recent detailed experimental work in this region. The questions to which we have hoped to provide answers are

(1) Which patterns of behavior of the amplitude of the $\Delta I=1$ staggering appear? Are these patterns related to the ones seen in diatomic molecules [20]?

(2) Can these patterns be interpreted in terms of the existing models [9,25–29], or in terms of any other theoretical description?

In Sec. II of the present paper the formalism of staggering is discussed, and is subsequently applied to the experimental data for octupole bands of light actinides in Sec. III. Section IV contains the relevant predictions of various algebraic models, while an interpretation of the experimental observations is given in Sec. V. Finally, Sec. VI contains the conclusions reached, as well as plans for future work.

II. FORMALISM

Traditionally the odd-even staggering ($\Delta I=1$ staggering) in octupole bands, as well as in gamma bands, has been estimated quantitatively through use of the expression [11]

$$\delta E(I)=E(I)-\frac{(I+1)E(I-1)+IE(I+1)}{2I+1}, \tag{1}$$

where $E(I)$ denotes the energy of the level with angular momentum I. This expression vanishes for

$$E(I)=E_0+AI(I+1), \tag{2}$$

but not for

$$E(I)=E_0+AI(I+1)+B[I(I+1)]^2. \tag{3}$$

Therefore, it is suitable for measuring deviations from the pure rotational behavior.

Recently, however, a new measure of the magnitude of staggering effects has been introduced [15] in the study of $\Delta I=2$ staggering of nuclear superdeformed bands. In this case the experimentally determined quantities are the γ-ray transition energies between levels differing by two units of angular momentum ($\Delta I=2$). For these the symbol

$$E_{2,\gamma}(I)=E(I+2)-E(I) \tag{4}$$

is used. The deviation of the γ-ray transition energies from the rigid rotator behavior is then measured by the quantity [15]

$$\Delta E_{2,\gamma}(I)=\frac{1}{16}[6E_{2,\gamma}(I)-4E_{2,\gamma}(I-2)-4E_{2,\gamma}(I+2)+E_{2,\gamma}(I-4)+E_{2,\gamma}(I+4)]. \tag{5}$$

Using the rigid rotator expression of Eq. (2) one can easily see that in this case $\Delta E_{2,\gamma}(I)$ vanishes. In addition, the perturbed rigid rotator expression of Eq. (3) gives vanishing $\Delta E_{2,\gamma}(I)$. These properties are due to the fact that Eq. (5) is a (normalized) discrete approximation of the fourth derivative of the function $E_{2,\gamma}(I)$, i.e., essentially the fifth derivative of the function $E(I)$. Therefore, we conclude that Eq. (5) is a more sensitive probe of deviations from rotational behavior than Eq. (1).

By analogy, $\Delta I=1$ staggering in nuclei can be measured by the quantity

$$\Delta E_{1,\gamma}(I)=\frac{1}{16}[6E_{1,\gamma}(I)-4E_{1,\gamma}(I-1)-4E_{1,\gamma}(I+1)+E_{1,\gamma}(I-2)+E_{1,\gamma}(I+2)], \tag{6}$$

where

$$E_{1,\gamma}(I)=E(I+1)-E(I). \tag{7}$$

The transition energies $E_{1,\gamma}(I)$ are determined directly from experiment.

III. ANALYSIS OF EXPERIMENTAL DATA

We have applied the formalism described above to all octupole bands of light actinides for which at least 12 energy levels are known [30–36] and which show no backbending (i.e., bandcrossing) [37] behavior. These nuclei are listed in Table I, along with the relevant values of the R_4 ratio,

$$R_4=\frac{E(4)}{E(2)}, \tag{8}$$

a well-known characteristic of collective behavior. Several nuclei ($^{222-226}$Ra, $^{224-228}$Th) are rotational or near-rotational (having $10/3\geqslant R_4\geqslant 2.7$), while others ($^{218-222}$Rn, ^{220}Ra, $^{220-222}$Th) are vibrational or near-

TABLE I. Nuclei included in the study and their $R_4=E(4)/E(2)$ ratios [Eq. (8)].

Nucleus	R_4	Nucleus	R_4	Nucleus	R_4
^{218}Rn	2.014	^{218}Ra	1.905	^{220}Th	2.035
^{220}Rn	2.214	^{220}Ra	2.298	^{222}Th	2.399
^{222}Rn	2.408	^{222}Ra	2.715	^{224}Th	2.896
		^{224}Ra	2.970	^{226}Th	3.136
		^{226}Ra	3.127	^{228}Th	3.235

vibrational (having $2.4 \geqslant R_4 \geqslant 2$). A special case is ^{218}Ra, for which it has been argued [31] that it is an example of a new type of transitional nuclei, in which the octupole deformation dominates over all other types of deformation.

The staggering results for $^{218-222}$Rn, $^{218-226}$Ra, and $^{220-228}$Th, are shown in Figs. 1, 2, and 3, respectively. In all cases the experimental errors are of the size of the symbol used for the experimental point and therefore are not visible. The following observations can be made:

(1) In all cases the shapes appearing are consistent with the following pattern: $\Delta I = 1$ staggering starts from large values at low I, it gradually decreases down to zero, then it starts increasing again, then it decreases down to zero and starts raising again. In other words, figures resembling beats appear. The most complete ''beat'' figures appear in the cases of ^{220}Ra, ^{224}Ra, ^{222}Th, as well as in the cases of ^{218}Ra, ^{222}Ra, ^{226}Ra.

(2) In all cases within the first ''beat'' [from the beginning up to the first zero of $\Delta E_{1,\gamma}(I)$] the minima appear at odd I, indicating that in this region the odd levels are slightly raised in comparison to the even levels. Within the second ''beat'' [i.e., between the first and the second zero of $\Delta E_{1,\gamma}(I)$], the opposite holds: the minima appear at even I, indicating that in this region the odd levels are slightly lowered in comparison to the even levels. Within the third ''beat'' [after the second zero of $\Delta E_{1,\gamma}(I)$] the situation occurring within the first ''beat'' is repeated. (Notice that ^{220}Th is not an exception, since what is seen in the figure is the second ''beat,'' starting from $I = 6$.)

(3) In the case of ^{222}Rn the decrease of the staggering with increasing I, in the region for which experimental data exist, is very slow, giving the impression of almost constant staggering. One can get a similar impression from parts of the patterns shown, as, for example, in the cases of ^{220}Ra (in the region $I = 12-20$), ^{222}Ra (for $I = 9-17$), ^{224}Ra (for $I = 10-16$), ^{226}Ra (for $I = 14-20$), ^{222}Th (for $I = 10-18$).

These observations bear considerable similarities to $\Delta I = 1$ staggering patterns found in rotational bands of diatomic molecules. In particular:

(1) Staggering patterns of almost constant amplitude have been found in some rotational bands of the AgH [20] molecule.

(2) Staggering patterns resembling the ''beat'' structure have been seen in several bands of the AgH molecule [20].

The following comments are also in place:

(1) In all cases bands not influenced by bandcrossing effects [37] have been considered, in order to make sure that the observed effects are ''pure'' single-band effects. The only exception is ^{220}Th, which shows signs of bandcrossings at 10^+ and 13^-, which, however, do not influence the relevant staggering pattern, which is shown in Fig. 3(a) for reasons of completeness. A special case is ^{218}Ra, which shows a rather irregular dependence of $E(I)$ on I. As we have already mentioned, it has been argued [31] that this nucleus is an example of a new type of transitional nuclei in which the octupole deformation dominates over all other types of deformation.

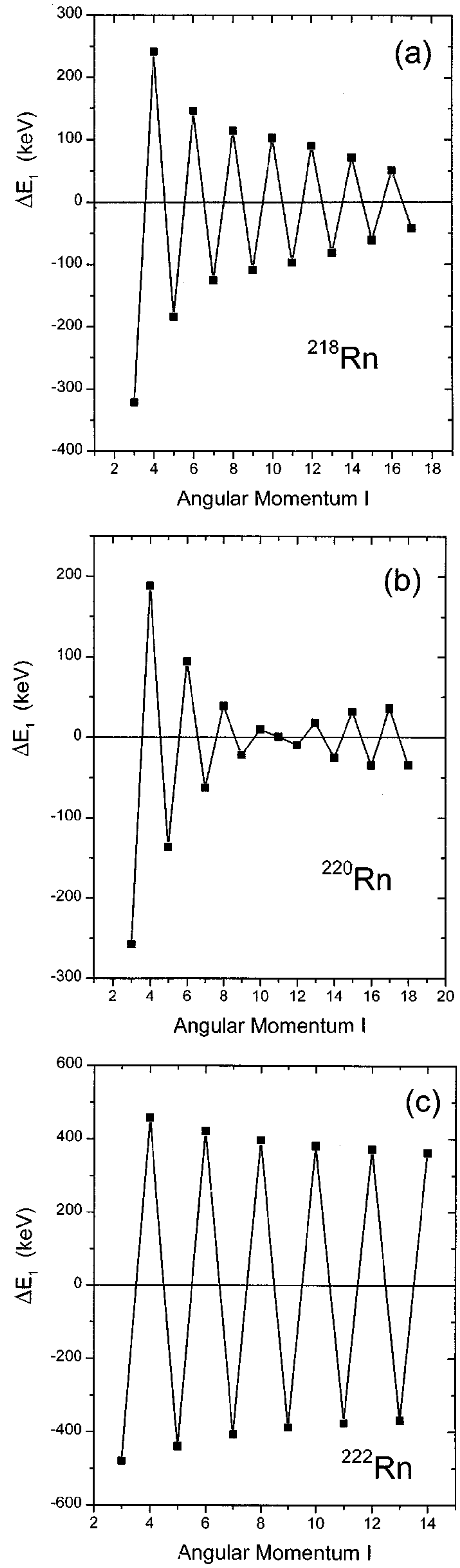

FIG. 1. $\Delta E_1(I)$ (in keV), calculated from Eq. (6), for octupole bands of (a) ^{218}Rn [30], (b) ^{220}Rn [30], and (c) ^{222}Rn [30]. The experimental error in all cases is of the order of the symbol used for the experimental point and therefore is not seen. See Sec. III for discussion.

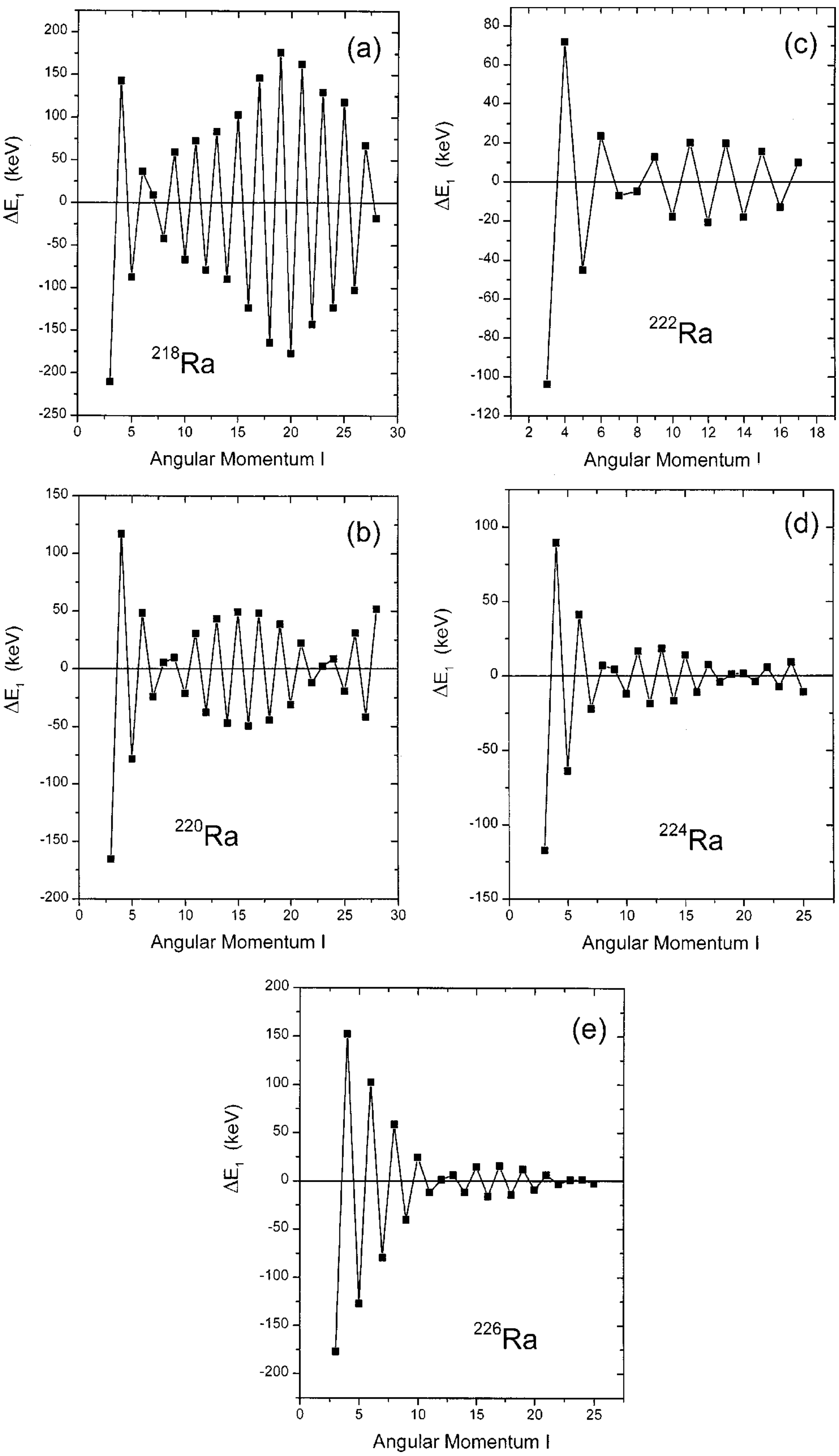

FIG. 2. Same as Fig. 1, but for (a) ^{218}Ra [31], (b) ^{220}Ra [32], (c) ^{222}Ra [30], (d) ^{224}Ra [30], and (e) ^{226}Ra [30].

(2) The same ''beat'' pattern appears in both rotational and vibrational nuclei. The only slight difference which can be observed, is that the first vanishing of the staggering amplitude seems to occur at higher I for the rotational isotopes than for their vibrational counterparts. Indeed, within the Ra and Th series of isotopes under study, the I at which the first vanishing of the staggering amplitude occurs seems to be an increasing function of R_4, i.e., an increasing function of the

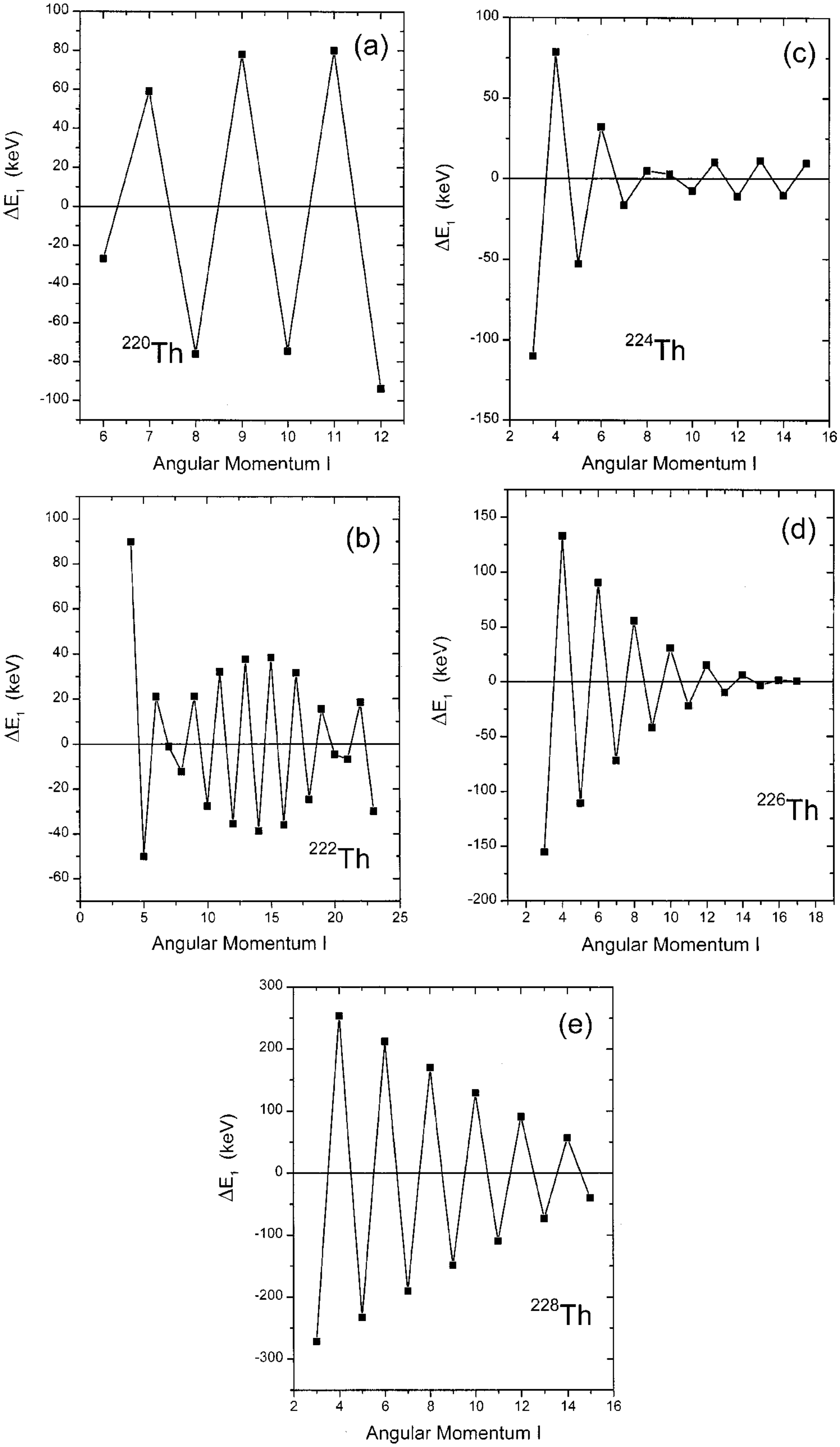

FIG. 3. Same as Fig. 1, but for (a) ^{220}Th [32], (b) ^{222}Th [33], (c) ^{224}Th [34], (d) ^{226}Th [35], and (e) ^{228}Th [36].

quadrupole collectivity.

(3) The present findings are partially consistent with older works [2–4]. The limited sets of data of that time were reaching only up to the I at which the first vanishing of the staggering amplitude occurs. It was then reasonable to assume that the staggering amplitude decreases down to zero and remains zero afterwards, since no experimental evidence for ''beat'' patterns existed at that time.

IV. ALGEBRAIC MODELS

As we have seen in the previous section, certain $\Delta I=1$ staggering patterns occur in the octupole bands of the light actinides. Before attempting any interpretation of these results, it is instructive to examine what kind of staggering patterns are predicted by various algebraic models of nuclear structure describing such bands. As we have already mentioned, these models are related to the description of octupole degrees of freedom, which are responsible for the presence of octupole bands, i.e., bands with a sequence of levels with $I^\pi=0^+,1^-,2^+,3^-,4^+,5^-,\ldots$ [2–4]. These bands are thought to be present in cases in which the nucleus acquires a shape with octupole deformation, i.e., a pearlike shape [5,6].

A. The spf-interacting boson model

In the spf-IBM [25], which possesses a u(11) symmetry, *s*, *p*, and *f* bosons (i.e., bosons with angular momentum 0, 1, and 3, respectively) are used. Octupole bands are described in the su(3) limit, which corresponds to the chain

$$\mathrm{u}(11)\supset\mathrm{u}(10)\supset\mathrm{su}(3)\supset\mathrm{o}(3)\supset\mathrm{o}(2). \tag{9}$$

The relevant basis is

$$|N,N_b,\omega_b,(\lambda_b,\mu_b),K_b,I,M\rangle, \tag{10}$$

where N is the total number of bosons labeling the irreducible representations (irreps) of u(11), N_b is the total number of negative parity bosons (p and f) labeling the irreps of u(10), ω_b is the ‘‘missing’’ quantum number in the decomposition u(10)⊃su(3),(λ_b,μ_b) are the Elliott quantum numbers [38] labeling the irreps of su(3), K_b is the ‘‘missing’’ quantum number in the decomposition su(3)⊃o(3) [38], I is the angular momentum quantum number labelling the irreps of o(3), M is the z component of the angular momentum labeling the irreps of o(2). The energy eigenvalues are given by

$$E(N_b,\lambda_b,\mu_b,I)=\alpha+\beta N_b+\gamma N_b^2+\kappa C(\lambda_b,\mu_b)+\kappa' I(I+1), \tag{11}$$

where

$$C(\lambda,\mu)=\lambda^2+\mu^2+\lambda\mu+3\lambda+3\mu. \tag{12}$$

It is clear that positive parity states occur when N_b is even, while negative parity states occur when N_b is odd. In the case of N being even, the ground-state band is sitting in the $(3N,0)$ irrep, while the odd levels of negative parity are sitting in the $(3N-3,0)$ irrep. Then from Eq. (6) one obtains

$$\Delta E(I)=\begin{cases}-(\beta+\gamma(2N-1)+18\kappa N), & \text{for } I=\text{even},\\ +(\beta+\gamma(2N-1)+18\kappa N), & \text{for } I=\text{odd}.\end{cases} \tag{13}$$

In the case of N being odd, the ground-state band is sitting in the $(3N-3,0)$ irrep, while the odd levels of negative parity are sitting in the $(3N,0)$ irrep. Then from Eq. (6) one has

$$\Delta E(I)=\begin{cases}+(\beta+\gamma(2N-1)+18\kappa N) & \text{for } I=\text{even},\\ -(\beta+\gamma(2N-1)+18\kappa N) & \text{for } I=\text{odd}.\end{cases} \tag{14}$$

Since N is a constant for a given nucleus, expressing the number of valence nucleon pairs counted from the nearest closed shells [39], we see that $\Delta I=1$ staggering of constant amplitude is predicted.

B. The spdf-interacting boson model

In the spdf-interacting boson model [25,26], which possesses a u(16) symmetry, *s, p, d*, and *f* bosons (i.e., bosons with angular momentum 0, 1, 2, and 3, respectively) are taken into account. Octupole bands are described in the su(3) limit, which corresponds to the chain

$$\mathrm{u}(16)\supset\mathrm{u}_a(6)\otimes\mathrm{u}_b(10)\supset\mathrm{su}_a(3)\otimes\mathrm{su}_b(3)\supset\mathrm{su}(3)\supset\mathrm{o}(3)\supset\mathrm{o}(2). \tag{15}$$

The relevant basis is

$$|N,N_a,N_b,\omega_b,(\lambda_a,\mu_a),(\lambda_b,\mu_b),(\lambda,\mu),K,I,M\rangle, \tag{16}$$

where N is the total number of bosons labeling the irreps of u(16), N_a is the number of positive parity bosons labeling the irreps of $\mathrm{u}_a(6)$, and N_b is the number of negative parity bosons labelling the irreps of $\mathrm{u}_b(10)$. The rest of the quantum numbers are analogous to those appearing in the basis of the u(11) model, described above. su(3) is the algebra obtained by adding the corresponding generators of $\mathrm{su}_a(3)$ and $\mathrm{su}_b(3)$. The energy eigenvalues are given by

$$E(N_b,\lambda_a,\mu_a,\lambda_b,\mu_b,\lambda,\mu,I)=\alpha+\beta N_b+\gamma N_b^2+\kappa_a C(\lambda_a,\mu_a)+\kappa_b C(\lambda_b,\mu_b)+\kappa C(\lambda,\mu)+\kappa' I(I-1), \tag{17}$$

with $C(\lambda,\mu)$ defined as in Eq. (12).

The ground-state band is sitting in the $(2N,0)_a$ irrep (which contains N bosons of positive parity and no bosons of negative parity), while the odd levels of negative parity are sitting in the $(2N-2,0)_a(3,0)_b(2N+1,0)$ band (which contains $N-1$ bosons of positive parity and one boson of negative parity). Then from Eq. (6) one has

$$\Delta E(I)=\begin{cases}+[\beta+\gamma-2k_a(4N+1)+18k_b+4k(N+1)] & \text{for } I=\text{even},\\ -[\beta+\gamma-2k_a(4N+1)+18k_b+4k(N+1)] & \text{for } I=\text{odd}.\end{cases} \tag{18}$$

Therefore, $\Delta I=1$ staggering of constant amplitude is predicted, since N is a constant for a given nucleus, representing the number of valence nucleon pairs counted from the nearest closed shells [39].

Another limit of the spdf-IBM in which octupole bands occur is the o(4) limit [26], which corresponds to the chain

$$\mathrm{u}(16)\supset \mathrm{u}(4)_a\otimes \mathrm{u}(4)_b\supset \mathrm{sp}(4)_a\otimes \mathrm{sp}(4)_b\supset \mathrm{su}(2)_a\otimes \mathrm{su}(2)_b \supset \mathrm{o}(3)\supset \mathrm{o}(2), \tag{19}$$

and owes its name to the isomorphism

$$\mathrm{su}(2)_a\otimes \mathrm{su}(2)_b\approx \mathrm{o}(4). \tag{20}$$

The relevant basis is

$$|N,(n_1,n_2,n_3,n_4),(n'_{1a},n'_{2a}),(n'_{1b},n'_{2b}),\nu,j_a,j_b,I,M\rangle, \tag{21}$$

where N is the total number of bosons labeling the irreps of u(16), (n_1,n_2,n_3,n_4) are labeling the irreps of $\mathrm{u}(4)_a$ and $\mathrm{u}(4)_b$, (n'_{1a},n'_{2a}) and (n'_{1b},n'_{2b}) are labeling the irreps of $\mathrm{sp}(4)_a$ and $\mathrm{sp}(4)_b$, respectively, ν denotes the three missing quantum numbers required in this case, j_a and j_b label the irreps of $\mathrm{su}(2)_a$ and $\mathrm{su}(2)_b$ respectively, while I and M have the same meaning as before. The energy eigenvalues are given by

$$\begin{aligned}E(N,n_1,n_2,n_3,n_4,n'_{1a},n'_{2a},n'_{1b},n'_{2b},\nu,j_a,j_b,I,M)\\ =E_0-2A[j_a(j_a+1)+j_b(j_b+1)]+(B+A)I(I+1)\\ =E_0-A[\omega(\omega+2)+(\omega')^2]+(B+A)I(I+1),\end{aligned} \tag{22}$$

where (ω,ω') are labeling the irreps of o(4) and are connected to j_a and j_b through the relations

$$\omega=j_a+j_b, \quad \omega'=|j_a-j_b|. \tag{23}$$

The lowest lying irrep is the irrep $(3N,0)$, which contains states of positive parity and states of negative parity together, i.e., it contains the states $0^+,1^-,2^+,3^-,4^+,5^-, \ldots,$ up to the state with $I=3N$. It is clear that in this case Eq. (6) gives a vanishing result, i.e., no $\Delta I=1$ staggering occurs in this limit.

C. The vector boson model

In the vector boson model (VBM) [27–29], the collective states are described in terms of two distinct kinds of vector bosons, whose creation operators $\boldsymbol{\xi}^+$ and $\boldsymbol{\eta}^+$ are o(3) vectors and in addition transform according to two independent su(3) irreducible representations (irreps) of the type $(\lambda,\mu)=(1,0)$, i.e., they are two distinct bosons of angular momentum 1. Octupole bands are described in the su(3) limit of the VBM, which corresponds to the chain

$$\mathrm{u}(6)\supset \mathrm{su}(3)\otimes \mathrm{u}(2)\supset \mathrm{so}(3)\otimes \mathrm{u}(1). \tag{24}$$

The relevant basis is

$$|N,(\lambda,\mu),(N,T),K,I,T_0\rangle, \tag{25}$$

where N is the total number of bosons labeling the irreps of u(6), (λ,μ) are the Elliott quantum numbers [38] labeling the irreps of su(3), N and T are the quantum numbers labelling the irreps of u(2), K is the ‘‘missing’’ quantum number in the su(3)$\supset$so(3) decomposition [38], I is the angular momentum quantum number labeling the irreps of so(3), and T_0 is the pseudospin projection quantum number labeling the irreps of u(1). The algebras su(3) and u(2) are mutually complementary [40–42], their irreps (λ, μ) and (N, T) being related by

$$N=\lambda+2\mu, \quad T=\lambda/2. \tag{26}$$

The energy eigenvalues are given by

$$\begin{aligned}E(N,\lambda,\mu,K,I,T_0=T)=aN+a_6N(N+5)+a_3C(\lambda,\mu)\\ +b_3I(I+1)+a_1\frac{\lambda^2}{4},\end{aligned} \tag{27}$$

with $C(\lambda,\mu)$ defined as in Eq. (12).

The ground-state band is sitting in the $(0,\mu)=(0,N/2)$ irrep of su(3), while the odd levels of negative parity are sitting in the $(2,\mu-1)=(2,N/2-1)$ irrep. Then from Eq. (6) one obtains

$$\Delta E(I)=\begin{cases}+(6a_3+a_1), & \text{for } I=\text{even},\\ -(6a_3+a_1), & \text{for } I=\text{odd}.\end{cases} \tag{28}$$

Therefore, $\Delta I=1$ staggering of constant amplitude is predicted.

D. The nuclear vibron model

As we have already mentioned, an alternative interpretation of the low-lying negative parity states appearing in the light actinides has been given following the assumption that alpha clustering is important in this region [9,10]. An algebraic model appropriate for the description of clustering effects in nuclei is the nuclear vibron model [9], which uses s and d bosons for the description of nuclear collectivity, plus s' and p bosons for taking into account the distance separating the center of the cluster from the center of the remaining nucleus. The chain corresponding to the su(3) limit of this model is

$$\mathrm{u}(6)\otimes\mathrm{u}(4)\supset\mathrm{su}_a(3)\otimes\mathrm{u}_b(3)\supset\mathrm{su}_a(3)\otimes\mathrm{su}_b(3)\supset\mathrm{su}(3)\supset\mathrm{o}(3)\supset\mathrm{o}(2), \tag{29}$$

where the subscript a labels the subalgebras of u(6), while the subscript b labels the subalgebras of u(4). The relevant basis is

$$|N,M,(\lambda_a,\mu_a),n_p,(\lambda,\mu),\chi,I,M\rangle, \tag{30}$$

where N is the number of the s and d bosons related to the u(6) algebra, M is the number of the s' and p bosons related to the u(4) algebra, (λ_a,μ_a) are the Elliott quantum numbers [38] related to $\mathrm{su}_a(3)$, n_p is the number of p bosons, (λ,μ) are the Elliott quantum numbers related to su(3), χ is the Vergados ''missing'' quantum number [43] in the decomposition su(3)⊃o(3), while I and M represent the angular momentum and its z component respectively, as usual. The energy eigenvalues are given by

$$E(n_p,\lambda_a,\mu_a,\lambda,\mu,I)=\epsilon_p n_p+\alpha_p n_p(n_p+3)+\kappa_d C(\lambda_a,\mu_a)+\kappa C(\lambda,\mu)+\kappa' I(I+1), \tag{31}$$

with $C(\lambda,\mu)$ defined as in Eq. (12).

The ground-state band is characterized by $(\lambda_a,\mu_a)=(2N,0)$, $n_p=0$, $(\lambda,\mu)=(2N,0)$ [i.e., it contains N bosons of positive parity and no p boson of negative parity], while the negative parity band is characterized by $(\lambda_a,\mu_a)=(2N,0)$, $n_p=1$, $(\lambda,\mu)=(2N+1,0)$ (i.e., it contains N bosons of positive parity plus one p boson of negative parity). Then from Eq. (6) one has

$$\Delta E(I)=\begin{cases}+[\epsilon_p+4\alpha_p+4\kappa(N+1)] & \text{for } I=\text{even},\\ -[\epsilon_p+4\alpha_p+4\kappa(N+1)] & \text{for } I=\text{odd}.\end{cases} \tag{32}$$

Therefore, $\Delta I=1$ staggering of constant amplitude is predicted.

E. Discussion

We conclude that the various algebraic models, describing low-lying negative parity bands in terms of octupole deformation [25–29] or in terms of alpha clustering [9], predict in their su(3) limits odd-even staggering ($\Delta I=1$ staggering) of constant amplitude. In all cases the staggering results from the fact that the negative parity states belong to an irrep different from the one in which the positive parity states composing the ground-state band sit.

It should be noticed, as already remarked in Sec. III, that the experimental data indicate that the value of I at which the first vanishing of the staggering amplitude occurs increases as a function of R_4, i.e., as the rotational limit is approached. The higher the value of I at which the first vanishing occurs, the more smooth the decrease of the staggering as a function of I is. We see, therefore, that as the rotational limit is approached, the experimental data approach more and more the constant staggering prediction provided by the various algebraic models. The best example is provided by ^{228}Th, the most rotational among the nuclei studied here.

As far as limits of algebraic models different from the su(3) limit are concerned, no staggering occurs in the o(4) limit of the spdf-IBM, which has been fully worked out [26]. Working out the details of other non-su(3) limits, like the ones of the vector boson model mentioned in Ref. [27], is an interesting open problem.

V. INTERPRETATION OF THE EXPERIMENTAL OBSERVATIONS

Although the results of the previous section are sufficient for providing an explanation for $\Delta I=1$ staggering in the cases in which this appears as having almost constant amplitude, it is clear that some additional thinking is required for the many cases in which the experimental results show a ''beat'' pattern, as in Sec. III has been exhibited.

A simple explanation for the appearance of beat patterns can be given by the following assumptions:

(1) It is clear that in each nucleus the even levels form the ground-state band, which starts at zero energy, while the odd levels form a separate negative parity band, which starts at some higher energy. Let us call E_0 the bandhead energy of the negative parity band.

(2) It is reasonable to try to describe the ground-state band by an expression like

$$E_+(I)=AI(I+1)-B(I(I+1))^2+C(I(I+1))^3+\cdots, \tag{33}$$

where the subscript + reminds us of the positive parity of these levels. Such expansions in terms of powers of $I(I+1)$ have been long used for the description of nuclear collective bands [44]. They also occur if one considers [45] Taylor expansions of the energy expressions provided by the variable moment of inertia model [46] and the $\mathrm{su}_q(2)$ model [47]. Notice that fits to experimental data [44] indicate that one always has $A>0,B>0,C>0,\ldots,$ while A is usually three orders of magnitude larger than B, B is 3 orders of magnitude larger than C, etc. Equation (33) has been long used in molecular spectroscopy as well, under the name of Dunham expansion [48].

(3) In a similar way, it is reasonable to try to describe the negative parity levels by an expression like

$$E_-(I)=E_0+A'I(I+1)-B'(I(I+1))^2+C'(I(I+1))^3+\cdots, \tag{34}$$

where the subscript − reminds us of the negative parity of these levels, while E_0 is the above-mentioned bandhead energy. In analogy to the previous case one expects to have $A'>0,B'>0,C'>0,\ldots$.

(4) In the above expansions it is reasonable to assume that $A>A',B>B',C>C',\ldots$. This assumption is in agreement with earlier work [49–51], in which the Coriolis couplings between the lowest $K=0$ negative parity band and higher negative parity bands with $K\neq 0$ are taken into account, resulting in an increase of the moment of inertia of the lowest $K=0$ negative parity band [52]. This argument means

that the coefficient A' in Eq. (34), which is inversely proportional to the moment of inertia of the negative parity band, should be smaller than the coefficient A in Eq. (33), which is inversely proportional to the moment of inertia of the positive parity band. In analogy to the relation $A>A'$, which we just justified, one can assume $B>B', C>C', \ldots$. This last argument is admitedly a weak one, which is however driving to interesting results, as we shall soon see.

Using Eqs. (33) and (34) in Eqs. (6) and (7) we find the following results:

$$\Delta E(I)=E_0-(A-A')(I^2+2I+2)+(B-B')\left(I^4+4I^3+13I^2+18I+\frac{23}{2}\right)-(C-C')\left(I^6+6I^5+33I^4+92I^3+\frac{357}{2}I^2+\frac{333}{2}I+68\right)+45C'(I+1)+\cdots, \quad \text{for} \quad I=\text{even}, \tag{35}$$

$$\Delta E(I)=-E_0+(A-A')(I^2+2I+2)-(B-B')\left(I^4+4I^3+13I^2+18I+\frac{23}{2}\right)+(C-C')\left(I^6+6I^5+33I^4+92I^3+\frac{357}{2}I^2+\frac{333}{2}I+68\right)-45C'(I+1)+\cdots, \quad \text{for} \quad I=\text{odd}. \tag{36}$$

A sample staggering pattern drawn using these formulas is shown in Fig. 4. On these results the following comments can be made:

(1) The expression for odd I is the opposite of the expression with even I. This explains why in Fig. 4 the staggering points for even I and the staggering points for odd I form two lines which are reflection symmetric with respect to the horizontal axis.

(2) For even I the behavior of the staggering amplitude is as follows: At low I it starts from a positive value, because of the presence of E_0. As I increases, the second term, which is essentialy proportional to I^2, becomes important. [E_0 is expected to be much larger than $(A-A')$.] This term is negative (since $A>A'$), thus it decreases the amplitude down to negative values. At higher values of I the third term, which is essentially proportional to I^4, becomes important. (Remember that usually B is 3 orders of magnitude smaller than A [44].) This term is positive (since $B>B'$), thus it increases the amplitude up to positive values. (The behavior up to this point can be seen in Fig. 4.) At even higher values of I the fourth term, which is essentially proportional to I^6, becomes important. (Remember that usually C is 3 orders of magnitude smaller than B [44].) This term is negative (since $C>C'$), thus it decreases the amplitude again down to negative values, and so on.

(3) For odd I the behavior of the staggering amplitude is exactly the opposite of the one described in (2) for even I. The amplitude starts from a negative value and then becomes consequently positive (because of the second term), negative (because of the third term), again positive (because of the fourth term), and so on. The first three steps of this behavior can be seen in Fig. 4.

(4) When drawing the staggering figure one jumps from an even I to an odd I, then back to an even I, then back to an odd I, and so on. It is clear therefore that a beat pattern appears, as it is seen in Fig. 4.

The following additional comments are also in place:

(1) In the case of a single band (i.e., in the case of $A=A'$, $B=B'$, $C=C'$, etc.), the first contribution to the staggering measure $\Delta E(I)$ is the last term in Eqs. (35), (36), which comes from the $C(I(I+1))^3$ term in the energy expansion [see Eqs. (33), (34)]. This is understandable: Since Eq. (6) is a discrete approximation of the fifth derivative of the function $E(I)$, as has already been remarked, the terms up to $B(I(I+1))^2$ are ''killed'' by the derivative, while the $C(I(I+1))^3$ term gives a contribution linear in I.

(2) The last term in Eqs. (35), (36) does not influence significantly the behavior of the staggering pattern, since C is usually six orders of magnitude smaller than A and 3 orders of magnitude smaller than B [44].

(3) One could argue that the above reasoning is valid only for the case of rotational or near-rotational bands, for which the expansions of Eqs. (33), (34) are known to be adequate (although one should be reminded at this point that the VMI model describes quite well not only rotational, but also transitional and even vibrational nuclei). One can attempt to mend this problem by adding to the expansions of Eqs. (33) and (34) a linear term, in the spirit of the Ejiri formula [53], the variable anharmonic vibrator model [54], and the u(5) and o(6) limits of the interacting boson model [55]

$$E_+(I)=A_1I+AI(I+1)-B(I(I+1))^2+C(I(I+1))^3+\cdots, \tag{37}$$

$$E_-(I)=E_0+A_1'I+A'I(I+1)-B'(I(I+1))^2+C'(I(I+1))^3+\cdots. \tag{38}$$

Then Eqs. (35) and (36) get modified as follows:

$$\Delta E(I)=E_0-(A_1-A_1')\left(I+\frac{1}{2}\right)-(A-A')(I^2+2I+2)+(B-B')\left(I^4+4I^3+13I^2+18I+\frac{23}{2}\right)-\cdots, \quad \text{for} \quad I=\text{even}, \tag{39}$$

$$\Delta E(I)=-E_0+(A_1-A_1')\left(I+\frac{1}{2}\right)+(A-A')(I^2+2I+2)-(B-B')\left(I^4+4I^3+13I^2+18I+\frac{23}{2}\right)+\cdots, \quad \text{for} \quad I=\text{odd}. \tag{40}$$

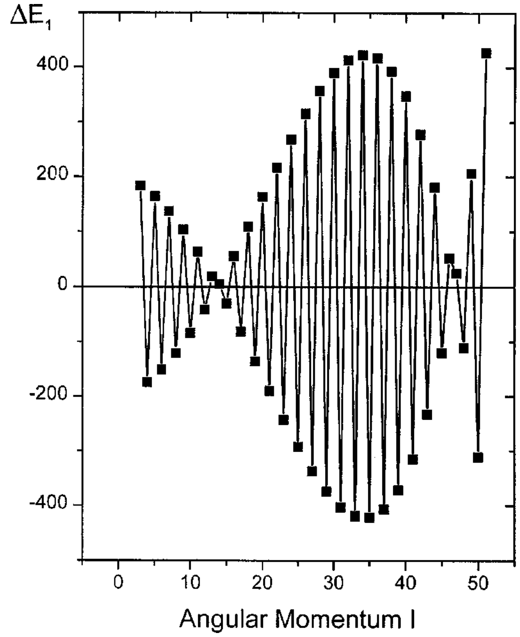

FIG. 4. $\Delta E_1(I)$, calculated from Eq. (6), using for the levels with even I the expansion of Eq. (33) with $A=10$, $B=5\ 10^{-4}$, $C=0$, and for the levels with odd I the expansion of Eq. (34) with $E_0=200$, $A'=9$, $B'=10^{-4}$, $C'=0$. See Sec. V for discussion.

We see that the extra term, which is proportional to (A_1-A_1'), plays the same role as the term proportional to $(A-A')$ in shaping up the behavior of the staggering amplitude. Therefore the conclusions reached above for rotational nuclei apply equally well to vibrational and transitional nuclei as well.

(4) This type of explanation of the staggering patterns seems to be outside the realm of the form of the su(3) limits of the algebraic models presented above. Even if one decides to include higher-order terms of the type $(I(I+1))^2$, $(I(I+1))^3$, etc., in these models, by including in the Hamiltonian higher powers of the relevant Casimir operator, these terms will appear with the same coefficients for both the ground-state band and the negative parity band, even though these two bands belong to different irreps. The only possible contributions to the staggering will then come from terms like the last term in Eqs. (35) and (36), which comes from the term $(I(I+1))^3$, and similar terms coming from higher powers of $I(I+1)$. However, the term $(I(I+1))^3$ in the framework of the algebraic models already corresponds to six-body interactions [39], which are usually avoided in nuclear structure studies.

We conclude, therefore, that the beat pattern can be explained in terms of two Dunham expansions with slightly different sets of coefficients, one for the ground-state band with quadrupole deformation and another for the negative parity band in which in addition the octupole deformation appears. This is, however, a phenomenological finding, the microscopic origins of which should be searched for. On this open problem the following comments apply:

(1) As has been mentioned above, the Coriolis coupling between the lowest $K=0$ negative parity band and higher $K\neq 0$ negative parity bands [49–51] results in an increase of the moment of inertia of the lowest $K=0$ negative parity band [52], offering in this way an argument in favor of using different coefficients in the Dunham expansions for the negative parity states and the positive parity states of the octupole band. However, this argument holds for the coefficients of the $I(I+1)$ terms only. If Coriolis coupling leads to different coefficients for the rest of the terms of the Dunham expansion remains to be seen.

(2) Nuclei with octupole deformation (pear-shaped nuclei) are supposed to be described by double-well potentials, the relative displacement of the negative parity levels and the positive parity levels being attributed to the tunneling through the barrier separating the wells [5,6,56]. The relative displacement vanishes in the limit in which the barrier separating the two wells becomes infinitely high. It should be examined if the details of the relevant potentials [5,6,56] give rise to a ''beating'' behavior of the relevant displacement.

(3) The coupling between the quadrupole modes and the octupole modes can also give rise to relative displacement of the negative parity levels and the positive parity levels of the octupole band [57]. In this case the octupole deformation can be parametrized in the way described in Refs. [58–60]. It should be examined if ''beating'' patterns appear in this case. Work in this direction is in progress.

VI. DISCUSSION

We have demonstrated that octupole bands in the light actinides exhibit $\Delta I=1$ staggering (odd-even staggering), the amplitude of which shows a beat behavior. The same pattern appears in both vibrational and rotational nuclei, forcing us to modify the traditional belief that in octupole bands the staggering pattern is gradually falling down to zero as a function of the angular momentum I and then remains there.

It has also been demonstrated that the su(3) limits of various algebraic models, including octupole degrees of freedom [25–29] or based on the assumption that alpha clustering is important in this region [9], predict $\Delta I=1$ staggering of amplitude constant as a function of the angular momentum I. Although this description becomes reasonable in the rotational limit, it cannot explain the beat patterns appearing in both the rotational and the vibrational regions. The detailed study of limits other than the su(3) ones for these models remains an interesting open problem.

A simple explanatation of the beat behavior has been given by describing the even I levels of the ground-state band and the odd I levels of the negative parity band by two Dunham expansions [48] [expansions in powers of $I(I+1)$] with slightly different sets of coefficients, the difference in the coefficients being attributed to Coriolis couplings of the negative parity band to other negative parity bands. However, the microscopic origins of the beat behavior need further elucidation, for example in the ways mentioned at the end of Sec. V.

The ''beat'' patterns found here in the octupole bands of the light actinides bear striking similarities to the ''beat'' patterns seen in the rotational bands of some diatomic molecules, like AgH [20]. It is expected that an explanation of the ''beat'' behavior in terms of two Dunham expansions with slightly different sets of coefficients should be equally applicable in this case.

It is also of interest to check if beat patterns appear in other kinds of bands as well. Preliminary results indicate that such patterns appear in some gamma bands (^{164}Er, ^{170}Yb,) as well as in a variety of negative parity bands. Further work in this direction is needed.

ACKNOWLEDGMENTS

P.P.R. acknowledges support from the Bulgarian Ministry of Science and Education under Contract No. Φ-547. N.M. has been supported by the Bulgarian National Fund for Scientific Research under Contract No. MU-E-02/98. D.B., C.D., and N.K. have been supported by the Greek Secretariat of Research and Technology under Contract No. PENED 95/1981.

[1] A. Bohr and B. R. Mottelson, *Nuclear Structure Vol. II: Nuclear Deformations* (World Scientific, Singapore, 1998).

[2] P. Schüler, Ch. Lauterbach, Y. K. Agarwal, J. de Boer, K. P. Blume, P. A. Butler, K. Euler, Ch. Fleischmann, C. Günther, E. Hauber, H. J. Maier, M. Marten-Tölle, Ch. Schandera, R. S. Simon, R. Tölle, and P. Zeyen, Phys. Lett. B **174**, 241 (1986).

[3] W. R. Phillips, I. Ahmad, H. Emling, R. Holzmann, R. V. F. Janssens, T.-L. Khoo, and M. W. Drigert, Phys. Rev. Lett. **57**, 3257 (1986).

[4] R. K. Sheline and P. C. Sood, Phys. Rev. C **34**, 2362 (1986).

[5] G. A. Leander, R. K. Sheline, P. Möller, P. Olanders, I. Ragnarsson, and A. J. Sierk, Nucl. Phys. **A388**, 452 (1982).

[6] G. A. Leander and R. K. Sheline, Nucl. Phys. **A413**, 375 (1984).

[7] I. Ahmad and P. A. Butler, Annu. Rev. Nucl. Part. Sci. **43**, 71 (1993).

[8] P. A. Butler and W. Nazarewicz, Rev. Mod. Phys. **68**, 349 (1996).

[9] H. J. Daley and F. Iachello, Ann. Phys. (N.Y.) **167**, 73 (1986).

[10] B. Buck, A. C. Merchant, and S. M. Perez, Phys. Rev. C **57**, R2095 (1998).

[11] W. Nazarewicz and P. Olanders, Nucl. Phys. **A441**, 420 (1985).

[12] D. Bonatsos, Phys. Lett. B **200**, 1 (1988).

[13] S. Flibotte, H. R. Andrews, G. C. Ball, C. W. Beausang, F. A. Beck, G. Belier, T. Byrski, D. Curien, P. J. Dagnall, G. de France, D. Disdier, G. Duchêne, Ch. Finck, B. Haas, G. Hackman, D. S. Haslip, V. P. Janzen, B. Kharraja, J. C. Lisle, J. C. Merdinger, S. M. Mullins, W. Nazaerwicz, D. C. Radford, V. Rauth, H. Savajols, J. Styczen, Ch. Theisen, P. J. Twin, J. P. Vivien, J. C. Waddington, D. Ward, K. Zuber, and S. Åberg, Phys. Rev. Lett. **71**, 4299 (1993).

[14] S. Flibotte, G. Hackman, I. Ragnarsson, Ch. Theisen, H. R. Andrews, G. C. Ball, C. W. Beausang, F. A. Beck, G. Bélier, M. A. Bentley, T. Byrski, D. Curien, G. de France, D. Disdier, G. Duchêne, B. Haas, D. S. Haslip, V. P. Janzen, P. M. Jones, B. Kharraja, J. A. Kuehner, J. C. Lisle, J. C. Merdinger, S. M. Mullins, E. S. Paul, D. Prévost, D. C. Radford, V. Rauch, J. F. Smith, J. Styczen, P. J. Twin, J. P. Vivien, J. C. Waddington, D. Ward, and K. Zuber, Nucl. Phys. **A584**, 373 (1995).

[15] B. Cederwall, R. V. F. Janssens, M. J. Brinkman, I. Y. Lee, I. Ahmad, J. A. Becker, M. P. Carpenter, B. Crowell, M. A. Deleplanque, R. M. Diamond, J. E. Draper, C. Duyar, P. Fallon, L. P. Farris, E. A. Henry, R. G. Henry, J. R. Hughes, T. L. Khoo, T. Lauritsen, A. O. Macchiavelli, E. Rubel, F. S. Stephens, M. A. Stoyer, W. Satula, I. Wiedenhoever, and R. Wyss, Phys. Rev. Lett. **72**, 3150 (1994).

[16] P. J. Twin, B. M. Nyakó, A. H. Nelson, J. Simpson, M. A. Bentley, H. W. Cranmer-Gordon, P. D. Forsyth, D. Howe, A. R. Mokhtar, J. D. Morrison, J. F. Sharpey-Schafer, and G. Sletten, Phys. Rev. Lett. **57**, 811 (1986).

[17] P. J. Nolan and P. J. Twin, Annu. Rev. Nucl. Part. Sci. **38**, 533 (1988).

[18] R. V. F. Janssens and T. L. Khoo, Annu. Rev. Nucl. Part. Sci. **41**, 32 (1991).

[19] G. Herzberg, *Molecular Spectra and Molecular Structure, Vol. I: Spectra of Diatomic Molecules* (Van Nostrand, Toronto, 1950).

[20] P. Raychev, J. Maruani, and S. Drenska, Phys. Rev. A **56**, 2759 (1997).

[21] D. Bonatsos, C. Daskaloyannis, S. B. Drenska, G. A. Lalazissis, N. Minkov, P. P. Raychev, and R. P. Roussev, Phys. Rev. A **54**, R2533 (1996).

[22] D. Bonatsos, C. Daskaloyannis, S. B. Drenska, N. Karoussos, J. Maruani, N. Minkov, P. P. Raychev, and R. P. Roussev, Phys. Rev. A **60**, 253 (1999).

[23] I. M. Pavlichenkov, Phys. Lett. **53B**, 35 (1974).

[24] L. P. Marinova, P. P. Raychev, and J. Maruani, Mol. Phys. **82**, 1115 (1994).

[25] J. Engel and F. Iachello, Phys. Rev. Lett. **54**, 1126 (1985).

[26] J. Engel and F. Iachello, Nucl. Phys. **A472**, 61 (1987).

[27] A. Georgieva, P. Raychev, and R. Roussev, J. Phys. G **8**, 1377 (1982).

[28] A. Georgieva, P. Raychev, and R. Roussev, J. Phys. G **9**, 521 (1983).

[29] A. Georgieva, P. Raychev, and R. Roussev, Bulg. J. Phys. **12**, 147 (1985).

[30] J. F. C. Cocks, P. A. Butler, K. J. Cann, P. T. Greenlees, G. D. Jones, S. Asztalos, P. Bhattacharyya, R. Broda, R. M. Clark, M. A. Deleplanque, R. M. Diamond, P. Fallon, B. Fornal, P. M. Jones, R. Julin, T. Lauritsen, I. Y. Lee, A. O. Macchiavelli, R. W. MacLeod, J. F. Smith, F. S. Stephens, and C. T. Zhang, Phys. Rev. Lett. **78**, 2920 (1997).

[31] N. Schulz, V. Vanin, M. Aïche, A. Chevallier, J. Chevallier, J. C. Sens, Ch. Briançon, S. Cwiok, E. Ruchowska, J. Fernandez-Niello, Ch. Mittag, and J. Dudek, Phys. Rev. Lett. **63**, 2645 (1989).

[32] A. Artna-Cohen, Nucl. Data Sheets **80**, 157 (1997).

[33] Y. A. Akovali, Nucl. Data Sheets **77**, 271 (1996).

[34] A. Artna-Cohen, Nucl. Data Sheets **80**, 227 (1997).

[35] Y. A. Akovali, Nucl. Data Sheets **77**, 433 (1996).
[36] A. Artna-Cohen, Nucl. Data Sheets **80**, 723 (1997).
[37] M. J. A. de Voigt, J. Dudek, and Z. Szymanski, Rev. Mod. Phys. **55**, 949 (1983).
[38] J. P. Elliott, Proc. R. Soc. London, Ser. A **245**, 128 (1958).
[39] F. Iachello and A. Arima, *The Interacting Boson Model* (Cambridge University Press, Cambridge, 1987).
[40] M. Moshinsky and C. Quesne, J. Math. Phys. **11**, 1631 (1970).
[41] G. Couvreur, J. Deenen, and C. Quesne, J. Math. Phys. **24**, 779 (1983).
[42] C. Quesne, J. Phys. A **18**, 2675 (1985).
[43] J. D. Vergados, Nucl. Phys. **A111**, 681 (1968).
[44] F. X. Xu, C. S. Wu, and J. Y. Zeng, Phys. Rev. C **40**, 2337 (1989).
[45] D. Bonatsos, E. N. Argyres, S. B. Drenska, P. P. Raychev, R. P. Roussev, and Yu. F. Smirnov, Phys. Lett. B **251**, 477 (1990).
[46] M. A. J. Mariscotti, G. Scharff-Goldhaber, and B. Buck, Phys. Rev. **178**, 1864 (1969).
[47] P. P. Raychev, R. P. Roussev, and Yu. F. Smirnov, J. Phys. G **16**, L137 (1990).
[48] J. L. Dunham, Phys. Rev. **41**, 721 (1932).
[49] K. Neergård and P. Vogel, Nucl. Phys. **A145**, 33 (1970).
[50] K. Neergård and P. Vogel, Nucl. Phys. **A149**, 217 (1970).
[51] P. Vogel, Phys. Lett. **60B**, 431 (1976).
[52] S. G. Rohoziński and W. Greiner, Phys. Lett. **128B**, 1 (1983).
[53] H. Ejiri, M. Ishihara, M. Sakai, K. Katori, and T. Inamura, J. Phys. Soc. Jpn. **24**, 1189 (1968).
[54] D. Bonatsos and A. Klein, Phys. Rev. C **29**, 1879 (1984).
[55] D. Bonatsos, C. Daskaloyannis, A. Faessler, P. P. Raychev, and R. P. Roussev, Phys. Rev. C **50**, 497 (1994).
[56] H. J. Krappe and U. Wille, Nucl. Phys. **A124**, 641 (1969).
[57] S. G. Rohoziński, M. Gajda, and W. Greiner, J. Phys. G **8**, 787 (1982).
[58] I. Hamamoto, B. Mottelson, H. Xie, and X. Z. Zhang, Z. Phys. D: At., Mol. Clusters **21**, 163 (1991).
[59] I. Hamamoto, X. Z. Zhang, and H. X. Xie, Phys. Lett. B **257**, 1 (1991).
[60] S. G. Rohoziński, J. Phys. G **16**, L173 (1990).