Cluster states in nuclei as representations of a U($\nu+1$) group

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We propose a description of cluster states in nuclei in terms of representations of unitary algebras U($\nu+1$), where ν is the number of space degrees of freedom. Within this framework, a variety of situations including both vibrational and rotational spectra, soft and rigid configurations, identical and nonidentical constituents can be described. As an example, we show how the method can be used to study α clustering configurations in ^{12}C with point group symmetry $\mathcal{D}_{3h}$.

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The purpose of this Brief Report is to point out that the algebra U($\nu+1$), which has been suggested to be the spectrum generating algebra for a quantum mechanical problem with ν space degrees of freedom [1], might provide a framework for a unified description of cluster states in nuclei. We note that the main properties of clustering in nuclei are (i) the softness of the cluster configuration which makes nuclei appear more like liquid structures rather than rigid molecular structures in which the constituents sit at some definite location in space, (ii) the near equality of vibrational and rotational energies which does not allow a clearcut distinction between these two types of motion, (iii) the fact that the constituents are not pointlike objects but particles with a spatial extent comparable to that of the overall structure, and (iv) the fact that the constituents are often identical which implies that permutation symmetry must be imposed. A unified description of clustering in nuclei should be able to accomodate all these properties.

To illustrate the uselfuness of the algebra U($\nu+1$) in describing the variety of observed situations, we consider the specific case of a cluster composed of three particles [a description of two-body cluster configurations in nuclei in terms of U(4) was suggested long ago [2] and has been used to describe resonances in heavy ion scattering [3]]. For a three-body problem, the number of space degrees of freedom (after removal of the center of mass) is $\nu=3n-3=6$. [We do not consider in this article constituents with an internal structure. For such situations the algebraic structure must be enlarged to U($\nu+1$)$\otimes$U(Ω) where Ω is the number of internal degrees of freedom.] The space degrees of freedom can be taken as the Jacobi coordinates $\vec{\rho}=(\vec{r}_1-\vec{r}_2)/\sqrt{2}$ and $\vec{\lambda}=(\vec{r}_1+\vec{r}_2-2\vec{r}_3)/\sqrt{6}$, where $\vec{r}_i$ $(i=1,2,3)$ are the coordinates of the three particles. The corresponding algebra is U(7). The algebra of U(7) is constructed by introducing two vector bosons b_ρ, b_λ together with an auxiliary scalar boson s. It was introduced in [4] where it was used to describe three-quark configurations in baryons. The 49 bilinear products of creation and annihilation operators generate the Lie algebra U(7),

$$b^\dagger_{\rho,m}, b^\dagger_{\lambda,m}, s^\dagger \equiv c^\dagger_\alpha \quad (m=0,\pm 1) \quad (\alpha=1,\ldots,7),$$

$$\mathcal{G}: G_{\alpha\beta}=c^\dagger_\alpha c_\beta \quad (\alpha,\beta=1,\ldots,7). \tag{1}$$

The creation and annihilation operators for vector bosons ($b^\dagger_{\rho,m}$, $b^\dagger_{\lambda,m}$ and $b_{\rho,m}$, $b_{\lambda,m}$) represent the second quantized form of the Jacobi coordinates and their canonically conjugate momenta, while the auxiliary scalar boson is introduced in order to construct the spectrum generating algebra. (The method of embedding the problem in a larger dimensional space [1] is similar to that used in Kaluza-Klein theories of particle physics.) The energy levels can be obtained by diagonalizing the Hamiltonian

$$\begin{aligned} H=&H_0+\epsilon_s s^\dagger\tilde{s}-\epsilon_p(b^\dagger_\rho\cdot\tilde{b}_\rho+b^\dagger_\lambda\cdot\tilde{b}_\lambda)+u_0(s^\dagger s^\dagger\tilde{s}\tilde{s})-u_1 s^\dagger(b^\dagger_\rho\cdot\tilde{b}_\rho+b^\dagger_\lambda\cdot\tilde{b}_\lambda)\tilde{s}\\ &+v_0[(b^\dagger_\rho\cdot b^\dagger_\rho+b^\dagger_\lambda\cdot b^\dagger_\lambda)\tilde{s}\tilde{s}+s^\dagger s^\dagger(\tilde{b}_\rho\cdot\tilde{b}_\rho+\tilde{b}_\lambda\cdot\tilde{b}_\lambda)]+\sum_{l=0,2}c_l[(b^\dagger_\rho\times b^\dagger_\rho-b^\dagger_\lambda\times b^\dagger_\lambda)^{(l)}\cdot(\tilde{b}_\rho\times\tilde{b}_\rho-\tilde{b}_\lambda\times\tilde{b}_\lambda)^{(l)}\\ &+4(b^\dagger_\rho\times b^\dagger_\lambda)^{(l)}\cdot(\tilde{b}_\lambda\times\tilde{b}_\rho)^{(l)}]+c_1(b^\dagger_\rho\times b^\dagger_\lambda)^{(1)}\cdot(\tilde{b}_\lambda\times\tilde{b}_\rho)^{(1)}+\sum_{l=0,2}w_l(b^\dagger_\rho\times b^\dagger_\rho+b^\dagger_\lambda\times b^\dagger_\lambda)^{(l)}\cdot(\tilde{b}_\rho\times\tilde{b}_\rho+\tilde{b}_\lambda\times\tilde{b}_\lambda)^{(l)}, \end{aligned} \tag{2}$$

within the space of the totally symmetric representations [N] of U(7). The coefficients ϵ_s, ϵ_p, u_0, u_1, v_0, c_0, c_1, c_2, w_0 and w_2 parametrize the interactions. The Hamiltonian H is the most general Hamitonian that preserves angular momentum and parity, transforms as a scalar under permutations (we consider here the case of three identical particles) and is

at most quadratic (two-body interactions). Associated with the Hamiltonian H, there are transition operators, T. Electromagnetic transition rates and form factors can all be calculated by considering the matrix elements of the operator

$$T=\mathrm{e}^{-iq\beta D_{\lambda,z}/X_D},$$

$$D_{\lambda,z}=(b_\lambda^\dagger\times\tilde{s}-s^\dagger\times\tilde{b}_\lambda)_z^{(1)}, \tag{3}$$

which is the algebraic image of the operator $\exp(iqr_{3z})$ obtained from the full operator $\Sigma_{i=1}^{3}e^{i\vec{q}\cdot\vec{r}_i}$ by choosing the momentum transfer $\vec{q}$ in the z direction and considering identical particles (the coefficient X_D is a normalization factor).

The Hamiltonian of Eq. (2) has two dynamic symmetries corresponding to the breakings of U(7) onto U(6) and SO(7)

$$\mathrm{U}(7)\supset\begin{cases}\mathrm{U}(6),\\ \mathrm{SO}(7).\end{cases} \tag{4}$$

When the Hamiltonian contains only Casimir operators of these chains, the eigenvalue problem can be solved in closed analytic form. The corresponding solutions describe two situations sometimes encountered in the three body problem: (i) six-dimensional vibrational spectra U(6), and (ii) an unusual situation which we call ω-unstable or SO(7) limit. Both situations will be described in a longer publication. Here instead, as an example of application of the algebraic method, we discuss another situation that is appropriate to three particles at the vertices of an equilateral triangle. The spectrum of an equilateral triangle configuration can be obtained from the Hamiltonian of Eq. (2) by setting some coefficients equal to zero and taking specific combinations of others [4]

$$\begin{aligned}H=&H_0+\xi_1(s^\dagger s^\dagger-b_\rho^\dagger\cdot b_\rho^\dagger-b_\lambda^\dagger\cdot b_\lambda^\dagger)(\tilde{s}\tilde{s}-\tilde{b}_\rho\cdot\tilde{b}_\rho-\tilde{b}_\lambda\cdot\tilde{b}_\lambda)\\&+\xi_2[(b_\rho^\dagger\cdot b_\rho^\dagger-b_\lambda^\dagger\cdot b_\lambda^\dagger)(\tilde{b}_\rho\cdot\tilde{b}_\rho-\tilde{b}_\lambda\cdot\tilde{b}_\lambda)+4(b_\rho^\dagger\cdot b_\lambda^\dagger)\\&\times(\tilde{b}_\lambda\cdot\tilde{b}_\rho)]+\xi_3(b_\rho^\dagger\tilde{b}_\rho+b_\lambda^\dagger\tilde{b}_\lambda)^{(1)}\cdot(b_\rho^\dagger\tilde{b}_\rho+b_\lambda^\dagger\tilde{b}_\lambda)^{(1)}\\&+\xi_4(b_\rho^\dagger\tilde{b}_\lambda-b_\lambda^\dagger\tilde{b}_\rho)^{(0)}\cdot(b_\lambda^\dagger\tilde{b}_\rho-b_\rho^\dagger\tilde{b}_\lambda)^{(0)}.\end{aligned} \tag{5}$$

This spectrum does not correspond to a dynamic symmetry, since it cannot be written in terms of invariants of a chain of algebras originating from U(7). However, an approximate expression for the energy levels can be obtained by making use of the method of intrinsic or coherent states (valid in the limit of large N). The energy eigenvalues are then given by [4,5]

$$\begin{aligned}E(v_1,v_2^l,L,K,M)=&E_0+Av_1+Bv_2+CL(L+1)\\&+D(K\pm 2l)^2,\end{aligned} \tag{6}$$

where $A\approx 4N\xi_1$, $B\approx 2N\xi_2$, $C=\xi_3/2$ and $D=\xi_4/3$. The quantum numbers have the following meaning: v_1, v_2 are vibrational quantum numbers; for three identical particles one of the vibration (v_1) is singly degenerate, while the other (v_2) is doubly degenerate; $l=v_2,v_2-2,\ldots,1$ or 0 is the vibrational angular momentum of the doubly degenerate vibration; L is the angular momentum, M its projection on a laboratory fixed axis and K its projection on a body fixed axis. We note the particular angular momentum composition of the rotation-vibration bands. The vibrationless ground state band $(v_1,v_2^l)=(0,0^0)$ has $K=3n$ $(n=0,1,2,\ldots)$ with $L=0,2,4,\ldots$ for $K=0$ and $L=K,K+1,K+2,\ldots$ for $K\neq 0$. The parity is given by $P=(-)^K$. The stretching vibration $(1,0^0)$ contains the same angular momenta $L^P=0^+$, 2^+, 3^-, $4^\pm$, ..., as the ground state band, while the bending vibration $(0,1^1)$ has $K=3n+1,3n+2$ $(n=0,1,2,\ldots)$ with $L=K,K+1,K+2,\ldots$. The angular momentum content of the bending vibration is then 1^-, $2^\pm$, $3^\pm$, ... Since we do not consider the excitation of the α particles themselves, the wave functions describing the relative motion have to be symmetric. As a consequence, the relative sign in the last term of Eq. (6) is such that $|K\pm 2l|=3m$, a multiple of 3 [5]. [The energy formula obtained from the Hamiltonian H of Eq. (5) contains a Coriolis term which do not discuss here, since a detailed treatment of this term requires the use of the full Hamiltonian of Eq. (2), rather than the simplified form of Eq. (5)]. In Fig. 1 we show the energy spectrum corresponding to Eq. (6). The importance of this figure is the particular nature of the rotation-vibration spectrum of a triangular configuration with $\mathcal{D}_{3h}$ symmetry. If a physical system is claimed to be composed of three identical structureless particles at the vertices of an equilateral triangle, then its spectrum *must* be as in Fig. 1. The algebraic framework produces this spectrum automatically by an appropriate choice of parameters.

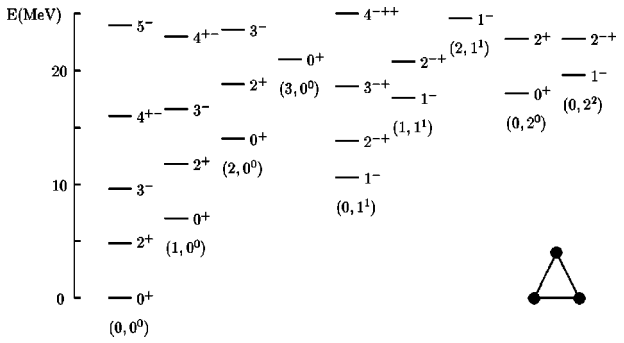

FIG. 1. Spectrum of an equilateral triangle configuration (shown in the inset) calculated using Eq. (6) with $A=7.0$, $B=9.0$, $C=0.8$, and $D=0.0$ MeV (only the levels with $E\leqslant 25$ MeV are shown). The levels are characterized by angular momentum and parity L^P, and the vibrational labels (v_1,v_2^l). Note the doubling and tripling of rotational states. The degeneracies are removed by using a value $D\neq 0$.

Another consequence of using the compact algebra U($\nu+1$) as a spectrum generating algebra is that one can evaluate all observables in exact form. For example, by taking matrix elements of the operator T between the eigenstates of H obtained by matrix diagonalization, one can evaluate form factors. When the Hamiltonian has a dynamic symmetry these can be derived in closed form. Although the Hamiltonian of Eq. (5) does not correspond to a dynamic symmetry, the form factors can still be obtained in explicit form in the limit of large N. For transitions among the lowest states they are given by

$$F(0_1^+ \to 0_1^+; q) = j_0(q\beta),$$

$$F(0_1^+ \to 2_1^+; q) = \frac{1}{2}\sqrt{5} j_2(q\beta),$$

$$F(0_1^+ \to 3_1^-; q) = -i\sqrt{\frac{35}{8}} j_3(q\beta),$$

$$F(0_1^+ \to 4_1^+; q) = \frac{9}{8} j_4(q\beta),$$

$$F(0_1^+ \to 0_2^+; q) = -\chi_1 q\beta j_1(q\beta),$$

$$F(0_1^+ \to 1_1^-; q) = -i\chi_2 \frac{1}{2}\sqrt{3} q\beta j_2(q\beta). \qquad (7)$$

Here q is the momentum transfer and β is the distance of the particles from the center (the first three form factors were already given in [6]). The last two form factors correspond to vibrational excitations. The coefficients χ_1 and χ_2 are proportional to the intrinsic matrix elements for each type of vibration (v_1 and v_2). Electromagnetic transition rates can be calculated from the $B(EL)$ values, which in turn can be obtained from the long wavelength limit of the form factors. In the case in which the constituents of the cluster are extended objects (as in nuclei) the form factors and $B(EL)$ values can be obtained by folding the point-like distribution with the charge distribution (and eventually magnetic moment distribution) of the constituents. In the case of clusters composed of α particles, the folding can be done in a straightforward way, since the charge distribution of the α particle can be taken to a very good approximation as $\exp(-\alpha r^2)$. The form factors for an extended distribution are then obtained from those in Eq. (7) by multiplying by $\exp(-q^2/4\alpha)$. They are a crucial ingredient in understanding whether a cluster configuration is present or not. When N is finite (the situation encountered in nuclei) the energy spectrum and form factors can be evaluated numerically using a computer program written by one of us [7]. In this case, vibrational bands are no longer decoupled, but instead show an appreciable mixing between them and, as a result, the spectrum is considerably distorted from the energy formula of Eq. (6).

The formalism introduced here can be used to study cluster states in ^{12}C. It was suggested long ago [8,9] that ^{12}C in its ground state can be viewed as three α particles at the vertices of an equilateral triangle (point group $\mathcal{D}_{3h}$). The experimental spectrum of ^{12}C is shown in Fig. 2, where it is compared with that given by Eq. (6). One can see that this spectrum is indeed similar (if not identical) to that of a triangular configuration. The crucial point is the sequence of angular momenta in the ground state rotational band: 0^+, 2^+, 3^-, 4^+, ... This sequence is typical of a triangular configuration. A linear configuration would not have negative parity states, while a shell-model configuration would not have the 3^- state as a member of the rotational band but rather as an octupole vibration, i.e., it would not form a rotational sequence with the 0^+, 2^+, 4^+ states. However,

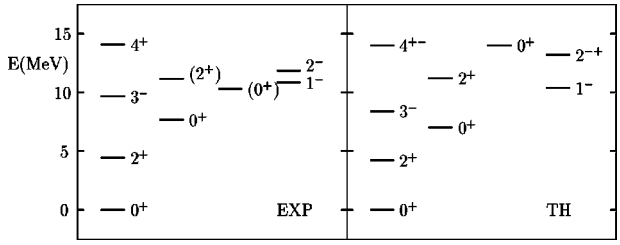

FIG. 2. Comparison between the low-lying experimental spectrum of ^{12}C [12] and that calculated using Eq. (6) with $A=7.0$, $B=9.0$, $C=0.7$, and $D=0.0$ MeV. States with uncertain spin-parity assignment are in parentheses.

the rotational spectrum does not follow precisely what expected from a triangular configuration [oblate top, $D<0$ in Eq. (6)] but it shows rather a spherical or slightly prolate top with $\mathcal{D}_{3h}$ symmetry. The spectrum also shows an excited 0^+ state at 7.65 MeV and an excited 1^- state at 10.84 MeV which could be interpreted as bandheads of the vibrational (stretching and doubly degenerate bending) excitations. Whether or not this is the case or rather those states represent other types of configurations, such as three α particles on a line as suggested by several authors, remains an open question. To settle this question uniquely one would have to identify the rotational sequences built on top of them which have a characteristic pattern for triangular configurations and another pattern for linear configurations. In particular the nature of the 2^+ state at 11.16 MeV and 2^- state at 11.83 MeV, which could form the rotational excitation of the doubly degenerate vibration, should be further investigated (the role played by the 2^+ state in determining the cluster structure of ^{12}C has been emphasized before [10]). We have also calculated form factors and electromagnetic transition rates [11]. All members of the ground rotational band are well described by Eq. (7), as well as the shape of the form factors leading to the 0^+ state at 7.65 MeV and the 1^- state at 10.84 MeV. This analysis will be presented in a forthcoming publication [11]. The result of the simultaneous investigation of spectra, transition rates and form factors done within U(7) is that an α clustering structure (albeit not a rigid one) with $\mathcal{D}_{3h}$ symmetry is a good description of the ground state configuration of ^{12}C. However, in order to make this conclusion stronger, we suggest to readdress the problem of α clustering in ^{12}C by a remeasurement of the properties of the high-lying states by (α,α') and (e,e') inelastic scattering. These experiments were done long ago and can benefit from new and improved techniques. We have predictions for all form factors, transition rates and energies of cluster states in the $\mathcal{D}_{3h}$ configuration. They can be obtained from us upon request.

In conclusion, we have proposed a description of cluster states in nuclei in terms of the group U($\nu+1$) and shown that within this algebraic structure one can describe many situations. In particular, for the three-body problem, one can recover the case of three particles at the vertices of a triangle, a configuration of interest in ^{12}C. We have shown that U(7) contains the main properties of clustering in nuclei: the softness of the cluster configuration, the near equality of vibrational and rotational energies, the spatial extension of the constituents and the permutation symmetry. We can also de-

scribe the situation of three particles on a line (not discussed here) and of vibrational spectra, in other words the method is flexible enough that it can accomodate many situations encountered in nuclei. We have also constructed the algebra appropriate to four-body problems, U(10), where additional geometric arrangements can occur, such as four particles at the vertices of a tetrahedron (point group $\mathcal{T}_d$) and used it to study cluster configurations in ^{16}O. In other words, all cluster structures up to four-body clusters can be studied with the algebraic method. The importance of using U($\nu+1$) for cluster states lies in the possibility of describing the variety of situations encountered in nuclei where clusters are not rigid structures but rather liquidlike structures arising from the nature of the nucleon-nucleon force (spin-isospin) and the shell structure. The unitary algebra U($\nu+1$) can also be of interest in the description of other quantum mechanical systems with nonrigid structure, such as atomic clusters, floppy molecules, and trimers making the method of broad applicability to a large class of problems.

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