

Collective states in nuclei and many-body random interactions

R. Bijker¹ and A. Frank^{1,2}

¹*Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, Apartado Postal 70-543, 04510 México, D.F., México*

²*Centro de Ciencias Físicas, Universidad Nacional Autónoma de México, Apartado Postal 139-B, Cuernavaca, Morelos, México*

(Received 17 February 2000; published 1 June 2000)

Low-lying collective states in nuclei are investigated in the framework of the interacting boson model using an ensemble of random many-body interactions. It is shown that whenever the number of bosons is sufficiently large compared to the rank of the interactions, the spectral properties are characterized by a dominance of $L^P=0^+$ ground states and the occurrence of both vibrational and rotational band structures. This indicates that these features represent a general and robust property of the collective model space.

PACS number(s): 21.10.Re, 21.60.Ev, 21.60.Fw, 24.60.Lz

I. INTRODUCTION

Random matrix ensembles provide a powerful tool to study generic spectral properties of complex many-body systems [1]. Most applications in the literature have centered on global characteristics such as first neighbor energy distributions, which typically involve states with the same quantum numbers (angular momentum, parity, isospin, ...). Recently, the relation between low-lying states in even-even nuclei with different quantum numbers was examined using Hamiltonians with random interactions in the nuclear shell model (SM) [2–5] and the interacting boson model (IBM) [6,7]. These studies have given rise to several surprising results. In both cases it was found that for a large variety of conditions there is a dominance ($\geq 60\%$) of $L^P=0^+$ ground states despite the random nature of the interactions. In addition, in the SM strong evidence was found for the occurrence of pairing properties [4], and in the IBM for both vibrational and rotational band structures [6]. These results are not only based on energies, but also involve the behavior of the wave functions via the pair transfer amplitudes in the case of pairing, and the quadrupole transitions for the collective bands. The use of random interactions (both in size and in sign) show that these regular features arise for a much wider class of Hamiltonians than are usually considered to be realistic. These results are in qualitative agreement with the empirical observations of very robust features in the low-lying spectra of medium and heavy even-even nuclei and a tripartite classification in terms of a seniority, vibrator, and rotor regime [8,9]. The conventional wisdom in nuclear structure physics is that the observed properties of nuclei can be explained by specific features of the SM (or IBM) Hamiltonian. The studies with random interactions, however, seem to imply that some of the generic characteristics of these systems may already be encoded in the corresponding shell model (or sd boson model) space. This is particularly striking in the case of the IBM, for which the model space corresponds to a drastic truncation of the original (shell model) Hilbert space to that composed of like-nucleon pairs with angular momentum $L=0$ and $L=2$ [10,11]. The selection of such a restricted subspace seems to impose strong constraints on the possible spectral properties.

These considerations lead naturally to the question of what are the specific causes of this behavior, given that the

ingredients of the calculations are the preservation of fundamental symmetries of the Hamiltonian (hermiticity, rotational invariance, time-reversal invariance), the one- and two-body nature of the Hamiltonian, a given number of active particles, and the structure of the model space. In Ref. [5] it was shown that the preponderance of $L^P=0^+$ ground states in the nuclear shell model is not due to the time-reversal symmetry of the interactions. The purpose of this paper is to address explicitly the role of the particle number and the rank of the random many-body interactions on the systematics of collective states in nuclei.

II. RANDOM INTERACTIONS IN THE IBM

To study the global features of low-lying collective states in nuclei we carry out an analysis of the IBM with random interactions. In the IBM, collective nuclei are described as a system of N interacting monopole and quadrupole bosons [10]. We consider all possible one-, two-, and three-body interactions. The one-body Hamiltonian contains the boson energies

$$H_1 = \epsilon_0 s^\dagger s + \epsilon_2 \sum_m d_m^\dagger d_m. \quad (1)$$

The two-body interactions can be expressed as

$$H_2 = \sum_{L=0,2,4} \sum_{i \leq j} \zeta_{Lij} \frac{P_{L_i}^\dagger \cdot \tilde{P}_{L_j} + P_{L_j}^\dagger \cdot \tilde{P}_{L_i}}{1 + \delta_{ij}}, \quad (2)$$

with $\tilde{P}_{LM} = (-1)^{L-M} P_{L,-M}$. Here P_L^\dagger denotes the creation operator of a pair of bosons coupled to angular momentum L

$$P_{0_1}^\dagger = \frac{1}{\sqrt{2}} (s^\dagger \times s^\dagger)^{(0)},$$

$$P_{0_2}^\dagger = \frac{1}{\sqrt{2}} (d^\dagger \times d^\dagger)^{(0)},$$

$$P_{2_1}^\dagger = (s^\dagger \times d^\dagger)^{(2)},$$

$$\begin{aligned}
P_{2_2}^\dagger &= \frac{1}{\sqrt{2}}(d^\dagger \times d^\dagger)^{(2)}, \\
P_{4_1}^\dagger &= \frac{1}{\sqrt{2}}(d^\dagger \times d^\dagger)^{(4)}.
\end{aligned} \tag{3}$$

Similarly, the three-body interactions are given by

$$H_3 = \sum_{L=0,2,3,4,6} \sum_{i \leq j} \xi_{Lij} \frac{P_{L_i}^\dagger \cdot \tilde{P}_{L_j} + P_{L_j}^\dagger \cdot \tilde{P}_{L_i}}{1 + \delta_{ij}}, \tag{4}$$

with

$$\begin{aligned}
P_{0_1}^\dagger &= \frac{1}{\sqrt{6}}(s^\dagger \times s^\dagger \times s^\dagger)^{(0)}, \\
P_{0_2}^\dagger &= \frac{1}{\sqrt{2}}(s^\dagger \times d^\dagger \times d^\dagger)^{(0)}, \\
P_{0_3}^\dagger &= \frac{1}{\sqrt{6}}(d^\dagger \times d^\dagger \times d^\dagger)^{(0)}, \\
P_{2_1}^\dagger &= \frac{1}{\sqrt{2}}(s^\dagger \times s^\dagger \times d^\dagger)^{(2)}, \\
P_{2_2}^\dagger &= \frac{1}{\sqrt{2}}(s^\dagger \times d^\dagger \times d^\dagger)^{(2)}, \\
P_{2_3}^\dagger &= \frac{1}{\sqrt{6}}(d^\dagger \times d^\dagger \times d^\dagger)^{(2)}, \\
P_{3_1}^\dagger &= \frac{1}{\sqrt{6}}(d^\dagger \times d^\dagger \times d^\dagger)^{(3)}, \\
P_{4_1}^\dagger &= \frac{1}{\sqrt{2}}(s^\dagger \times d^\dagger \times d^\dagger)^{(4)}, \\
P_{4_2}^\dagger &= \frac{1}{\sqrt{6}}(d^\dagger \times d^\dagger \times d^\dagger)^{(4)}, \\
P_{6_1}^\dagger &= \frac{1}{\sqrt{6}}(d^\dagger \times d^\dagger \times d^\dagger)^{(6)}.
\end{aligned} \tag{5}$$

The coefficients ϵ_L , ζ_{Lij} , and ξ_{Lij} correspond to the two one-body, seven two-body, and 17 three-body matrix elements, respectively. They are chosen independently from a Gaussian distribution of random numbers with zero mean and variance v^2 as

$$\begin{aligned}
\langle \epsilon_L \epsilon_{L'} \rangle &= \delta_{LL'} 2 v^2, \\
\langle \zeta_{Lij} \zeta_{L'i'j'} \rangle &= \delta_{LL'} (1 + \delta_{ij,i'j'}) v^2,
\end{aligned}$$

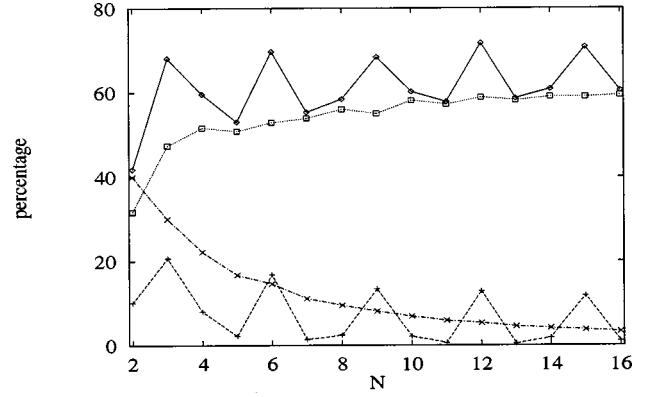


FIG. 1. Percentage of $L^P=0^+$ ground states as a function of the boson number N for $H=H_2$ for which the energy ratio of Eq. (7) is given by $0 < R < 1$ (dashed line), $R \geq 1$ (dotted line), and $R > 0$ (solid line). The dashed-dotted line shows the percentage of $L^P=0^+$ states in the model space.

$$\begin{aligned}
\langle \xi_{Lij} \xi_{L'i'j'} \rangle &= \delta_{LL'} (1 + \delta_{ij,i'j'}) v^2, \\
\langle \epsilon_L \zeta_{Lij} \rangle &= \langle \epsilon_{L'} \xi_{L'ij'} \rangle = \langle \zeta_{Lij} \xi_{L'i'j'} \rangle = 0.
\end{aligned} \tag{6}$$

The choice of the ensembles is such that they are invariant under orthogonal basis transformations. The variance of the Gaussian distribution v^2 sets the overall energy scale. The ensemble defined by Eq. (6) for H_k is called the k -body random ensemble (k -BRE) [12]. For two-body interactions $H=H_2$ it reduces to the T(wo)BRE [12,13]. When the number of bosons is equal to the rank of the interactions $N=k$, the Hamiltonian matrix is entirely random and the ensemble coincides with the Gaussian orthogonal ensemble (GOE). For $N > k$ the many-body matrix elements of H_k are correlated via the appropriate reduction formulas and depend, in principle, on all random k -body matrix elements.

III. RESULTS

In Ref. [6] we used random one- and two-body interactions with $N=16$ to study the systematics of low-lying collective states in the IBM. Here we wish to study how these results depend on the boson number and the rank of the random interactions.

We first analyze the dependence on the total number of bosons. Hereto we take the Hamiltonian H_2 of Eq. (2) with random two-body matrix elements. In all calculations we make 1000 runs. For each set of randomly generated two-body matrix elements we calculate the entire energy spectrum and the $B(E2)$ values between the yrast states. In Fig. 1 we show the percentage of $L^P=0^+$ ground states as a function of N (solid line). For $N=2$ the Hamiltonian matrix is a real-symmetric random matrix. For each value of the angular momentum L the ensemble corresponds to GOE, whose level distribution is a semicircle with radius $\sqrt{4dv^2}$ and width $\sqrt{(d+1)v^2}$ [1]. In this case, the percentage of ground states for a given value of L is determined by the dimension d of the Hamiltonian matrix: $d=2$ for $L=0,2$ and $d=1$ for $L=4$. For $3 \leq N \leq 16$ the situation is completely

TABLE I. Percentage of ground states with angular momentum L and corresponding widths of TBRE level distributions. The results are obtained for 1000 runs and $N=16$ bosons. The widths are divided by $N(N-1)$.

L	Dim	TBRE	Width	L	Dim	TBRE	Width
0	30	60.5%	0.45	17	23	0.0%	0.45
2	51	12.9%	0.44	18	31	0.7%	0.46
3	21	0.0%	0.43	19	16	0.0%	0.46
4	64	0.0%	0.43	20	23	1.3%	0.47
5	35	0.0%	0.42	21	11	0.0%	0.48
6	70	0.1%	0.43	22	16	0.8%	0.49
7	42	0.0%	0.42	23	7	0.0%	0.51
8	71	0.4%	0.43	24	11	0.9%	0.52
9	44	0.0%	0.42	25	4	0.0%	0.54
10	67	0.1%	0.43	26	7	1.1%	0.56
11	42	0.0%	0.42	27	2	0.0%	0.58
12	60	0.2%	0.43	28	4	0.7%	0.60
13	37	0.0%	0.43	29	1	0.0%	0.63
14	51	0.3%	0.44	30	2	0.6%	0.64
15	30	0.0%	0.44	32	1	18.6%	0.71
16	41	0.8%	0.45				

different. The ensemble is now TBRE. The dominant angular momentum of the ground state is determined by the shapes of the level distributions as a function of the angular momentum, in particular by the tails, i.e., the higher moments, of the distributions (all have the same centroid). The distribution whose tail extends furthest is the most likely to provide the ground state. For a semicircular (GOE) or a Gaussian distribution (TBRE in the nuclear shell model [12–14]) the shape is completely determined by the width. In these two cases, the dominance of $L^P=0^+$ ground states can be correlated to the widths of the distributions [5,7]. However, for a system of interacting bosons the TBRE distribution of eigenvalues is neither semicircular (except for $N=k$) nor Gaussian [15]. There is no relation between the width (second moment) and higher moments of the distribution, which determine the dominant angular momentum of the ground state. Table I shows that the width increases with angular momentum, whereas the most likely value of the ground state angular momentum is $L^P=0^+$. In fact, the probability that the ground state has a certain value of the angular momentum is not really fixed by the full distribution of eigenvalues, but rather by that of the lowest one. Work is in progress to elucidate the form of these distributions in a schematic exactly solvable model [16].

Despite the different shapes of the TBRE level distributions for fermions and bosons we find, just as in the fermion case [2–5], a dominance ($\sim 60\%$) of $L^P=0^+$ ground states in the IBM with $3 \leq N \leq 16$. This fraction is large compared to the percentage of $L^P=0^+$ states in the model space (solid and dashed-dotted lines in Fig. 1). The oscillations with maxima at $N=3n$ (multiple of 3) are due to the “unphysical” region of parameter space for which the energy ratio,

$$R = \frac{E(4_1^+) - E(0_1^+)}{E(2_1^+) - E(0_1^+)}, \quad (7)$$

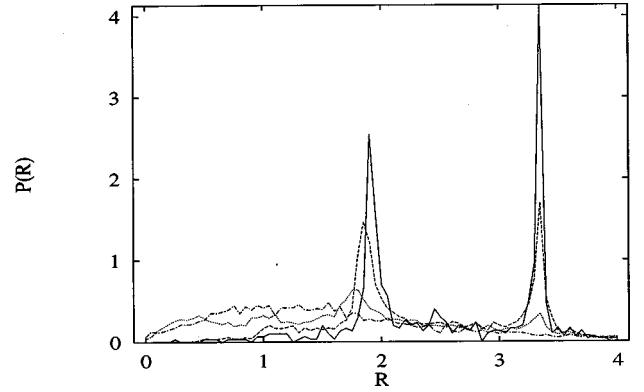


FIG. 2. Probability distributions $P(R)$ of the energy ratio R of Eq. (7) with $\int P(R)dR=1$ in the IBM with random two-body interactions for $N=3$ (dash-dotted), 6 (dotted), 10 (dashed), and 16 (solid).

is less than 1 (dashed line) corresponding to a level sequence $0_1^+, 4_1^+, 2_1^+$, rather than $0_1^+, 2_1^+, 4_1^+$ for $R>1$ (dotted line). The enhancement for $N=3n$ can be attributed to the existence of a 0^+ state in which all d bosons are organized into $n_\Delta=N/3$ triplets. This state has the $U(5)$ quantum numbers $|N, n_d, v, n_\Delta, L\rangle = |N, N, 0, N/3, 0\rangle$ and can become the ground state if the vibrational spectrum is turned “upside down.”

For the cases with a $L^P=0^+$ ground state we present in Fig. 2 the probability distribution $P(R)$ of the energy ratio R of Eq. (7). This energy ratio has very characteristic values for the harmonic vibrator and the rotor, $R=2$ and $R=10/3$, respectively. The Hamiltonian matrix of H_2 depends on seven independent random two-body matrix elements. For small values of N there is little correlation among the matrix elements of H , and as a consequence the probability distribution $P(R)$ shows little structure for $N=3$ (dash-dotted curve). For increasing values of N there is a correspondingly higher correlation between the different matrix elements of H , which results in the development of two peaks in $P(R)$. We first see the development of a maximum at $R \sim 1.9$ for $N=6$ (dotted curve), followed by another one at $R \sim 3.3$ for $N=10$ (dashed curve). For $N=16$ the probability distribution $P(R)$ has two very pronounced peaks, one at $R \sim 1.95$ and a narrower one at $R \sim 3.35$ (solid curve). These values correspond almost exactly to those for the harmonic vibrator and the rotor. The two maxima correspond to the two basic phases that characterize the collective region: a spherical one with $R \sim 2.0$ and an axially deformed one with $R \sim 3.3$. There is no peak for γ -unstable nuclei [$SO(6)$ limit], since this requires that the matrix element of $\zeta_{212}[(s^\dagger \times d^\dagger)^{(2)} \cdot (\tilde{d} \times \tilde{d})^{(2)} + \text{H.c.}]$ vanishes identically, effectively corresponding to a zero-measure case for the random sample. Any other value of $\zeta_{212} (\neq 0)$ gives rise to an axially symmetric rotor [17,18].

In a second calculation we take the Hamiltonian H_3 of Eq. (4) with random three-body interactions. Figure 3 shows the same qualitative behavior as Fig. 2 although the peak structure is far less pronounced. For $N=16$ we see again two maxima at the vibrator and rotor values of the energy ratio R . The case of three-body interactions in the IBM is of special

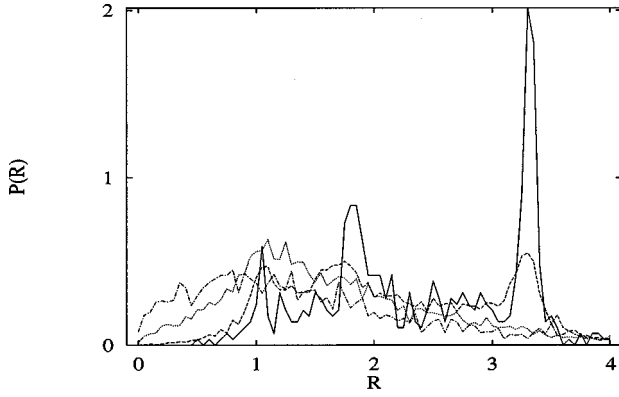


FIG. 3. As in Fig. 2, but for random three-body interactions.

interest, since it can give rise to stable triaxial deformations [19], which are absent in the case of Hamiltonians including one- and two-body interactions only. We note, however, that in the neutron-proton version of the IBM, triaxial deformation can be obtained from Hamiltonians with one- and two-body interactions only [20]. Figure 3 shows no clear sign of a ‘‘triaxial’’ peak (e.g., a triaxially deformed rotor with $\gamma = 30^\circ$ has $R = 8/3$), nor of a ‘‘ γ unstable’’ one with $R = 5/2$.

After these two model studies, we now turn to a more realistic case. It has been shown [10] that the phenomenology of low-lying collective states in nuclei is well described by an IBM Hamiltonian consisting of both one- and two-body interactions

$$H_{12} = \frac{1}{N} \left[H_1 + \frac{1}{N-1} H_2 \right]. \quad (8)$$

In order to remove the N dependence of the matrix elements of k -body interactions, we have scaled H_k by $\prod_{i=1}^k (N+1-i)$. In Fig. 4 we show the corresponding probability distribution $P(R)$ of the energy ratio R of Eq. (7) for different values of the number of bosons. The results are very similar to those of Fig. 2 which were obtained with pure two-body interactions. With increasing values of N the many-body matrix elements of H_{12} become increasingly correlated, which

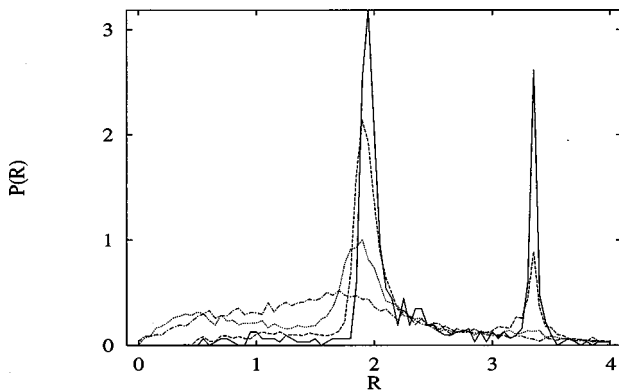
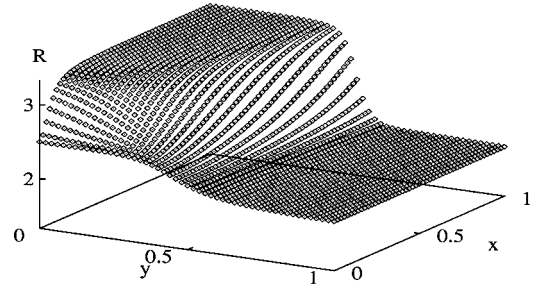


FIG. 4. As in Fig. 2, but for random one- and two-body interactions.

FIG. 5. The energy ratio R of Eq. (7) as a function of x and y in the consistent Q formulation of the IBM.

results in the development of two maxima in $P(R)$. The curve for $N = 16$ is identical to the calculation discussed in Ref. [6]. The occurrence of two basic phases for the collective region is further exemplified in Fig. 5 in which we plot the energy ratio R for the consistent- Q formulation [21] of the IBM

$$H = \epsilon \hat{n}_d - \kappa \hat{Q}(\chi) \cdot \hat{Q}(\chi),$$

$$\hat{Q}_\mu(\chi) = (s^\dagger \tilde{d} + d^\dagger s)_\mu^{(2)} + \chi (d^\dagger \tilde{d})_\mu^{(2)}, \quad (9)$$

with realistic values of the interactions, i.e., a positive d boson energy ($\epsilon > 0$) and an attractive quadrupole-quadrupole interaction ($\kappa > 0$). The results in Fig. 5 are plotted as a function of the scaled parameters $x = -2\chi/\sqrt{7}$ and $y = \epsilon/[\epsilon + 4\kappa(N-1)]$, which have been used as control parameters in a study of phase transitions in the IBM [18,22,23]. For $y = 1$ we recover the vibrational or $SU(5)$ limit of the IBM, whereas for $y = 0$ and $x = 1$ one finds the rotational or $SU(3)$ limit, and for $y = 0$ and $x = 0$ the γ unstable or $SO(6)$ limit. We clearly see two planes corresponding to $R \sim 2.0$ and $R \sim 3.3$, respectively, which are separated by a sharp transitional region, in agreement with the observation in Refs. [22,23] that the collective region is characterized by two phases (spherical and deformed) connected by a sharp phase transition.

In order to investigate the effect of higher order interactions we now add three-body interactions to the Hamiltonian

$$H_{123} = \frac{1}{N} \left[H_1 + \frac{1}{N-1} \left[H_2 + \frac{1}{N-2} H_3 \right] \right]. \quad (10)$$

In this case the Hamiltonian matrix depends on 26 independent random matrix elements (two one-body, seven two-body, and 17 three-body). Therefore, for a fixed value of N there is less correlation between the N -body matrix elements of H_{123} than for H_{12} , which results in broader peaks in the probability distribution $P(R)$. A comparison of Figs. 4 and 6 shows that the probability distribution $P(R)$ behaves in a very similar way, and that the addition of three-body interactions does not change the results in a significant way. When N is sufficiently large compared to the maximum rank k of the interactions (2 and 3, respectively) the results become independent of k .

This result is qualitatively very similar to that of Ref. [12], in which the transition from a Gaussian to a semicircular

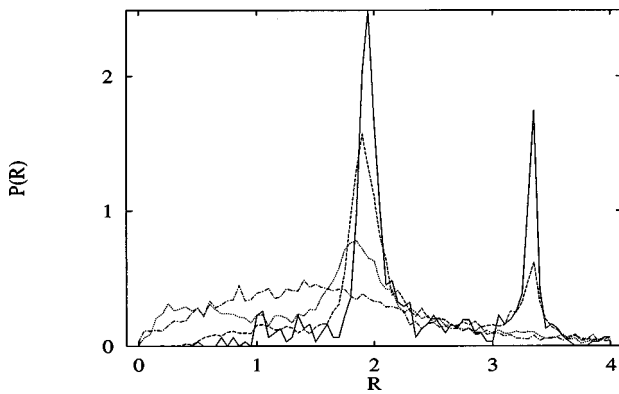


FIG. 6. As in Fig. 2, but for random one-, two-, and three-body interactions.

lar level distribution was studied in the nuclear shell model for a fixed particle number $N=7$ with increasing values of the rank $2 \leq k \leq 7$. The characteristic features of the ensemble depend on the ratio of the number of particles and the rank of the interactions. For N sufficiently large compared to k there is a saturation, and the properties of the ensemble no longer depend on k .

IV. SUMMARY AND CONCLUSIONS

In summary, we have studied global properties of low-lying collective levels using the interacting boson model with random interactions. In particular, we addressed the dependence of the dominance of $L^P=0^+$ ground states and the occurrence of vibrational and rotational band structures on the boson number N and the rank k of the interactions.

Just as for the nuclear shell model it was found that despite the randomness of the interactions (both in size and sign) the ground state has $L^P=0^+$ in approximately 60% of the cases. The oscillation in the percentage of $L^P=0^+$ ground states with N was shown to be entirely due to cases in which the level sequence is given by $0^+, 4^+, 2^+$ ($R < 1$). For the cases with $R > 1$ there is a very smooth dependence on N .

The vibrational and rotational band structures appear gradually as N/k increases. For $N \sim k$ there is little or no evidence for such bands. As N grows we first see evidence for the development of vibrational structure, followed later by the appearance of rotational bands. If N increases further these band structures become more and more pronounced.

Essentially the same behavior is found for random two- and three-body interactions. In realistic applications to collective nuclei the IBM Hamiltonian consists of a combination of one- and two-body interactions. A study with random ensembles of one- and two-body interactions shows similar results to the case of pure two-body terms. The inclusion of random three-body interactions does not significantly change the basic features.

In conclusion, we find that the dominance of $L^P=0^+$ ground states and the occurrence of vibrational and rotational features are independent of the boson number, as long as N is sufficiently large compared with the maximum rank of the interactions. We can conclude that these features represent general and robust properties of the interacting boson model space, and are a consequence of the many-body dynamics, which enters via the reduction formulas for the N -body matrix elements of k -body interactions (angular momentum coupling, coefficients of fractional parentage, etc.). Since the structure of the model space is completely determined by the corresponding degrees of freedom, these results emphasize the importance of the selection of the relevant degrees of freedom. In this context, a relevant question is whether vibrational and rotational collective behavior can be directly observed in the shell model with random interactions if an appropriate truncation of the (shell model) Hilbert space is carried out.

It is important to stress that these properties do not arise as an artifact of a particular model of nuclear structure. In empirical studies of the low-lying collective states of medium and heavy even-even nuclei very regular and robust features have been observed, such as the tripartite classification into seniority, anharmonic vibrator, and rotor regimes [8,9], and the systematics of excitation energy and $M1$ strength of the scissors mode [24]. Finally, we remark that the use of random interactions to study the generic behavior of low-lying states has also found useful applications in many-body quantum systems of a different nature, such as quantum dots or small metallic particles [25].

ACKNOWLEDGMENTS

It is a pleasure to thank Rick Casten, Jorge Flores, Franco Iachello, Stuart Pittel, and Thomas Seligman for illuminating discussions. This work was supported in part by DGAPA-UNAM under Project No. IN101997 and by CONACyT under Projects No. 32416-E and 32397-E.

- [1] See, e.g., the reviews T. A. Brody, J. Flores, J. B. French, P. A. Mello, A. Pandey, and S. S. M. Wong, *Rev. Mod. Phys.* **53**, 385 (1981); T. Guhr, A. Müller-Groeling, and H. A. Weidenmüller, *Phys. Rep.* **299**, 189 (1998).
 [2] C. W. Johnson, G. F. Bertsch, and D. J. Dean, *Phys. Rev. Lett.* **80**, 2749 (1998).
 [3] C. W. Johnson, *Rev. Mex. Fis.* **45 S2**, 25 (1999).
 [4] C. W. Johnson, G. F. Bertsch, D. J. Dean, and I. Talmi, *Phys. Rev. C* **61**, 014311 (2000).

- [5] R. Bijker, A. Frank, and S. Pittel, *Phys. Rev. C* **60**, 021302 (1999).
 [6] R. Bijker and A. Frank, *Phys. Rev. Lett.* **84**, 420 (2000).
 [7] R. Bijker, A. Frank, and S. Pittel, *Proceedings of XXIII Symposium on Nuclear Physics, Oaxtepec, Morelos, México, 2000*, *Rev. Mex. Fis.* (to be published).
 [8] R. F. Casten, N. V. Zamfir, and D. S. Brenner, *Phys. Rev. Lett.* **71**, 227 (1993).
 [9] N. V. Zamfir, R. F. Casten, and D. S. Brenner, *Phys. Rev. Lett.*

- 72**, 3480 (1994).
- [10] F. Iachello and A. Arima, *The Interacting Boson Model* (Cambridge University, Cambridge, 1987).
- [11] F. Iachello and I. Talmi, *Rev. Mod. Phys.* **59**, 339 (1987).
- [12] J. B. French and S. S. M. Wong, *Phys. Lett.* **33B**, 449 (1970); **35B**, 5 (1971).
- [13] O. Bohigas and J. Flores, *Phys. Lett.* **34B**, 261 (1971).
- [14] A. Gervois, *Nucl. Phys.* **A184**, 507 (1972).
- [15] V. K. B. Kota and V. Potbhare, *Phys. Rev. C* **21**, 2637 (1980).
- [16] D. Kusnezov, R. Bijker, and A. Frank (work in progress).
- [17] J. N. Ginocchio and M. Kirson, *Phys. Rev. Lett.* **44**, 1744 (1980).
- [18] A. E. L. Dieperink, O. Scholten, and F. Iachello, *Phys. Rev. Lett.* **44**, 1747 (1980).
- [19] P. van Isacker and J. Q. Chen, *Phys. Rev. C* **24**, 684 (1981).
- [20] A. E. L. Dieperink and R. Bijker, *Phys. Lett.* **116B**, 77 (1982).
- [21] D. D. Warner and R. F. Casten, *Phys. Rev. Lett.* **48**, 1385 (1982).
- [22] F. Iachello, N. V. Zamfir, and R. F. Casten, *Phys. Rev. Lett.* **81**, 1191 (1998).
- [23] R. F. Casten, D. Kusnezov, and N. V. Zamfir, *Phys. Rev. Lett.* **82**, 5000 (1999).
- [24] N. Pietralla, P. von Brentano, R.-D. Herzberg, U. Kneissl, N. Lo Iudice, H. Maser, H. H. Pitz, and A. Zilges, *Phys. Rev. C* **58**, 184 (1998); J. Enders, H. Kaiser, P. von Neumann-Cosel, C. Rangacharyulu, and A. Richter, *ibid.* **59**, R1851 (1999).
- [25] C. Mejía-Monasterio, J. Richert, T. Rupp, and H. A. Weidenmüller, *Phys. Rev. Lett.* **81**, 5189 (1998).