Momentum space integral equations for three charged particles: Nondiagonal kernels

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Standard solution methods are known to be applicable to Faddeev-type momentum space integral equations for three-body transition amplitudes, not only for purely short-range interactions but also, after suitable modifications, for potentials possessing Coulomb tails provided the total energy is below the three-body threshold. For energies above that threshold, however, long-range Coulomb forces have been suspected to give rise to such severe singularities in the kernels, even of the modified equations, that their compactness properties are lost. Using the rigorously equivalent formulation in terms of an effective-two-body theory we prove that, for all energies, the nondiagonal kernels occurring in the integral equations which determine the transition amplitudes for all binary collision processes, possess on and off the energy shell only integrable singularities, provided all three particles have charges of the same sign, i.e., all Coulomb interactions are purely repulsive. Hence, after a few iterations these kernels become compact. The case of the diagonal kernels is dealt with in a subsequent paper.

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I. INTRODUCTION

Since pioneering work of Faddeev [1], the three-body quantum scattering theory has become a powerful tool for the investigation of many different processes in various areas of physics. However, one major obstacle which has impeded its wider-spread application, in particular to atomic reactions, has remained, viz., the question of how to incorporate long-range Coulomb forces into the three-body scattering formalism.

From the principle point of view, Dollard's [2,3] time-dependent approach to *N*-particle scattering with Coulomb-like potentials, which applies in particular also to three charged particles, represents a formal solution by providing a mathematically rigorous definition of the relevant MØLLER operators. But progress towards the implementation of this result into a practical approach has been slow [4]. For further reviews of time-dependent approaches see, e.g., [5,6].

Also most stationary approaches, based on integral equations, for taking into account the Coulomb interactions have remained formal so far. For instance, in the approach proposed by Noble [7] (see also Bencze [8]), the three-body integral equations are rewritten in a such a way that all Coulomb potentials are included in what had before been the ''unperturbed'' Green's function. Thereby, the latter is changed into the three-body Coulomb Green's function which then enters the kernels of the new integral equations (as well as those of auxiliary three-body equations for quantities which in the original formulation had been ordinary two-body *T*-operators). That is, all unpleasant features related to the Coulomb interactions are hidden in the unknown three-body Coulomb Green's function. Clearly, the problem of calculating the latter is not any simpler than the initial problem of solving for the full three-body Green's function. Merkuriev's approach [9,10] is based on the same idea, except that there the Coulomb potentials are split by means of suitable cutoff functions into ''inner'' and ''outer'' parts, and only the latter are incorporated into the—formerly free—three-body Green's function. Not surprisingly, the kernels of the Faddeev-type integral equations for the Green's function for the cutoff Coulomb plus short-range potentials have similar compactness properties as those for short-range potentials alone and, thus, can be treated by conventional methods. But for the determination of the auxiliary Green's function containing the ''outer'' Coulomb potential parts, again only formal integral equations have been proposed and shown to possess compact kernels provided that all Coulomb interactions are repulsive [10]. Their explicit solution appears to be very difficult, and in any case has not yet been attempted. (Note, however, that this proposition in Ref. [10] regarding the compactness property of the kernels contradicts a claim made in [9], namely that compactness has been proved for repulsive as well as attractive Coulomb potentials.) For completeness we mention that, as stated in [10], the uniqueness of the solutions of the *differential* Faddeev equations for Coulomb-like potentials has been proved in a special class of functions, again only under the assumption that all three particles have charges of equal sign (repulsive Coulomb potentials). But in this context it should be kept in mind that the boundary condition to be imposed on the solutions of the differential equations used in Ref. [10] was not

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complete. Indeed, the missing part was derived only later by the present authors [11].

An important practical result has been derived by Veselova [12,13]. When considering the Faddeev integral equations with screened Coulomb potentials at energies below the breakup threshold, she succeeded to single out from the kernel that term which in the zero-screening limit yields the so-called two-particle or center-of-mass Coulomb singularity, in such a form that it could be inverted explicitly. In this way, modified three-body integral equations with compact kernels were obtained. But this inversion procedure was only shown to work for energies below the breakup threshold. At energies above that threshold, three-particle singularities have been suspected to appear [12,10] which nobody has succeeded to handle till now.

Because of the aforementioned difficulties to derive proper equations for the kernels of *three-body transition operators* which are valid for all energies and are well suited for practical calculations, it appears more promising to split the problem into several independent parts. An obvious first step consists in developing integral equations for *effective-two-body transition amplitudes* which describe all possible binary processes, i.e., processes in which a projectile impinges on a two-particle bound state leading again to a two-body final state [(in-)elastic and rearrangement collisions, or quite generally so-called $2\rightarrow2$ reactions]. The search for appropriate equations for breakup amplitudes describing $2\rightarrow3$ reactions, or for three-body equations for amplitudes describing $3\rightarrow3$ processes, is deferred to a later stage.

An approach along these lines has been developed in [14,15]. Starting from the Alt-Grassberger-Sandhas (AGS) integral equations for the three-body transition operators [16], they can be reduced exactly by means of the so-called quasiparticle approach to a set of coupled, multichannel, Lippmann-Schwinger-type equations for effective-two-body (i.e., binary) transition amplitudes. By using the screening method, this formulation allowed the isolation, and subsequent extraction, of the leading (in the limit of vanishing screening radius) Coulomb singularity which then could be inverted explicitly. After application of an appropriate renormalization procedure, the various screened binary amplitudes have been shown by Alt and Sandhas [17] to coincide, in the zero-screening limit, with the corresponding amplitudes as resulting from Dollard's time-dependent theory, in particular also for energies above the three-body threshold. In fact, the unique relation between amplitudes as defined in the time-dependent and in the stationary screening and renormalization approach could be established also for the breakup ($2\rightarrow3$) amplitudes, but only for the case of two charged and one neutral particles. Thus, for the latter case, as has been stated in [10], from the mathematical point of view the screening and renormalization approach provides a proof of the compactness of the corresponding (three-body) Faddeev or AGS integral equations, in a special class of functions (this statement represents another contradiction with [9] where it is claimed without proof that even for this special case singularities occur which reflect the noncompactness of the corresponding kernels). We mention that these, and many other aspects of charged-particle scattering are described in detail in a recent pedagogical review [18].

In spite of the success of the screening and renormalization approach, not only as a method for proving the existence of various quantities of interest but also as a practical computational tool, it appears highly desirable to also investigate the effective-two-body AGS equations directly for *unscreened* Coulomb potentials. Quite generally, the question of compactness of the kernels occurring therein depends on the analytical properties of their constituents, which are the so-called ''effective potentials'' and ''effective free propagators.'' The latter are known to have only a pole singularity ''at the on-shell point'' (besides the three-body cut). For the effective potentials, however, no thorough investigation of their singularities has been performed up to now. The aim of this series of papers is to overcome that deficiency. The present paper, in particular, deals with the nondiagonal effective potentials which are the driving terms for all possible rearrangements of the three particles in $2\rightarrow2$ processes. Throughout the investigation it is assumed that all Coulomb potentials are repulsive, i.e., that the charges of all three particles are of the same sign. The new result is that the singularity in the momentum-transfer plane, which is the leading and, therefore, the most dangerous one, is an integrable branch point located off the energy shell. Hence, it can never coincide, for values of the momenta in the integration region, with the pole of the effective free propagator. Consequently, the leading singularities of the nondiagonal kernels are integrable.

A forthcoming paper deals with the singularity structure of the diagonal kernels. There it will be shown that, if the charges of all three particles are of the same sign, nonintegrable singularities appear only on the energy shell, and coincide below the breakup threshold with those considered by Veselova [12]. They can, however, be explicitly singled out and inverted as has been done by Alt and Sandhas [15]. Moreover, the off-the-energy-shell singularities of the diagonal kernels turn out to be integrable. These, together with the present results imply that after a few iterations the (suitably modified) effective-two-body AGS equations become integral equations with compact kernels.

The plan of the paper is as follows. In Sec. II we briefly recapitulate the relevant definitions and equations of the effective-two-body formulation of the three-body scattering theory. For the convenience of those readers who are not interested in mathematical details we summarize in Sec. III the results obtained. Detailed proofs of the assertions are deferred to Sec. IV. There we investigate the leading singularity of the various contributions to the nondiagonal effective potentials. Combining these results with those concerning the singularities of the effective free propagator allows us to find the leading singularities of the (off- and on-shell) nondiagonal kernels. A summary is given in Sec. V. Various auxiliary results are collected in the appendices. In particular, Appendix A provides some of the frequently used transformation formulas of Jacobi variables belonging to different groupings of the particles. In Appendix B we investigate the singularity structure of the residue function of the effective free propagator. An auxiliary theorem describing the singular behavior of the off-shell Coulomb-modified form factor in the on-shell limit is proved in Appendix C. Appendix D contains a brief recapitulation of the asymptotic behavior of the three-charged-particle scattering wave function, in the asymptotic regions needed. Finally, an important integral is evaluated in Appendix E.

We choose units such that $\hbar = c = 1$. Moreover, unit vectors are denoted by a hat, i.e., $\hat{\mathbf{v}} = \mathbf{v}/v$.

II. EFFECTIVE-TWO-BODY ALT-GRASSBERGER-SANDHAS (AGS) EQUATIONS

For the convenience of the reader we briefly recapitulate in this section some basic notions of the effective-two-body formulation of the three-particle theory within the framework of the three-particle AGS integral equations approach [16].

Consider three distinguishable particles with masses m_ν and charges e_ν, $\nu = 1,2,3$. We use the standard notation: on a one-body quantity an index α characterizes the particle α, on a two-body quantity the pair of particles $(\beta\gamma)$, with $\beta,\gamma \neq \alpha$, and finally on a three-body quantity the two-fragment partition $\alpha + (\beta\gamma)$ describing free particles α and $(\beta\gamma)$. Throughout we work in the total center-of-mass system. Jacobi coordinates are introduced as follows: $\mathbf{k}_\alpha(\mathbf{r}_\alpha)$ is the relative momentum (coordinate) between particles β and γ, and $\mu_\alpha = m_\beta m_\gamma/(m_\beta + m_\gamma)$ their reduced mass; $\mathbf{q}_\alpha(\boldsymbol{\rho}_\alpha)$ denotes the relative momentum (coordinate) between particle α and the center of mass of the pair $(\beta\gamma)$, the corresponding reduced mass being defined as $M_\alpha = m_\alpha(m_\beta + m_\gamma)/(m_\alpha + m_\beta + m_\gamma)$.

The Hamiltonian of the three-body system is

$$H = H_0 + V = H_0 + \sum_{\nu=1}^{3} V_\nu, \tag{1}$$

with

$$H_0 = K_\alpha^2/2\mu_\alpha + Q_\alpha^2/2M_\alpha \tag{2}$$

being the free three-body Hamiltonian. $\mathbf{K}_\alpha$ and $\mathbf{Q}_\alpha$ are the momentum operators with eigenvalues $\mathbf{k}_\alpha$ and $\mathbf{q}_\alpha$, respectively. Moreover,

$$V_\alpha = V_\alpha^S + V_\alpha^C \tag{3}$$

is the full interaction between particles β and γ, consisting of a short-range (V_α^S) and a Coulombic part,

$$V_\alpha^C(\mathbf{r}_\alpha) = \frac{e_\beta e_\gamma}{r_\alpha}. \tag{4}$$

In this paper we assume that the charges of all three particles are of equal sign, i.e., all three pairwise Coulomb potentials are repulsive.

As usual, we define the channel interaction for channel α as the sum of the interactions between particle α and each of the particles β and γ,

$$\overline{V}_\alpha = \sum_{\nu=1}^{3} \overline{\delta}_{\nu\alpha} V_\nu = \overline{V}_\alpha^S + \overline{V}_\alpha^C = \sum_{\nu=1}^{3} \overline{\delta}_{\nu\alpha} V_\nu^S + \sum_{\nu=1}^{3} \overline{\delta}_{\nu\alpha} V_\nu^C. \tag{5}$$

Here, $\overline{\delta}_{\beta\alpha} = 1 - \delta_{\beta\alpha}$ is the anti-Kronecker-Copelli symbol. This allows an alternative decomposition of H,

$$H = H_\alpha + \overline{V}_\alpha, \tag{6}$$

where $H_\alpha = H_0 + V_\alpha$ is the channel Hamiltonian.

The transition from the three-body to the effective-two-body theory can be effected, e.g., by splitting each of the subsystem interactions into the sum of a separable part plus a (possibly nonseparable) remainder. In order not to unnecessarily complicate the resulting equations we assume that each pair of particles can support one and only one (S-wave) bound state. Such a restriction is most simply accounted for by choosing the short-range pair potentials as purely separable potentials of rank one:

$$V_\alpha^S = |\chi_\alpha\rangle \Lambda_\alpha \langle \chi_\alpha|, \quad \alpha = 1,2,3. \tag{7}$$

Here, $|\chi_\alpha\rangle$ is the so-called form factor and Λ_α the strength parameter. This provides an obvious decomposition of the full interaction (3) into a separable (V_α^S) and a nonseparable (V_α^C) part. Note that the generalization of the formalism to arbitrary (but sufficiently smooth) short-range interactions does not cause any problem since the latter can always be approximated to arbitrary accuracy by a sum of separable terms. And it is easily seen that inclusion of an arbitrary but finite number of bound states (recall that attractive Coulomb potentials are excluded) will not change our final results.

We introduce some additional notation. Let $z = E + i0$, with E being the total energy of the three-body system. Furthermore, denote by $\overline{\mathbf{q}}_\alpha$ $(\overline{\mathbf{q}}'_\beta)$ the on-shell relative momentum of the two fragments in channel α (β), and, e.g., by $-B_\alpha < 0$ the binding energy of the bound pair $(\beta\gamma)$ (we preclude zero-energy bound states in all subsystems). Then, energy conservation requires

$$E = \overline{q}_\alpha^2/2M_\alpha - B_\alpha = \overline{q}'^2_\beta/2M_\beta - B_\beta. \tag{8}$$

Consider a collision initiating in channel α. If the incident kinetic energy $\overline{q}_\alpha^2/2M_\alpha$ is large enough, then four different transitions $\alpha \to \nu$ are possible: $\nu = \alpha$ corresponds to elastic scattering, $\nu = \beta$ or γ to rearrangement processes, and $\nu = 0$ to the breakup reaction leading to three particles in continuum. One of the most important and useful aspects of the effective-two-body formulation of the three-body theory is that the resulting equations couple only the transition amplitudes for all binary processes. (The breakup amplitudes can be obtained from the two-fragment amplitudes by quadrature, or alternatively from a separate set of integral equations [16].) They have the structure of coupled Lippmann-Schwinger-type equations and are given by [16]

$$\mathcal{T}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)=\mathcal{V}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)+\sum_{\nu=1}^{3}\int\frac{d\mathbf{q}''_\nu}{(2\pi)^3}\mathcal{K}_{\beta\nu}(\mathbf{q}'_\beta,\mathbf{q}''_\nu;z)\times\mathcal{T}_{\nu\alpha}(\mathbf{q}''_\nu,\mathbf{q}_\alpha;z). \tag{9}$$

Here, $\mathcal{T}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$ is the off-shell reaction amplitude corresponding to the transition from channel α to channel β since, *a priori*, $q'_\beta\neq\overline{q}'_\beta$ and $q_\alpha\neq\overline{q}_\alpha$. The kernels are defined as

$$\mathcal{K}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z):=\mathcal{V}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)\mathcal{G}_{0;\alpha}(\mathbf{q}_\alpha;z). \tag{10}$$

The effective potentials can be written in two equivalent ways,

$$\mathcal{V}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z):=\langle\mathbf{q}'_\beta,\chi_\beta|G^C(z)-\delta_{\beta\alpha}G^C_\alpha(z)|\chi_\alpha,\mathbf{q}_\alpha\rangle \tag{11a}$$

$$=\langle\mathbf{q}'_\beta,\chi_\beta|G^C_\beta(z)U^C_{\beta\alpha}(z)G^C_\alpha(z)|\chi_\alpha,\mathbf{q}_\alpha\rangle. \tag{11b}$$

Note that as a result of assumption (7), namely that the short-range interactions are described by separable potentials of rank one, they contain only pure Coulombic quantities. Representation (11a) uses the resolvent of the three-particle Coulomb Hamiltonian $H^C=H_0+V^C=H_0+\Sigma_{\nu=1}^3V^C_\nu$,

$$G^C(z)=(z-H^C)^{-1}, \tag{12}$$

besides the resolvent of the Coulomb channel Hamiltonian,

$$G^C_\alpha(z)=(z-H_0-V^C_\alpha)^{-1}. \tag{13}$$

The "auxiliary three-body transition operators," $U^C_{\beta\alpha}$ in Eq. (11b), are defined in terms of Eq. (12) and the Coulomb part of the channel interaction (5) as

$$U^C_{\beta\alpha}=\overline{\delta}_{\beta\alpha}G_0^{-1}+\sum_\nu\overline{\delta}_{\beta\nu}\overline{\delta}_{\alpha\nu}V^C_\nu+\overline{V}^C_\beta G^C\overline{V}^C_\alpha, \tag{14}$$

where

$$G_0(z)=(z-H_0)^{-1} \tag{15}$$

is the resolvent of the three-free particle Hamiltonian. Alternatively, they can be found as solutions of the equations

$$U^C_{\beta\alpha}=\overline{\delta}_{\beta\alpha}G_0^{-1}+\sum_\nu\overline{\delta}_{\beta\nu}T^C_\nu G_0U^C_{\beta\alpha} \tag{16a}$$

$$=\overline{\delta}_{\beta\alpha}G_0^{-1}+\sum_\nu\overline{\delta}_{\beta\nu}\overline{\delta}_{\nu\alpha}T^C_\nu+\sum_{\nu,\mu}\overline{\delta}_{\beta\nu}\overline{\delta}_{\nu\mu}\overline{\delta}_{\mu\alpha}T^C_\nu G_0T^C_\mu+\cdots. \tag{16b}$$

As usual, the subsystem Coulomb T-operator T^C_α is related to the Coulomb channel resolvent via

$$G^C_\alpha=G_0+G_0T^C_\alpha G_0. \tag{17}$$

Here, and throughout the following, any explicit z dependence is omitted unless required for clarity. Finally, the plane wave $|\mathbf{q}_\alpha\rangle$ in channel α is the eigenfunction of the momentum operator $\mathbf{Q}_\alpha$ to the eigenvalue $\mathbf{q}_\alpha$.

An important special case arises when only two of the three particles are charged and one is neutral. Such a situation is realized, e.g., in deuteron-induced nuclear reactions. In that case, the effective potential (11a) simplifies considerably. To be specific, let the neutral particle carry the index 3. Thus, only $V^C_3\neq0$, and the full three-body Coulomb resolvent reduces to the Coulomb resolvent for channel 3: $G^C(z)\equiv G^C_3(z)$. Using representation (11a) one finds

$$\mathcal{V}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)=\langle\mathbf{q}'_\beta,\chi_\beta|(1-\delta_{\beta\alpha}\delta_{\alpha3})G^C_3-\delta_{\beta\alpha}\overline{\delta}_{\alpha3}G_0|\chi_\alpha,\mathbf{q}_\alpha\rangle, \tag{18}$$

which is *exact* to all orders in the Coulomb potential (within the presently adopted model for the short-range interaction).

The effective free propagator describing the propagation of the noninteracting particles, α and $(\beta\gamma)$, is defined as

$$\mathcal{G}_{0;\alpha}(\mathbf{q}_\alpha;z)=\frac{S_\alpha(z-q^2_\alpha/2M_\alpha)}{z-q^2_\alpha/2M_\alpha+B_\alpha}, \tag{19}$$

with

$$S^{-1}_\alpha(\hat{z})=\langle\chi_\alpha|\hat{G}^C_\alpha(-B_\alpha)\hat{G}^C_\alpha(\hat{z})|\chi_\alpha\rangle. \tag{20}$$

Here,

$$\hat{G}^C_\alpha(\hat{z})=(\hat{z}-K^2_\alpha/2\mu_\alpha-V^C_\alpha)^{-1} \tag{21}$$

is the two-body Coulomb resolvent read in the two-particle space. For clarity, here and in the following all energy-dependent operators, when read in the two-particle space, are characterized by a hat.

We point out that, with a suitable choice of the normalization of the form factor, the bound state wave function for the pair $(\beta\gamma)$ is given as

$$|\psi_\alpha\rangle=\hat{G}^C_\alpha(-B_\alpha)|\chi_\alpha\rangle. \tag{22}$$

If $|\psi_\alpha\rangle$ is normalized to unity one has on the energy shell, i.e., for $z=E+i0$ and $q_\alpha=\overline{q}_\alpha$, or equivalently for $E-\overline{q}^2_\alpha/2M_\alpha=-B_\alpha$,

$$S_\alpha(-B_\alpha)=1. \tag{23}$$

III. LEADING SINGULARITIES OF THE NONDIAGONAL KERNELS $\mathcal{K}_{\beta\alpha}$: RESUMMÉ

The compactness of the effective-two-body AGS integral equations depends on the analytical properties of the effective potentials $\mathcal{V}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$ and propagators $\mathcal{G}_{0;\alpha}(\mathbf{q}_\alpha;z)$, which occur in the kernels $\mathcal{K}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$, Eq. (10), and in the inhomogeneous term of Eq. (9). In this section we pro-

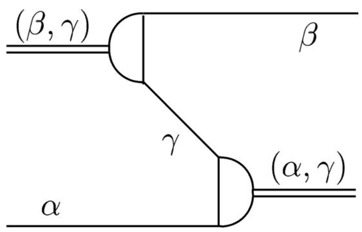

FIG. 1. Nondiagonal effective potential for neutral particles. Semicircles denote the neutral-particle form factors.

vide a synopsis of the results concerning the leading singularities of the nondiagonal parts of these kernels (i.e., for $\beta \neq \alpha$), the investigation of the diagonal part ($\beta=\alpha$) being deferred to a subsequent publication. The detailed investigation of the singularity structure is presented in the following section.

A. General remarks

Consider the nondiagonal effective potential $\mathcal{V}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$ which, according to Eq. (11a), is given as

$$\mathcal{V}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)=\langle \mathbf{q}'_\beta,\chi_\beta|G^C(z)|\chi_\alpha,\mathbf{q}_\alpha\rangle,\quad \text{with}\quad \beta\neq\alpha. \tag{24}$$

Its physical interpretation is that of the (on- or off-the-energy-shell) transfer amplitude of particle γ from the incoming $(\beta\gamma)$ to the outgoing bound state $(\alpha\gamma)$, while allowing for all possible successive Coulomb scatterings of the particles α, β, and γ, in the intermediate state as represented by the three-body Coulomb resolvent G^C.

If all intermediate-state Coulomb scatterings are neglected, i.e., if G^C is replaced by G_0, expression (24) reduces to the lowest-order particle-γ-transfer amplitude (''pole amplitude'')

$$\mathcal{V}^{(0)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)=\langle \mathbf{q}'_\beta,\chi_\beta|G_0(z)|\chi_\alpha,\mathbf{q}_\alpha\rangle. \tag{25}$$

This is nothing but the familiar effective potential pertaining to the scattering of uncharged particles which is exact within our simple model for the short-range interaction. Its diagrammatic representation is given in Fig. 1.

B. Leading singularity of $\mathcal{V}^{(0)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$

It is helpful to recapitulate the singular behavior of $\mathcal{V}^{(0)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$, $z=E+i0$, which is the effective potential in the absence of Coulomb interactions. This also serves to introduce some notation. The corresponding analytic expression is

$$\mathcal{V}^{(0)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)=-2\mu_\beta\frac{\chi^*_\beta(\mathbf{k}'_\beta)\chi_\alpha(\mathbf{k}_\alpha)}{\sigma_\beta(q'_\beta;z)}. \tag{26}$$

Here,

$$\mathbf{k}_\alpha=\epsilon_{\alpha\beta}(\mathbf{q}'_\beta+\lambda_{\beta\gamma}\mathbf{q}_\alpha)\quad\text{and}\quad \mathbf{k}'_\beta=\epsilon_{\beta\alpha}(\mathbf{q}_\alpha+\lambda_{\alpha\gamma}\mathbf{q}'_\beta) \tag{27}$$

are the relative momenta between particles β and γ in the vertex $(\beta\gamma)\rightarrow\beta+\gamma$, and between particles α and γ in the vertex $(\alpha\gamma)\rightarrow\alpha+\gamma$, respectively, expressed as linear combinations of the incoming and the outgoing momenta according to Eq. (A2). The following notation is used:

$$\lambda_{\nu\mu}=m_\nu/(m_\nu+m_\mu)=1-\lambda_{\mu\nu},\quad \nu\neq\mu; \tag{28}$$

$\epsilon_{\alpha\beta}=-\epsilon_{\beta\alpha}$ is the antisymmetric symbol with $\epsilon_{\alpha\beta}=+1$ if (α,β) is a cyclic ordering of the indices (1,2,3). Moreover,

$$\sigma_\beta(q'_\beta;z):=k'^2_\beta-2\mu_\beta\hat{z}_\beta=(\mathbf{q}_\alpha+\lambda_{\alpha\gamma}\mathbf{q}'_\beta)^2-2\mu_\beta\hat{z}_\beta, \tag{29}$$

with

$$\hat{z}_\beta\equiv\hat{E}_\beta+i0,\quad \hat{E}_\beta:=E-q'^2_\beta/2M_\beta. \tag{30}$$

That is, $\hat{E}_\beta$ is the energy parameter for the subsystem $(\alpha+\gamma)$. Note that for on-shell values of the momentum $q'_\beta=\overline{q'_\beta}$, cf. Eq. (8), one has

$$\hat{E}_\beta|_{q'_\beta=\overline{q'_\beta}}=-B_\beta; \tag{31}$$

thus, the deviation of $\hat{E}_\beta$ from $-B_\beta$, is a measure of the ''off-shellity.''

Similarly, we introduce

$$\sigma_\alpha(q_\alpha;z):=k^2_\alpha-2\mu_\alpha\hat{z}_\alpha=(\mathbf{q}'_\beta+\lambda_{\beta\gamma}\mathbf{q}_\alpha)^2-2\mu_\alpha\hat{z}_\alpha. \tag{32}$$

Here, $\hat{z}_\alpha$ is defined in analogy to Eq. (30) but in terms of α-channel quantities:

$$\hat{z}_\alpha\equiv\hat{E}_\alpha+i0,\quad \hat{E}_\alpha:=E-q^2_\alpha/2M_\alpha, \tag{33}$$

with $\hat{E}_\alpha$ being the energy parameter for the subsystem $(\beta+\gamma)$. One easily derives the relation

$$\sigma_\alpha(q_\alpha;z)/\mu_\alpha=\sigma_\beta(q'_\beta;z)/\mu_\beta, \tag{34}$$

which holds true in particular also for on-shell values of the momenta. The corresponding quantities are denoted by

$$\overline{\sigma}_\beta:=(\overline{\mathbf{q}}_\alpha+\lambda_{\alpha\gamma}\overline{\mathbf{q}'_\beta})^2+2\mu_\beta B_\beta,$$
$$\overline{\sigma}_\alpha:=(\overline{\mathbf{q}'_\beta}+\lambda_{\beta\gamma}\overline{\mathbf{q}}_\alpha)^2+2\mu_\alpha B_\alpha. \tag{35}$$

From Eq. (26) the familiar result follows, namely that the main singularity of $\mathcal{V}^{(0)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$ is a pole at

$$\sigma_\beta(q'_\beta;z)=0=\sigma_\alpha(q_\alpha;z). \tag{36}$$

For $z=E+i0, E\geqslant 0$, it is located in the region of integration over q_α, while for $E<0$ or for on-shell values of the momenta ($q'_\beta=\overline{q'_\beta}$ and $q_\alpha=\overline{q_\alpha}$) this pole is situated off the real axis in the complex q_α plane at Eq. (36) or at $\overline{\sigma}_\beta=0=\overline{\sigma}_\alpha$, respectively. Singularities like the pole of $\mathcal{V}^{(0)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$ at Eq. (36), the position of which depend on the energy, are termed ''dynamic'' singularities.

We point out that at the singular point (36), the angle between $\mathbf{q}'_\beta$ and $\mathbf{q}_\alpha$ is determined by the magnitudes q'_β and q_α, and by z.

Remark. $\mathcal{V}^{(0)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$ has additional singularities coming from the vertex functions $\chi_\beta(\mathbf{k}'_\beta)$ and $\chi_\alpha(\mathbf{k}_\alpha)$ which are, however, related to the characteristics of the short-range interactions V^S_β and V^S_α. As is well known, the latter are not dangerous because they are located in the unphysical region at $k'^2_\beta<0$ and $k^2_\alpha<0$, respectively. Since their position is independent of energy they are called ''static'' singularities. These facts can easily be checked for the case of a simple Yukawa-type S-wave form factor which in momentum space reads as

$$\chi_\alpha(\mathbf{k}_\alpha)=1/(k^2_\alpha+\beta^2_\alpha). \tag{37}$$

Here, the quantity β^{-1}_α represents a measure of the ''range'' of the interaction V^S_α. Singularities of this type are of no interest in the present context.

C. Leading singularity of $\mathcal{V}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$

Let us state the assertion in the following form.

Theorem. The leading (dynamic) singularity of the nondiagonal effective potential (24) with respect to the momentum transfer is in general a branch point at

$$\sigma_\beta(q'_\beta;z)=0=\sigma_\alpha(q_\alpha;z). \tag{38}$$

(i) Consider off-shell values of the momenta $q'_\beta\neq\overline{q'_\beta}$ and $q_\alpha\neq\overline{q_\alpha}$, satisfying

$$q'_\beta\neq\tilde{q}_\beta:=\sqrt{2M_\beta E}\quad\text{and}\quad q_\alpha\neq\tilde{q}_\alpha:=\sqrt{2M_\alpha E}, \tag{39}$$

which implies $\hat{E}_\beta\neq0$ and $\hat{E}_\alpha\neq0$. With $z=E+i0$, in the q'_β plane the locus of this branch point is determined by $q'^2_\beta+M_\beta(\mathbf{q}_\alpha+\lambda_{\alpha\gamma}\mathbf{q}'_\beta)^2/\mu_\beta=\tilde{q}^2_\beta$, and in the q_α plane by $q^2_\alpha+M_\alpha(\mathbf{q}'_\beta+\lambda_{\beta\gamma}\mathbf{q}_\alpha)^2/\mu_\alpha=\tilde{q}^2_\alpha$. In its vicinity, $\mathcal{V}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$ behaves as

$$\mathcal{V}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)\overset{\sigma_\beta(q'_\beta;z)\to0}{\sim}\frac{1}{\sigma_\beta(q'_\beta;z)^{1-i(\hat{\eta}_\alpha+\hat{\eta}_\beta)}}\qquad(q'_\beta\neq\tilde{q}_\beta,\overline{q'_\beta};\;\;q_\alpha\neq\tilde{q}_\alpha,\overline{q_\alpha}), \tag{40}$$

where

$$\hat{\eta}_\alpha\equiv\hat{\eta}_\alpha(\sqrt{2\mu_\alpha\hat{z}_\alpha})=e_\beta e_\gamma\mu_\alpha/\sqrt{2\mu_\alpha\hat{z}_\alpha}, \tag{41a}$$

$$\hat{\eta}_\beta\equiv\hat{\eta}_\beta(\sqrt{2\mu_\beta\hat{z}_\beta})=e_\gamma e_\alpha\mu_\beta/\sqrt{2\mu_\beta\hat{z}_\beta} \tag{41b}$$

are the Coulomb parameters pertaining to the particles β and γ, and α and γ, respectively.

(ii) The special points $q'_\beta=\tilde{q}_\beta$ and/or $q_\alpha=\tilde{q}_\alpha$ have to be treated separately. For this purpose we introduce the notations

$$\tilde{\mathbf{q}}'_\beta\equiv\tilde{q}_\beta\hat{\mathbf{q}}'_\beta\quad\text{and}\quad\tilde{\mathbf{q}}_\alpha\equiv\tilde{q}_\alpha\hat{\mathbf{q}}_\alpha. \tag{42}$$

Fixing the outgoing momentum at $q'_\beta=\tilde{q}_\beta$ we have $\hat{E}_\beta=0$ and hence [cf. Eqs. (29) and (30)] $\sigma_\beta(\tilde{q}_\beta;z)=k'^2_\beta-i0$, with $\mathbf{k}'_\beta$ defined as in Eq. (27) but as linear combination of $\tilde{\mathbf{q}}'_\beta$ and $\mathbf{q}_\alpha$. Then, if $q_\alpha\neq\tilde{q}_\alpha$, the leading singular behavior of the nondiagonal effective potential (24) is of the form

$$\mathcal{V}_{\beta\alpha}(\tilde{\mathbf{q}}'_\beta,\mathbf{q}_\alpha;z)\overset{k'_\beta\to0}{\sim}\frac{C(k'_\beta)}{k'^{\,2-2i\hat{\eta}_\alpha+i\eta_\gamma}_\beta}\qquad(q_\alpha\neq\tilde{q}_\alpha,\overline{q_\alpha}), \tag{43}$$

with

$$\lim_{k'_\beta\to0}C(k'_\beta)=0. \tag{44}$$

In other words, $\mathcal{V}_{\beta\alpha}(\tilde{\mathbf{q}}'_\beta,\mathbf{q}_\alpha;z)$ is actually less singular than described by the exponent in Eq. (43). Here, $\eta_\gamma=e_\alpha e_\beta\mu_\gamma/k_\gamma$ with k_γ is considered as linear combination of $\tilde{\mathbf{q}}'_\beta$ and $\mathbf{q}_\alpha$. Let us put $z=E+i0$. In the q'_β-plane, this branch point is located at $q'_\beta=\tilde{q}_\beta$. In the q_α plane, its locus is determined by $k'^2_\beta\equiv(\mathbf{q}_\alpha+\lambda_{\alpha\gamma}\tilde{\mathbf{q}}'_\beta)^2=0$ which for $E\geqslant0$, i.e., for real $\tilde{q}_\beta$, can be on the positive real q_α axis, while for $E<0$ it is always located at complex values of q_α. An analogous result holds for $q'_\beta\neq\tilde{q}_\beta$ but $q_\alpha=\tilde{q}_\alpha$ when $\hat{E}_\alpha=0$ and $\sigma_\alpha(\tilde{q}_\alpha;z)=k^2_\alpha-i0$, with $\mathbf{k}_\alpha$ defined as in Eq. (27) but with $\tilde{\mathbf{q}}_\alpha$ instead of $\mathbf{q}_\alpha$. In that case, the singular behavior in the limit $\sigma_\alpha(\tilde{q}_\alpha;z)\to0$ is of the form

$$\mathcal{V}_{\beta\alpha}(\mathbf{q}'_\beta,\tilde{\mathbf{q}}_\alpha;z)\overset{k_\alpha\to0}{\sim}\frac{C'(k_\alpha)}{k^{\,2-2i\hat{\eta}_\beta+i\eta_\gamma}_\alpha}\qquad(q'_\beta\neq\tilde{q}_\beta,\overline{q'_\beta}), \tag{45}$$

again with

$$\lim_{k_\alpha\to0}C'(k_\alpha)=0 \tag{46}$$

and η_γ as defined above but with k_γ being considered now as linear combination of $\mathbf{q}'_\beta$ and $\tilde{\mathbf{q}}_\alpha$. And, finally, if $q_\alpha=\tilde{q}_\alpha$ and $q'_\beta=\tilde{q}_\beta$, the nondiagonal effective potential (24) behaves for $\sigma_\beta(\tilde{q}_\beta;z)=k'^2_\beta=\mu_\beta k^2_\alpha/\mu_\alpha\to0$ as

$$\mathcal{V}_{\beta\alpha}(\tilde{\mathbf{q}}'_\beta,\tilde{\mathbf{q}}_\alpha;z)\overset{k'_\beta\to0}{\sim}D(k'_\beta)k'^{-2}_\beta\quad\text{for}\;\;E=0, \tag{47}$$

with

$$\lim_{k'_\beta\to0}D(k'_\beta)=0. \tag{48}$$

We note that, on account of the linear relations (A1) or (A2), for $q_\alpha=\tilde{q}_\alpha$ and $q'_\beta=\tilde{q}_\beta$ the limiting values $k'_\beta=0=k_\alpha$ can be reached only for $\tilde{q}_\beta=\tilde{q}_\alpha=0$, i.e., for $E=0$.

(iii) An important special case arises when either the incoming and/or the outgoing momentum equals its on-shell value, that is, when $\mathbf{q}_\alpha\to\overline{\mathbf{q}_\alpha}$ (i.e., $\hat{E}_\alpha\to-B_\alpha$) and/or $\mathbf{q}'_\beta\to\overline{\mathbf{q}'_\beta}$ (i.e., $\hat{E}_\beta\to-B_\beta$), cf. Eq. (8). Denote the Coulomb parameters for the bound pairs $(\beta\gamma)$ and $(\alpha\gamma)$ by $\eta^{(bs)}_\alpha$ and $\eta^{(bs)}_\beta$, respectively. They are given explicitly as

$$\eta_\alpha^{(bs)} = e_\beta e_\gamma \mu_\alpha / \sqrt{2\mu_\alpha B_\alpha}, \quad \eta_\beta^{(bs)} = e_\alpha e_\gamma \mu_\alpha / \sqrt{2\mu_\beta B_\beta}. \tag{49}$$

Then the leading singular behavior of the effective potential can be obtained from the previous results with the substitutions $\hat{\eta}_\alpha \to -i\eta_\alpha^{(bs)}$ and/or $\hat{\eta}_\beta \to -i\eta_\beta^{(bs)}$. For instance, the behavior of the fully on-shell effective potential in the vicinity of the leading singularity at

$$\overline{\sigma}_\beta = 0 = \overline{\sigma}_\alpha \tag{50}$$

is again a branch point of the form

$$\mathcal{V}_{\beta\alpha}(\overline{\mathbf{q}'_\beta}, \overline{\mathbf{q}}_\alpha; E+i0) \overset{\sigma_\beta(\overline{q'_\beta};z)\to 0}{\sim} \frac{1}{\overline{\sigma}_\beta^{\,1-\eta_\alpha^{(bs)}-\eta_\beta^{(bs)}}}. \tag{51}$$

Its position is always off the real axis in the complex q_α- or q_β-plane. Note that if, e.g., $\mathbf{q}_\alpha = \overline{\mathbf{q}}_\alpha$ then the cases (45) and (47) cannot occur since $\overline{q}_\alpha > \tilde{q}_\alpha$ (because of our requirement $B_\nu > 0$ for $\nu = 1,2,3$); analogously for $\mathbf{q}'_\beta = \overline{\mathbf{q}'_\beta}$.

(iv) The branch point singularities at the positions (38) and (50), respectively, arise solely from the Coulomb modifications of the initial- and final-state form factors while Coulomb interactions of the three particles in the intermediate state only alter the strength of the singularity but not its type or position.

Note. The assertions of this theorem are valid if the charges of all three particles are of the same sign (all Coulomb potentials are repulsive).

Corollary. If only two of the three particles, say 1 and 2, are charged, and particle 3 is neutral, the leading singularity of the effective potential is weaker. Explicitly one has

$$\mathcal{V}_{\beta\alpha}(\mathbf{q}'_\beta, \mathbf{q}_\alpha; z) \overset{\sigma_\beta(q'_\beta;z)\to 0}{\sim} \frac{\delta_{\beta 3}}{\sigma_\beta(q'_\beta;z)^{1-i\hat{\eta}_\beta}} + \frac{\delta_{\alpha 3}}{\sigma_\beta(q'_\beta;z)^{1-i\hat{\eta}_\alpha}} + \frac{\overline{\delta}_{\beta 3}\overline{\delta}_{\alpha 3}}{\sigma_\beta(q'_\beta;z)}, \tag{52a}$$

$$\mathcal{V}_{\beta\alpha}(\tilde{\mathbf{q}}'_\beta, \mathbf{q}_\alpha; z) \overset{k'_\beta\to 0}{\sim} k'^0_\beta \delta_{\beta 3} + \frac{\delta_{\alpha 3}}{k'^{\,2-2i\hat{\eta}_\alpha}_\beta} + \frac{\overline{\delta}_{\beta 3}\overline{\delta}_{\alpha 3}}{k'^2_\beta}, \tag{52b}$$

$$\mathcal{V}_{\beta\alpha}(\mathbf{q}'_\beta, \tilde{\mathbf{q}}_\alpha; z) \overset{k_\alpha\to 0}{\sim} \frac{\delta_{\beta 3}}{k_\alpha^{2-2i\hat{\eta}_\beta}} + k^0_\alpha \delta_{\alpha 3} + \frac{\overline{\delta}_{\beta 3}\overline{\delta}_{\alpha 3}}{k^2_\alpha}, \tag{52c}$$

$$\mathcal{V}_{\beta\alpha}(\tilde{\mathbf{q}}'_\beta, \tilde{\mathbf{q}}_\alpha; z) \overset{k'_\beta\to 0}{\sim} k'^0_\beta (\delta_{\beta 3} + \delta_{\alpha 3}) + \frac{\overline{\delta}_{\beta 3}\overline{\delta}_{\alpha 3}}{k'^2_\beta}. \tag{52d}$$

Comment. Comparison with the effective potential $\mathcal{V}^{(0)}_{\beta\alpha}(\mathbf{q}'_\beta, \mathbf{q}_\alpha; z)$ for neutral particles shows that quite generally the sole additional effect of the Coulomb interactions consists in converting the pole of the latter into a branch point, without shifting the position of the singularity.

D. Singular behavior of the kernel $\mathcal{K}_{\beta\alpha}(\mathbf{q}'_\beta, \mathbf{q}_\alpha; E+i0)$

Given the leading singularity of the nondiagonal effective potential $\mathcal{V}_{\beta\alpha}(\mathbf{q}'_\beta, \mathbf{q}_\alpha; E+i0)$, the singularity structure of the kernel $\mathcal{K}_{\beta\alpha}(\mathbf{q}'_\beta, \mathbf{q}_\alpha; E+i0)$, Eq. (10), for $\beta \neq \alpha$ can be elucidated. Integration over the right-hand variable, presently denoted by $\mathbf{q}_\alpha$, is implied in Eq. (9); $\mathbf{q}'_\beta$ is a vector-valued parameter. The leading singularities of the kernel are the branch point that originates from $\mathcal{V}_{\beta\alpha}(\mathbf{q}'_\beta, \mathbf{q}_\alpha; E+i0)$ and is located as described in Sec. III C, and the pole of the effective propagator $\mathcal{G}_{0;\alpha}(\mathbf{q}_\alpha; E+i0)$, Eq. (19), at the "on-shell point" $q_\alpha = \overline{q}_\alpha$. Note that, according to Appendix B, for the repulsive Coulomb potentials considered the numerator function $S_\alpha(E+i0-q_\alpha^2/2M_\alpha)$ of $\mathcal{G}_{0;\alpha}$ is not dangerous and, hence, will not give rise to any problem when the integration over $\mathbf{q}_\alpha$ is performed.

(i) Since the singularity of $\mathcal{V}_{\beta\alpha}(\mathbf{q}'_\beta, \mathbf{q}_\alpha; E+i0)$ can lie on the integration contour only for off-shell values of the momenta (i.e., for $\mathbf{q}'_\beta \neq \overline{\mathbf{q}'_\beta}$ and $\mathbf{q}_\alpha \neq \overline{\mathbf{q}}_\alpha$), it can never coincide with the propagator pole.

(ii) If $\mathbf{q}'_\beta$ equals its on-shell value $\overline{\mathbf{q}'_\beta}$, the leading singularity of $\mathcal{K}_{\beta\alpha}(\overline{\mathbf{q}'_\beta}, \mathbf{q}_\alpha; z)$ is, for all physically accessible values of $\mathbf{q}_\alpha$, the propagator pole. For, the leading singularity of $\mathcal{V}_{\beta\alpha}(\overline{\mathbf{q}'_\beta}, \mathbf{q}_\alpha; E+i0)$ at $(\overline{\mathbf{q}'_\beta} + \lambda_{\beta\gamma}\mathbf{q}_\alpha)^2 + 2\mu_\alpha B_\alpha = 0$ is always located outside the integration contour and, hence, is harmless. An analogous situation prevails if $\mathbf{q}_\alpha$ equals its on-shell value $\overline{\mathbf{q}}_\alpha$, or if $\mathbf{q}'_\beta = \overline{\mathbf{q}'_\beta}$ and $\mathbf{q}_\alpha = \overline{\mathbf{q}}_\alpha$.

Summarizing we have the result that leading singularities of the nondiagonal kernels are integrable, for momenta both off and on the energy shell, and can thus be treated by standard methods. This concludes the overview of the singularity structure of the nondiagonal kernel.

IV. PROOFS OF THE ASSERTIONS

A. Decomposition of $\mathcal{V}_{\beta\alpha}(\mathbf{q}'_\beta, \mathbf{q}_\alpha; z)$

As has already been pointed out and will become clear soon, the Coulomb interactions in the initial and final vertices play a special role. It, therefore, proves advantageous to work with the representation (11b) where the corresponding Coulomb channel resolvents are already factored out. Indeed, the resolvent G_α^C (G_β^C) describes the propagation of the three-particle system (α,β,γ) with allowance for Coulomb scatterings to all orders between particles β and γ after the virtual decay $(\beta\gamma) \to \beta + \gamma$ of the initial bound state $(\beta\gamma)$ [of α and γ before the virtual recombination $\alpha + \gamma \to (\alpha\gamma)$ leading to the formation of the final bound state $(\alpha\gamma)$].

First we note that

$$G_\alpha^C(z)|\chi_\alpha, \mathbf{q}_\alpha\rangle = |\mathbf{q}_\alpha\rangle \hat{G}_\alpha^C(\hat{z}_\alpha)|\chi_\alpha\rangle, \tag{53}$$

with $\hat{G}_\alpha^C(\hat{z}_\alpha)$ defined in Eq. (21). Denoting by $\hat{G}_0(\hat{z}_\alpha)$ the corresponding free two-body resolvent, the so-called "off-shell Coulomb-modified form factor" (i.e., off the two-body energy shell) is introduced as

$$\phi_\alpha(\mathbf{k}_\alpha; \hat{z}_\alpha) \equiv \langle \mathbf{k}_\alpha | \phi_\alpha(\hat{z}_\alpha)\rangle := \langle \mathbf{k}_\alpha | \hat{G}_0^{-1}(\hat{z}_\alpha)\hat{G}_\alpha^C(\hat{z}_\alpha)|\chi_\alpha\rangle. \tag{54}$$

Thus, using the explicit definition (14) of $U^C_{\beta\alpha}$, Eq. (11b) can be rewritten as

$$\mathcal{V}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)=\langle\mathbf{q}'_\beta,\phi_\beta|G_0+G_0V^C_\gamma G_0 + G_0\overline{V}^C_\beta G^C\overline{V}^C_\alpha G_0|\phi_\alpha,\mathbf{q}_\alpha\rangle,\quad \gamma\neq\alpha,\beta, \tag{55}$$

$$=\mathcal{V}^{(a)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)+\mathcal{V}^{(b)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z) + \tilde{\mathcal{V}}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z), \tag{56}$$

with

$$\mathcal{V}^{(a)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z):=\langle\mathbf{q}'_\beta,\phi_\beta(\hat{z}^*_\beta)|G_0(z)|\phi_\alpha(\hat{z}_\alpha),\mathbf{q}_\alpha\rangle, \tag{57}$$

$$\mathcal{V}^{(b)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z):=\langle\mathbf{q}'_\beta,\phi_\beta(\hat{z}^*_\beta)|G_0(z) \times V^C_\gamma G_0(z)|\phi_\alpha(\hat{z}_\alpha),\mathbf{q}_\alpha\rangle, \quad \gamma\neq\alpha,\beta, \tag{58}$$

$$\tilde{\mathcal{V}}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z):=\langle\mathbf{q}'_\beta,\phi_\beta(\hat{z}^*_\beta)|G_0(z)\overline{V}^C_\beta G^C(z) \times\overline{V}^C_\alpha G_0(z)|\phi_\alpha(\hat{z}_\alpha),\mathbf{q}_\alpha\rangle. \tag{59}$$

The first term $\mathcal{V}^{(a)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$ describes the transfer mechanism of particle γ from the incoming bound state $(\beta\gamma)$, to the outgoing one composed of particles $(\alpha\gamma)$, with Coulomb scattering to all orders of particles β and γ in the initial vertex $(\beta\gamma)\to\beta+\gamma$, and of α and γ in the final vertex $(\alpha\gamma)\to\alpha+\gamma$, having been absorbed in the Coulomb-modified form factors $|\phi_\alpha\rangle$ and $|\phi_\beta\rangle$. The second contribution $\mathcal{V}^{(b)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$ contains an additional intermediate-state Coulomb interaction V^C_γ between the particles α and β which are unbound before and after the interaction described by V^C_γ. The last term, finally, comprises all intermediate-state Coulomb scatterings between the three particles as represented by the three-body Coulomb resolvent.

In the following we will show that, except for the special points $q_\alpha=\tilde{q}_\alpha$ and/or $q'_\beta=\tilde{q}_\beta$, the full effective potential $\mathcal{V}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$ has the same leading dynamic singularity in the momentum transfer plane as the simple, first term $\mathcal{V}^{(a)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$ in the representation (56). Hereby, leading singularity is defined as that singularity which results from the coincidence of the singularities of all the operators which are sandwiched between the states $\langle\mathbf{q}'_\beta,\phi_\beta|$ and $|\phi_\alpha,\mathbf{q}_\alpha\rangle$, with the singularities of the off-shell Coulomb-modified form factors $|\phi_\alpha\rangle$ and $|\phi_\beta\rangle$. In other words, we will prove the theorem that the replacement of $[G_0+G_0V^C_\gamma G_0 +G_0\overline{V}^C_\beta G^C\overline{V}^C_\alpha G_0]$ in Eq. (55) by G_0 changes neither the type (in the case under consideration, branch point), nor the location of the leading singularity. This goal will be achieved by showing that the second and the third term in the more detailed representation (56) have the same leading singularity in the momentum transfer plane as $\mathcal{V}^{(a)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$. The points $q'_\beta=\tilde{q}_\beta$ and $q_\alpha=\tilde{q}_\alpha$ are investigated separately.

B. Leading singularity of $\mathcal{V}^{(a)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$

We start by investigating the singular behavior of Eq. (57) which in the momentum space representation reads as

$$\mathcal{V}^{(a)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)=-2\mu_\beta\frac{\phi^*_\beta(\mathbf{k}'_\beta;\hat{z}^*_\beta)\phi_\alpha(\mathbf{k}_\alpha;\hat{z}_\alpha)}{\sigma_\beta(q'_\beta;z)}. \tag{60}$$

Evidently, its main singularity is located at $\sigma_\beta(q'_\beta;z)=0 =\sigma_\alpha(q_\alpha;z)$ as the free Green's function has a pole singularity there. In addition, $\mathcal{V}^{(a)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$ has also singularities coming from the Coulomb-modified vertex functions $\phi_\beta(\mathbf{k}'_\beta;\hat{z}_\beta)$ and $\phi_\alpha(\mathbf{k}_\alpha;\hat{z}_\alpha)$.

As shown in Appendix C, in the limit $\sigma_\alpha(q_\alpha;z)\equiv(k^2_\alpha -2\mu_\alpha\hat{z}_\alpha)\to0$, with $\hat{z}_\alpha=\hat{E}_\alpha+i0$, the off-shell Coulomb-modified form factor $\phi_\alpha(\mathbf{k}_\alpha;\hat{z}_\alpha)$ behaves as

$$\phi_\alpha(\mathbf{k}_\alpha;\hat{z}_\alpha)\overset{k^2_\alpha-2\mu_\alpha\hat{z}_\alpha\to0}{\sim}\begin{cases}(k^2_\alpha-2\mu_\alpha\hat{z}_\alpha)^{i\hat{\eta}_\alpha} & \text{for } \hat{E}_\alpha\neq0\\ k^2_\alpha & \text{for } \hat{E}_\alpha=0.\end{cases} \tag{61}$$

It, therefore, proves to be convenient to put

$$\phi_\alpha(\mathbf{k}_\alpha;\hat{z}_\alpha)=:(k^2_\alpha-2\mu_\alpha\hat{z}_\alpha)^{i\hat{\eta}_\alpha}\tilde{\phi}_\alpha(\mathbf{k}_\alpha;\hat{z}_\alpha)\quad\text{for }\hat{E}_\alpha\neq0, \tag{62a}$$

$$\phi_\alpha(\mathbf{k}_\alpha;0)=:k^2_\alpha\tilde{\phi}_\alpha(\mathbf{k}_\alpha;0)\quad\text{for }\hat{E}_\alpha=0, \tag{62b}$$

with the ''reduced Coulomb-modified form factor'' $\tilde{\phi}_\alpha(\cdot)$ being regular and nonvanishing at

$$k^2_\alpha-2\mu_\alpha\hat{z}_\alpha=0,\quad \hat{z}_\alpha=\hat{E}_\alpha+i0,\quad \forall\hat{E}_\alpha. \tag{63}$$

The Coulomb parameter $\hat{\eta}_\alpha$ is defined in Eq. (41a). Note that the on-shell case $\hat{E}_\alpha\to-\hat{B}_\alpha$, and hence $\hat{\eta}_\alpha\to-i\hat{\eta}^{(bs)}_\alpha$, is included in Eq. (62a). Though, also $\tilde{\phi}_\alpha(\mathbf{k}_\alpha;\hat{z}_\alpha)$ has singularities but they lie farther away from the physical region than the singularity of $\phi_\alpha(\mathbf{k}_\alpha;\hat{z}_\alpha)$ at Eq. (63), their specific location depending on the decay properties of the tail of the short-range potential. This can be checked explicitly for the case of a form factor of the type (37) (cf. Appendix C).

Analogously, for $\phi^*_\beta(\mathbf{k}'_\beta;\hat{z}^*_\beta)$, with $\hat{z}_\beta=\hat{E}_\beta+i0$, one defines

$$\phi^*_\beta(\mathbf{k}'_\beta;\hat{z}^*_\beta)=:(k'^2_\beta-2\mu_\beta\hat{z}_\beta)^{i\hat{\eta}_\beta}\tilde{\phi}^*_\beta(\mathbf{k}'_\beta;\hat{z}^*_\beta)\quad\text{for }\hat{E}_\beta\neq0, \tag{64a}$$

$$\phi^*_\beta(\mathbf{k}'_\beta;0)=:k'^2_\beta\tilde{\phi}^*_\beta(\mathbf{k}_\beta;0)\quad\text{for }\hat{E}_\beta=0, \tag{64b}$$

the Coulomb parameter having been defined in Eq. (41b). In Eq. (64a) we have made use of the fact that $\hat{\eta}^*_\beta(\sqrt{2\mu_\beta\hat{z}^*_\beta}) =-\hat{\eta}_\beta(\sqrt{2\mu_\beta\hat{z}_\beta})$ for both $\hat{E}_\beta>0$ and $\hat{E}_\beta<0$. Thus, its leading singularity is located at $\sigma_\beta(q'_\beta;z)\equiv k'^2_\beta-2\mu_\beta\hat{z}_\beta=0$. The singular points in Eqs. (62a), (62b), and (64a), (64b) coincide, on account of the identity (34). It is apparent that the leading singularities of both off-shell Coulomb-modified form factors $\phi_\beta(\mathbf{k}'_\beta;\hat{z}_\beta)$ and $\phi_\alpha(\mathbf{k}_\alpha;\hat{z}_\alpha)$ are in general branch points.

The coincidence of the singularities of the free Green's function and of the initial- and final-state Coulomb-modified form factors at Eq. (38) produces the leading singularity of the contribution $\mathcal{V}^{(a)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$. Its behavior in the vicinity of this singularity can be read off directly from Eq. (60). Apart from some trivial mass factor, it is given by

$$\mathcal{V}^{(a)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z) \overset{\sigma_\beta(q'_\beta;z)\to 0}{\sim} \frac{\tilde{\phi}^*_\beta(\mathbf{k}'_\beta;\hat{z}^*_\beta)\,\tilde{\phi}_\alpha(\mathbf{k}_\alpha;\hat{z}_\alpha)}{\sigma_\beta(q'_\beta;z)^{1-i(\hat{\eta}_\alpha+\hat{\eta}_\beta)}}$$
$$(q'_\beta\neq\tilde{q}_\beta,\overline{q'_\beta};\quad q_\alpha\neq\tilde{q}_\alpha,\overline{q_\alpha}), \tag{65a}$$

$$\mathcal{V}^{(a)}_{\beta\alpha}(\tilde{\mathbf{q}}'_\beta,\mathbf{q}_\alpha;z) \overset{k'_\beta\to 0}{\sim} k'^{\,2i\hat{\eta}_\alpha}_\beta\tilde{\phi}^*_\beta(\mathbf{0};0)\,\tilde{\phi}_\alpha(\mathbf{k}_\alpha;\hat{z}_\alpha)$$
$$(q_\alpha\neq\tilde{q}_\alpha,\overline{q_\alpha}), \tag{65b}$$

$$\mathcal{V}^{(a)}_{\beta\alpha}(\mathbf{q}'_\beta,\tilde{\mathbf{q}}_\alpha;z) \overset{k_\alpha\to 0}{\sim} k^{2i\hat{\eta}_\beta}_\alpha\tilde{\phi}^*_\beta(\mathbf{k}'_\beta;\hat{z}^*_\beta)\,\tilde{\phi}_\alpha(\mathbf{0};0)$$
$$(q_\beta\neq\tilde{q}_\beta,\overline{q'_\beta}), \tag{65c}$$

$$\mathcal{V}^{(a)}_{\beta\alpha}(\tilde{\mathbf{q}}'_\beta,\tilde{\mathbf{q}}_\alpha;z) \overset{k'_\beta\to 0}{\sim} k'^{2}_\beta\tilde{\phi}^*_\beta(\mathbf{0};0)\,\tilde{\phi}_\alpha(\mathbf{0};0). \tag{65d}$$

[Concerning the case (65d) recall the remark following Eq. (48).] It is obvious that the leading singular behavior of $\mathcal{V}^{(a)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$, if $\mathbf{q}'_\beta$ or $\mathbf{q}_\alpha$ or both equal their on-shell value $\overline{\mathbf{q}'_\beta}$ and $\overline{\mathbf{q}_\alpha}$, respectively, in the limit $\overline{\sigma_\beta}\to 0$ can be obtained from Eqs. (65a)–(65c) by the substitutions $i\hat{\eta}_\beta\to\eta^{(bs)}_\beta$ and/or $i\hat{\eta}_\alpha\to\eta^{(bs)}_\alpha$. We point out that on account of our assumption that all binding energies have nonzero values, one always has $\overline{q_\nu}\neq\tilde{q}_\nu$, $\nu=1,2,3$.

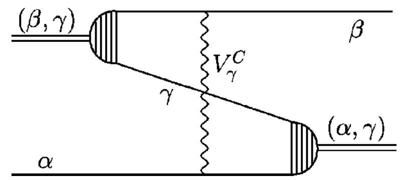

FIG. 2. Nondiagonal effective potential with single Coulombic rescattering in the intermediate state. Dashed semicircles denote the Coulomb-modified form factors.

Summarizing we have found that the leading dynamic singularity of $\mathcal{V}^{(a)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$ is in general a branch point at Eq. (38). The singularities of the reduced Coulomb-modified form factors, which yield further (the so-called static) singularities of $\mathcal{V}^{(a)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$, are located outside the physical regions $0\leqslant k'_\beta,k_\alpha<\infty$.

C. Leading singularity of $\mathcal{V}^{(b)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$

Theorem 1. The nondiagonal particle transfer amplitude $\mathcal{V}^{(b)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$ has the same leading singularity with respect to the momentum transfer as $\mathcal{V}^{(a)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$.

Proof of Theorem 1. Instead of considering the second term (58) in the decomposition (55), which is displayed in diagrammatic form in Fig. 2 and describes single rescattering in the intermediate state, we investigate the more general expression

$$\tilde{\mathcal{V}}^{(b)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)=\langle\mathbf{q}'_\beta,\phi_\beta(\hat{z}^*_\beta)|G_0(z)\times T^C_\gamma(z)G_0(z)|\phi_\alpha(\hat{z}_\alpha),\mathbf{q}_\alpha\rangle,$$
$$\gamma\neq\alpha,\beta, \tag{66}$$

which contains the Coulomb T-matrix T^C_γ instead of the Coulomb potential V^C_γ. The leading singularity of this expression is generated by the coincidence of the singularities of two free Green's functions with those of the two Coulomb-modified form factors ϕ_α and ϕ_β, and with the forward-scattering singularity of the Coulomb T-matrix. Explicitly we have

$$\tilde{\mathcal{V}}^{(b)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)=\int\frac{d\mathbf{k}}{(2\pi)^3}\frac{\phi^*_\beta(\mathbf{k}''_\beta;\hat{z}^*_\beta)\hat{T}^C_\gamma(\mathbf{k}''_\gamma,\mathbf{k}'_\gamma;\hat{z}_\gamma)\phi_\alpha(\mathbf{k}'_\alpha;\hat{z}_\alpha)}{[z-k''^2_\beta/2\mu_\beta-q'^2_\beta/2M_\beta][z-k'^2_\alpha/2\mu_\alpha-q^2_\alpha/2M_\alpha]}. \tag{67}$$

Here, the additional notations $\mathbf{k}''_\beta=\epsilon_{\alpha\beta}(\mathbf{k}+\lambda_{\gamma\alpha}\mathbf{q}'_\beta)$, $\mathbf{k}'_\alpha=\epsilon_{\beta\alpha}(\mathbf{k}+\lambda_{\gamma\beta}\mathbf{q}_\alpha)$, $\mathbf{k}''_\gamma=\epsilon_{\beta\alpha}(\mathbf{q}'_\beta+\lambda_{\beta\alpha}\mathbf{k})$, $\mathbf{k}'_\gamma=\epsilon_{\alpha\beta}(\mathbf{q}_\alpha+\lambda_{\alpha\beta}\mathbf{k})$, and $\hat{z}_\gamma=z-k^2/2M_\gamma$, have been introduced. For the Coulomb T-matrix, restricted to the two-body space, we use the integral representation which follows from Eq. (3′) of Schwinger [19] by partial integration:

$$\hat{T}^C_\gamma(\mathbf{k}''_\gamma,\mathbf{k}'_\gamma;\hat{z}_\gamma)=\frac{2\pi e_\alpha e_\beta\mu_\gamma}{\hat{z}_\gamma}\left(\hat{z}_\gamma-\frac{k''^2_\gamma}{2\mu_\gamma}\right)\left(\hat{z}_\gamma-\frac{k'^2_\gamma}{2\mu_\gamma}\right)\int_0^1 dt\frac{(t^2-1)t^{i\hat{\eta}_\gamma}}{\left[(\mathbf{k}''_\gamma-\mathbf{k}'_\gamma)^2t-\frac{\mu_\gamma}{2\hat{z}_\gamma}\left(\hat{z}_\gamma-\frac{k''^2_\gamma}{2\mu_\gamma}\right)\left(\hat{z}_\gamma-\frac{k'^2_\gamma}{2\mu_\gamma}\right)(1-t)^2\right]^2}, \tag{68}$$

where

$$\hat{\eta}_\gamma=e_\alpha e_\beta\mu_\gamma/\sqrt{2\mu_\gamma\hat{z}_\gamma} \tag{69}$$

is the Coulomb parameter, and $\hat{z}_\gamma=\hat{E}_\gamma+i0$ with $\hat{E}_\gamma$ being the relative kinetic energy, of particles α and β. In Eq. (68) it is assumed that $\hat{E}_\gamma\neq 0$. We mention that a representation that is valid in the vicinity of the zero energy region has been developed, e.g., in Ref. [20]. We first consider the case (39). Introducing the representation (68) in Eq. (67), and bearing in mind Eqs. (62a), (64a), and the identities $\hat{z}_\gamma-k''^2_\gamma/2\mu_\gamma=\hat{z}_\beta-k''^2_\beta/2\mu_\beta$ and $\hat{z}_\gamma-k'^2_\gamma/2\mu_\gamma=\hat{z}_\alpha-k'^2_\alpha/2\mu_\alpha$, one obtains

$$\tilde{\mathcal{V}}^{(b)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)=2\pi\mu_\gamma e_\alpha e_\beta\int\frac{d\mathbf{k}}{(2\pi)^3}\frac{1}{\hat{z}_\gamma}\frac{\tilde{\phi}^*_\beta(\mathbf{k}''_\beta;\hat{z}^*_\beta)\tilde{\phi}_\alpha(\mathbf{k}'_\alpha;\hat{z}_\alpha)}{[k''^2_\beta-2\mu_\beta\hat{z}_\beta]^{-i\hat{\eta}_\beta}[k'^2_\alpha-2\mu_\alpha\hat{z}_\alpha]^{-i\hat{\eta}_\alpha}}\times\int_0^1 dt\frac{(t^2-1)t^{i\hat{\eta}_\gamma}}{\left[(\mathbf{k}''_\gamma-\mathbf{k}'_\gamma)^2t-\frac{\mu_\gamma}{2\hat{z}_\gamma}\left(\hat{z}_\beta-\frac{k''^2_\beta}{2\mu_\beta}\right)\left(\hat{z}_\alpha-\frac{k'^2_\alpha}{2\mu_\alpha}\right)(1-t)^2\right]^2}. \tag{70}$$

Change of the integration variable from $\mathbf{k}$ to $\mathbf{k}'=\mathbf{k}''_\gamma-\mathbf{k}'_\gamma=\epsilon_{\beta\alpha}(\mathbf{q}'_\beta+\mathbf{q}_\alpha+\mathbf{k})$ allows us to write

$$k''^2_\beta-2\mu_\beta\hat{z}_\beta=k'^2-2\mathbf{k}'\cdot\mathbf{k}'_\beta+\sigma_\beta(q'_\beta;z), \tag{71}$$

$$k'^2_\alpha-2\mu_\alpha\hat{z}_\alpha=k'^2+2\mathbf{k}'\cdot\mathbf{k}_\alpha+\sigma_\alpha(q_\alpha;z). \tag{72}$$

Thus, making the scaling transformation

$$\mathbf{k}'=\sigma_\beta(q'_\beta;z)\mathbf{v} \tag{73}$$

and taking into account the identity (34) one finds that

$$\tilde{\mathcal{V}}^{(b)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)\overset{\sigma_\beta(q'_\beta;z)\to 0}{=}\frac{f_b(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)}{\sigma_\beta(q'_\beta;z)^{1-i(\hat{\eta}_\alpha+\hat{\eta}_\beta)}}\quad(q'_\beta\neq\tilde{q}_\beta,\overline{q'_\beta};\quad q_\alpha\neq\tilde{q}_\alpha,\overline{q_\alpha}), \tag{74}$$

where $f_b(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$ is a regular function at $\sigma_\beta(q'_\beta;z)=0$.

The behavior at the exceptional points $q'_\beta=\tilde{q}_\beta$ and/or $q_\alpha=\tilde{q}_\alpha$ can be studied by similar means. Quite generally it is obvious that by inserting the form factor behavior displayed in Eqs. (62b) and/or (64b) in $\tilde{\mathcal{V}}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$, Eq. (67), the number of singular denominators under the integral sign is reduced. Consequently, the behavior of the integral must be expected to become less singular. Indeed, an analogous reasoning shows that for $q'_\beta=\tilde{q}_\beta$ but $q_\alpha\neq\tilde{q}_\alpha,\overline{q_\alpha}$

$$\tilde{\mathcal{V}}^{(b)}_{\beta\alpha}(\tilde{\mathbf{q}}'_\beta,\mathbf{q}_\alpha;z)\overset{k'_\beta\to 0}{\sim}k'^{2i\hat{\eta}_\alpha}_\beta f'_b(\tilde{\mathbf{q}}'_\beta,\mathbf{q}_\alpha;z), \tag{75}$$

for $q_\alpha=\tilde{q}_\alpha$ but $q'_\beta\neq\tilde{q}_\beta,\overline{q'_\beta}$

$$\tilde{\mathcal{V}}^{(b)}_{\beta\alpha}(\mathbf{q}'_\beta,\tilde{\mathbf{q}}_\alpha;z)\overset{k_\alpha\to 0}{\sim}k^{2i\hat{\eta}_\beta}_\alpha f''_b(\mathbf{q}'_\beta,\tilde{\mathbf{q}}_\alpha;z), \tag{76}$$

and finally for $q'_\beta=\tilde{q}_\beta$ and $q_\alpha=\tilde{q}_\alpha$

$$\tilde{\mathcal{V}}^{(b)}_{\beta\alpha}(\tilde{\mathbf{q}}'_\beta,\tilde{\mathbf{q}}_\alpha;z)\overset{k'_\beta\to 0}{\sim}k'^{0}_\beta f'''_b(\mathbf{0},\mathbf{0};z) \tag{77}$$

with f'_b,f''_b,f'''_b being nonsingular at $\sigma_\beta(\tilde{q}_\beta;z)=0$ and/or $\sigma_\alpha(\tilde{q}_\alpha;z)=0$.

A look at the above derivation makes clear that, if the final and/or the initial momentum equals its on-shell value $\overline{\mathbf{q}'_\beta}$ and/or $\overline{\mathbf{q}_\alpha}$, respectively, the leading singular behavior of the effective potential in the limit $\overline{\sigma_\beta}\to 0$ follows from the appropriate expression by the substitutions $i\hat{\eta}_\beta\to\eta^{(bs)}_\beta$ and/or $i\hat{\eta}_\alpha\to\eta^{(bs)}_\alpha$. This concludes the proof of Theorem 1.

From this result the following conclusions can be drawn.

(i) $\mathcal{V}^{(b)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$, which is the Born part of $\tilde{\mathcal{V}}^{(b)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$, has likewise the structure (74)–(76), and therefore shows the same branch point singularity as $\mathcal{V}^{(a)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$; cf. Eqs. (65a)–(65c).

(ii) By adding $\mathcal{V}^{(a)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$ to $\tilde{\mathcal{V}}^{(b)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$ it follows that also the Coulomb channel resolvent G^C_γ, with $\gamma\neq\alpha,\beta$, when sandwiched between states $|\phi_\alpha(z),\mathbf{q}_\alpha\rangle$ and $|\phi_\beta(z),\mathbf{q}_\beta\rangle$, behaves in the vicinity of Eq. (38) effectively like G_0.

Corollary. For the case that only two of the three particles are charged and one is neutral, the results of Secs. IV B and IV C already provide the complete proof of the main Theorem, namely that the leading singularity of the exchange part ($\beta\neq\alpha$) of the full effective potential (18) at the point (38) is of the same type as that of the elementary particle exchange contribution $\mathcal{V}^{(a)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$. This is most easily seen by rewriting Eq. (18) for $\beta\neq\alpha$ more explicitly as

$$\mathcal{V}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)=\delta_{\alpha 3}\langle\mathbf{q}'_\beta,\chi_\beta|G_0(z)|\phi_\alpha(\hat{z}_\alpha),\mathbf{q}_\alpha\rangle+\delta_{\beta 3}\langle\mathbf{q}'_\beta,\phi_\beta(\hat{z}^*_\beta)|G_0(z)|\chi_\alpha,\mathbf{q}_\alpha\rangle+\bar{\delta}_{\alpha 3}\bar{\delta}_{\beta 3}\langle\mathbf{q}'_\beta,\chi_\beta|G^C_3|\chi_\alpha,\mathbf{q}_\alpha\rangle. \tag{78}$$

For the first two terms the result (65a)–(65c) can be taken over directly, keeping in mind that $\hat{\eta}_\beta=0$ in the first, and $\hat{\eta}_\alpha=0$ in the second term. Finally, as a special case of the above comment (ii), the third term in Eq. (78) is easily seen to behave, in the limit $\sigma_\beta(q'_\beta;z)\to 0$, like $\langle\mathbf{q}'_\beta,\chi_\beta|G_0|\chi_\alpha,\mathbf{q}_\alpha\rangle$, cf. Eq. (26).

D. Leading singularity of $\tilde{\mathcal{V}}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$

1. Statement of Theorem 2

Consider now the singular behavior of $\tilde{\mathcal{V}}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z) = \langle \mathbf{q}'_\beta,\phi_\beta|\mathcal{O}|\phi_\alpha,\mathbf{q}_\alpha\rangle$, with $\mathcal{O}:=G_0\overline{V}^C_\beta G^C\overline{V}^C_\alpha G_0$ [cf. Eq. (59)] which contains all possible Coulomb interactions: it begins with the incoming-channel interaction $\overline{V}^C_\alpha$, ends up with the outgoing-channel interaction $\overline{V}^C_\beta$, while G^C takes into account all possible Coulomb interactions between particles α, β, and γ, in the intermediate state.

It is to be suspected that straightforward generalization of Theorem 1 to terms describing multiple rescattering of arbitrary order in the intermediate state as they would follow from a Neumann series expansion of G^C, is not possible. For, the latter leads to products with an *infinite* number of operators T^C_ν (this is most easily seen by using $G^C = \delta_{\beta\alpha}G^C_\alpha + G^C_\beta U^C_{\beta\alpha}G^C_\alpha$ [cf. Eqs. (11a) and (11b)] and introducing there the expansion (16b)). This fact could cause problems with the convergence of the series near the singular point. However, it will be shown that near the leading singularity of the nondiagonal effective potential contribution $\tilde{\mathcal{V}}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$, even the operator $\mathcal{O}$ which contains the three-body Coulomb resolvent G^C may effectively be replaced by G_0.

Theorem 2. Even an infinite number of Coulomb rescatterings in the intermediate state of the nondiagonal effective potential $\mathcal{V}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$, as represented by the three-body Coulomb resolvent G^C, does change neither position nor character (provided $q'_\beta\neq\tilde{q}_\beta$ and $q_\alpha\neq\tilde{q}_\alpha$) of the leading singularity in the momentum transfer variable as compared to its lowest-order contribution $\mathcal{V}^{(a)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$, but alters its strength. However, at the special points $q'_\beta=\tilde{q}_\beta$ and/or $q_\alpha=\tilde{q}_\alpha$, the character of the singularity is different.

2. Preliminaries

The proof of the theorem will be based on the representation (55) of the nondiagonal effective potential $\mathcal{V}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$. Since it has already been shown in the preceding sections that both $\mathcal{V}^{(a)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$ and $\mathcal{V}^{(b)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$ have the same leading singularity (for $q_\alpha\neq\tilde{q}_\alpha, q'_\beta\neq\tilde{q}_\beta$), it remains to prove that the same holds true also for the third term $\tilde{\mathcal{V}}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$.

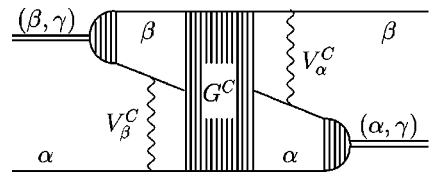

FIG. 3. Contribution (80) to the exact nondiagonal effective potential.

According to its definition (59), $\tilde{\mathcal{V}}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$ can be written as a sum of four terms

$$\tilde{\mathcal{V}}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z) = \sum_{\nu\neq\beta,\sigma\neq\alpha} \langle \mathbf{q}'_\beta,\phi_\beta|G_0V^C_\nu G^C \times V^C_\sigma G_0|\phi_\alpha,\mathbf{q}_\alpha\rangle, \quad \beta\neq\alpha. \tag{79}$$

Consider, for example, the term with $\sigma=\beta$ and $\nu=\alpha$,

$$\tilde{\mathcal{V}}^{(\alpha\beta)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z) := \langle \mathbf{q}'_\beta,\phi_\beta|G_0V^C_\alpha G^C V^C_\beta G_0|\phi_\alpha,\mathbf{q}_\alpha\rangle, \tag{80}$$

which is represented in diagrammatical form in Fig. 3. The leading singularity of $\tilde{\mathcal{V}}^{(\alpha\beta)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$ is generated by the coincidence of the poles of two G_0's, of the Coulomb potentials V^C_β and V^C_β, of the three-body Coulomb resolvent G^C, and of the branch point singularities which are, in general, present in the Coulomb-modified form factors ϕ_α and ϕ_β.

We first treat the case (39), i.e., $q'_\beta\neq\tilde{q}_\beta$ and $q_\alpha\neq\tilde{q}_\alpha$. In the momentum representation we have explicitly

$$\tilde{\mathcal{V}}^{(\alpha\beta)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z) = \int \frac{d\mathbf{k}''_\alpha}{(2\pi)^3}\int\frac{d\mathbf{q}''_\alpha}{(2\pi)^3}\int\frac{d\mathbf{k}''_\beta}{(2\pi)^3}\int\frac{d\mathbf{q}''_\beta}{(2\pi)^3}\frac{\tilde{\phi}^*_\beta[\boldsymbol{\epsilon}_{\beta\alpha}(\mathbf{q}''_\alpha+\lambda_{\alpha\gamma}\mathbf{q}'_\beta);\hat{z}^*_\beta]}{[(\mathbf{q}''_\alpha+\lambda_{\alpha\gamma}\mathbf{q}'_\beta)^2-2\mu_\beta\hat{z}_\beta]^{1-i\hat{\eta}_\beta}}V^C_\alpha[\mathbf{k}''_\alpha-\boldsymbol{\epsilon}_{\alpha\beta} \times(\mathbf{q}'_\beta+\lambda_{\beta\gamma}\mathbf{q}''_\alpha)]G^C(\mathbf{k}''_\alpha,\mathbf{q}''_\alpha;\mathbf{k}''_\beta,\mathbf{q}''_\beta;z)V^C_\beta[\mathbf{k}''_\beta-\boldsymbol{\epsilon}_{\beta\alpha}(\mathbf{q}_\alpha+\lambda_{\alpha\gamma}\mathbf{q}''_\beta)]\frac{\tilde{\phi}_\alpha[\boldsymbol{\epsilon}_{\alpha\beta}(\mathbf{q}''_\beta+\lambda_{\beta\gamma}\mathbf{q}_\alpha);\hat{z}_\alpha]}{[(\mathbf{q}''_\beta+\lambda_{\beta\gamma}\mathbf{q}_\alpha)^2-2\mu_\alpha\hat{z}_\alpha]^{1-i\hat{\eta}_\alpha}}. \tag{81}$$

In what follows we drop the argument $\hat{z}_\nu$ in the reduced Coulomb-modified form factors $\tilde{\phi}_\nu$ unless required for clarity. Since we assume that the charges of all three particles are of equal sign, i.e., that all Coulomb potentials are repulsive, the three-body Coulomb resolvent has the following spectral representation in coordinate space:

$$\langle \mathbf{r}'_\alpha,\boldsymbol{\rho}'_\alpha|G^C(z)|\mathbf{r}_\alpha,\boldsymbol{\rho}_\alpha\rangle = \int\frac{d\mathbf{k}^0_\alpha}{(2\pi)^3}\int\frac{d\mathbf{q}^0_\alpha}{(2\pi)^3}\frac{\Psi^{C(+)}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{r}'_\alpha,\boldsymbol{\rho}'_\alpha)\Psi^{C(+)*}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{r}_\alpha,\boldsymbol{\rho}_\alpha)}{\left[z-\dfrac{k^{0^2}_\alpha}{2\mu_\alpha}-\dfrac{q^{0^2}_\alpha}{2M_\alpha}\right]}, \tag{82}$$

and in momentum space

$$\langle \mathbf{k}'_\alpha,\mathbf{q}'_\alpha|G^C(z)|\mathbf{k}_\alpha,\mathbf{q}_\alpha\rangle=\int\frac{d\mathbf{k}^0_\alpha}{(2\pi)^3}\int\frac{d\mathbf{q}^0_\alpha}{(2\pi)^3}\frac{\Psi^{C(+)}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{k}'_\alpha,\mathbf{q}'_\alpha)\Psi^{C(+)*}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{k}_\alpha,\mathbf{q}_\alpha)}{\left[z-\frac{k^{0^2}_\alpha}{2\mu_\alpha}-\frac{q^{0^2}_\alpha}{2M_\alpha}\right]}. \tag{83}$$

Here, $\Psi^{C(+)}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{r}_\alpha,\boldsymbol{\rho}_\alpha)$ $[\Psi^{C(+)}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{k}_\alpha,\mathbf{q}_\alpha)]$ is the coordinate- [momentum-]space representation of the three-body Coulomb scattering wave function for three particles in continuum, with on-shell momenta $\mathbf{k}^0_\alpha$ and $\mathbf{q}^0_\alpha$, and total energy $k^{0^2}_\alpha/2\mu_\alpha+q^{0^2}_\alpha/2M_\alpha$ where the first part represents the relative kinetic energy of particles β and γ, and the second one the relative kinetic energy of particle α and the center of mass of the pair $(\beta\gamma)$.

3. Leading singularity of $\tilde{\mathcal{V}}^{(2)(\alpha\beta)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$

Before investigating $\tilde{\mathcal{V}}^{(\alpha\beta)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$, it proves helpful to first consider the simpler expression which results by substituting in Eq. (81) the free Green's function for the three-body Coulomb Green's function. In this way one obtains just one of the second-order terms of the effective potential, to be denoted by $\tilde{\mathcal{V}}^{(2)(\alpha\beta)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$. The momentum space representation of the spectral resolution of the free Green's function is

$$\begin{aligned}\langle \mathbf{k}''_\alpha,\mathbf{q}''_\alpha|G_0(z)|\mathbf{k}''_\beta,\mathbf{q}''_\beta\rangle&=\int\frac{d\mathbf{k}^0_\alpha}{(2\pi)^3}\int\frac{d\mathbf{q}^0_\alpha}{(2\pi)^3}\frac{\delta(\mathbf{k}''_\alpha-\mathbf{k}^0_\alpha)\,\delta(\mathbf{q}''_\alpha-\mathbf{q}^0_\alpha)}{\left[z-\frac{k^{0^2}_\alpha}{2\mu_\alpha}-\frac{q^{0^2}_\alpha}{2M_\alpha}\right]}\delta[\mathbf{k}''_\beta-(\epsilon_{\beta\alpha}\mu_\alpha\mathbf{q}^0_\alpha/M_\beta-\lambda_{\alpha\gamma}\mathbf{k}^0_\alpha)]\,\delta[\mathbf{q}''_\beta+(\epsilon_{\beta\alpha}\mathbf{k}^0_\alpha+\lambda_{\beta\gamma}\mathbf{q}^0_\alpha)]\\&=\int\frac{d\mathbf{q}^0_\beta}{(2\pi)^3}\int\frac{d\mathbf{q}^0_\alpha}{(2\pi)^3}\frac{\delta[\mathbf{k}''_\alpha-\epsilon_{\alpha\beta}(\mathbf{q}^0_\beta+\lambda_{\beta\gamma}\mathbf{q}^0_\alpha)]\,\delta(\mathbf{q}''_\alpha-\mathbf{q}^0_\alpha)}{\left[z-\frac{(\mathbf{q}^0_\beta+\lambda_{\beta\gamma}\mathbf{q}^0_\alpha)^2}{2\mu_\alpha}-\frac{q^{0^2}_\alpha}{2M_\alpha}\right]}\delta[\mathbf{k}''_\beta-\epsilon_{\beta\alpha}(\mathbf{q}^0_\alpha+\lambda_{\alpha\gamma}\mathbf{q}^0_\beta)]\,\delta(\mathbf{q}''_\beta-\mathbf{q}^0_\beta),\end{aligned} \tag{84}$$

where to arrive at the second equality use has been made of the relation $\mathbf{k}^0_\alpha=\epsilon_{\alpha\beta}(\mathbf{q}^0_\beta+\lambda_{\beta\gamma}\mathbf{q}^0_\alpha)$ [cf. Eq. (A2)] to induce a change of the integration variables. Inserting this expression into Eq. (81) with G^C replaced by G_0 yields

$$\begin{aligned}\tilde{\mathcal{V}}^{(2)(\alpha\beta)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)&=\int\frac{d\mathbf{q}^0_\beta}{(2\pi)^3}\int\frac{d\mathbf{q}^0_\alpha}{(2\pi)^3}\frac{\tilde{\phi}^*_\beta[\epsilon_{\beta\alpha}(\mathbf{q}^0_\alpha+\lambda_{\alpha\gamma}\mathbf{q}'_\beta)]}{[(\mathbf{q}^0_\alpha+\lambda_{\alpha\gamma}\mathbf{q}'_\beta)^2-2\mu_\beta\hat{z}_\beta]^{1-i\hat{\eta}_\beta}}\frac{4\pi e_\beta e_\gamma}{(\mathbf{q}^0_\beta-\mathbf{q}'_\beta)^2}\frac{1}{\left[z-\frac{(\mathbf{q}^0_\beta+\lambda_{\beta\gamma}\mathbf{q}^0_\alpha)^2}{2\mu_\alpha}-\frac{q^{0^2}_\alpha}{2M_\alpha}\right]}\\&\quad\times\frac{4\pi e_\alpha e_\gamma}{(\mathbf{q}^0_\alpha-\mathbf{q}_\alpha)^2}\frac{\tilde{\phi}_\alpha[\epsilon_{\alpha\beta}(\mathbf{q}^0_\beta+\lambda_{\beta\gamma}\mathbf{q}_\alpha)]}{[(\mathbf{q}^0_\beta+\lambda_{\beta\gamma}\mathbf{q}_\alpha)^2-2\mu_\alpha\hat{z}_\alpha]^{1-i\hat{\eta}_\alpha}}.\end{aligned} \tag{85}$$

Here, the explicit expressions for the Fourier transforms of the Coulomb potentials V^C_β and V^C_α have been introduced.

The leading singularity of $\tilde{\mathcal{V}}^{(2)(\alpha\beta)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$ is generated by the coincidence of the singularities of the integrand at

$$(\mathbf{q}^0_\beta+\lambda_{\beta\gamma}\mathbf{q}_\alpha)^2-2\mu_\alpha\hat{z}_\alpha=0, \tag{86}$$

$$(\mathbf{q}^0_\alpha+\lambda_{\alpha\gamma}\mathbf{q}'_\beta)^2-2\mu_\beta\hat{z}_\beta=0, \tag{87}$$

$$z-(\mathbf{q}^0_\beta+\lambda_{\beta\gamma}\mathbf{q}^0_\alpha)^2/2\mu_\alpha-q^{0\,2}_\alpha/2M_\alpha=0, \tag{88}$$

$$\boldsymbol{\Delta}^0_\alpha:=\mathbf{q}^0_\alpha-\mathbf{q}_\alpha=0, \tag{89}$$

$$\boldsymbol{\Delta}^0_\beta:=\mathbf{q}^0_\beta-\mathbf{q}'_\beta=0. \tag{90}$$

It is evident that the coincidence of these zeros of the denominators in Eq. (85) can produce a dangerous singularity of the integral. Changing the integration variables to $\boldsymbol{\Delta}^0_\alpha$ and $\boldsymbol{\Delta}^0_\beta$, expression (85) takes the form

$$\tilde{\mathcal{V}}^{(2)(\alpha\beta)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)=-2\mu_\alpha\int\frac{d\mathbf{\Delta}^0_\beta}{(2\pi)^3}\int\frac{d\mathbf{\Delta}^0_\alpha}{(2\pi)^3}\frac{\tilde{\phi}^*_\beta(\epsilon_{\beta\alpha}\mathbf{\Delta}^0_\alpha+\mathbf{k}'_\beta)}{[(\Delta^0_\alpha)^2+2\epsilon_{\beta\alpha}\mathbf{\Delta}^0_\alpha\cdot\mathbf{k}'_\beta+\sigma_\beta(q'_\beta;z)]^{1-i\hat{\eta}_\beta}}\frac{4\pi e_\beta e_\gamma}{(\Delta^0_\beta)^2}$$
$$\times\frac{1}{\left[\sigma_\alpha(q_\alpha;z)+\frac{2\mu_\alpha}{M_\alpha}\mathbf{\Delta}^0_\alpha\cdot\mathbf{q}_\alpha+\frac{\mu_\alpha}{M_\alpha}(\Delta^0_\alpha)^2+2\epsilon_{\alpha\beta}(\mathbf{\Delta}^0_\beta+\lambda_{\beta\gamma}\mathbf{\Delta}^0_\alpha)\cdot\mathbf{k}_\alpha+(\mathbf{\Delta}^0_\beta+\lambda_{\beta\gamma}\mathbf{\Delta}^0_\alpha)^2\right]}\frac{4\pi e_\alpha e_\gamma}{(\Delta^0_\alpha)^2}$$
$$\times\frac{\tilde{\phi}_\alpha(\epsilon_{\alpha\beta}\mathbf{\Delta}^0_\beta+\mathbf{k}_\alpha)}{[(\Delta^0_\beta)^2+2\epsilon_{\alpha\beta}\mathbf{\Delta}^0_\beta\cdot\mathbf{k}_\alpha+\sigma_\alpha(q_\alpha;z)]^{1-i\hat{\eta}_\alpha}}. \tag{91}$$

Here, $\mathbf{k}_\alpha$ and $\mathbf{k}'_\beta$ are defined in Eq. (27). Recall that the reduced form factors $\tilde{\phi}_\alpha$ and $\tilde{\phi}_\beta$ are nonsingular at Eqs. (86) and (87), respectively. Finally, making the substitutions

$$\mathbf{\Delta}^0_\alpha=\sigma_\beta(q'_\beta;z)\mathbf{v}_\alpha,\quad \mathbf{\Delta}^0_\beta=\sigma_\alpha(q_\alpha;z)\mathbf{v}_\beta, \tag{92}$$

where $\mathbf{v}_\nu,\nu=\alpha,\beta$, has the dimension of an inverse momentum, and recalling the identity (34), the desired result follows:

$$\tilde{\mathcal{V}}^{(2)(\alpha\beta)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)=\frac{J[\sigma_\beta(q'_\beta;z);\mathbf{q}'_\beta,\mathbf{q}_\alpha;z]}{[\sigma_\beta(q'_\beta;z)]^{1-i(\hat{\eta}_\alpha+\hat{\eta}_\beta)}}. \tag{93}$$

The integral in Eq. (91) which results after extraction of $\sigma_\beta(q'_\beta;z)$ has been denoted by $J[\sigma_\beta(q'_\beta;z);\mathbf{q}'_\beta,\mathbf{q}_\alpha;z]$. Since it remains finite at the point $\sigma_\beta(q'_\beta;z)=0$ we immediately obtain in the limit $\sigma_\beta(q'_\beta;z)$ going to zero:

$$\tilde{\mathcal{V}}^{(2)(\alpha\beta)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)\overset{\sigma_\beta(q'_\beta;z)\to0}{=}\frac{J(0;\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)}{[\sigma_\beta(q'_\beta;z)]^{1-i(\hat{\eta}_\alpha+\hat{\eta}_\beta)}}+o\left(\frac{1}{\sigma_\beta(q'_\beta;z)}\right), \tag{94}$$

where

$$J(0;\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)=-2\mu_\beta\left(\frac{\mu_\alpha}{\mu_\beta}\right)^{i\hat{\eta}_\alpha}\lim_{\sigma_\beta(q'_\beta;z)\to0}\int\frac{d\mathbf{v}_\beta}{(2\pi)^3}\int\frac{d\mathbf{v}_\alpha}{(2\pi)^3}\frac{\tilde{\phi}^*_\beta[\epsilon_{\beta\alpha}\sigma_\beta(q'_\beta;z)\mathbf{v}_\alpha+\mathbf{k}'_\beta]}{[1+2\epsilon_{\beta\alpha}\mathbf{v}_\alpha\cdot\mathbf{k}'_\beta+\sigma_\beta(q'_\beta;z)v^2_\alpha]^{1-i\hat{\eta}_\beta}}$$
$$\times\frac{4\pi e_\beta e_\gamma}{v^2_\beta}\frac{1}{D_0(\mathbf{q}'_\beta,\mathbf{q}_\alpha)}\frac{4\pi e_\alpha e_\gamma}{v^2_\alpha}\frac{\tilde{\phi}_\alpha[\epsilon_{\alpha\beta}\sigma_\alpha(q_\alpha;z)\mathbf{v}_\beta+\mathbf{k}_\alpha]}{[1+2\epsilon_{\alpha\beta}\mathbf{v}_\beta\cdot\mathbf{k}_\alpha+\sigma_\alpha(q_\alpha;z)v^2_\beta]^{1-i\hat{\eta}_\alpha}}. \tag{95}$$

Here, the abbreviation

$$D_0(\mathbf{q}'_\beta,\mathbf{q}_\alpha):=1+\frac{2\mu_\beta}{M_\alpha}\mathbf{v}_\alpha\cdot\mathbf{q}_\alpha+\frac{\mu_\alpha}{M_\alpha}\sigma_\beta(q'_\beta;z)v^2_\alpha+2\epsilon_{\alpha\beta}(\mathbf{v}_\beta+\lambda_{\alpha\gamma}\mathbf{v}_\alpha)\cdot\mathbf{k}_\alpha+\sigma_\alpha(q_\alpha;z)(\mathbf{v}_\beta+\lambda_{\alpha\gamma}\mathbf{v}_\alpha)^2 \tag{96}$$

has been introduced. [We point out that the order relation has the usual meaning: $f(x)=o(g(x))$ for $x\to x_0$, if $\lim_{x\to x_0}f(x)/g(x)=0$ $(g(x_0)\neq0)$.]

The same reasoning shows that for $q'_\beta=\tilde{q}_\beta$ but $q_\alpha\neq\tilde{q}_\alpha$, in the limit $\sigma_\beta(\tilde{q}_\beta;z)=k'^2_\beta=\mu_\beta\sigma_\alpha(q_\alpha;z)/\mu_\alpha\to0$, the leading singular behavior is given by

$$\tilde{\mathcal{V}}^{(2)(\alpha\beta)}_{\beta\alpha}(\tilde{\mathbf{q}}'_\beta,\mathbf{q}_\alpha;z)\overset{k'_\beta\to0}{=}\frac{1}{k'^{1-2i\hat{\eta}_\alpha}_\beta}J'(0;\tilde{\mathbf{q}}'_\beta,\mathbf{q}_\alpha;z), \tag{97}$$

where

$$J'(0;\tilde{\mathbf{q}}'_\beta,\mathbf{q}_\alpha;z)=2\mu_\beta\lim_{k'_\beta\to0}\int\frac{d\mathbf{v}_\beta}{(2\pi)^3}\int\frac{d\mathbf{v}_\alpha}{(2\pi)^3}\frac{4\pi e_\beta e_\gamma}{v^2_\beta}\frac{4\pi e_\alpha e_\gamma}{v^2_\alpha}\frac{\tilde{\phi}^*_\beta(\epsilon_{\beta\alpha}k'_\beta\mathbf{v}_\alpha+\mathbf{k}'_\beta)\tilde{\phi}_\alpha(\epsilon_{\alpha\beta}k'^2_\beta\mathbf{v}_\beta+\mathbf{k}_\alpha)}{D_1(\tilde{\mathbf{q}}'_\beta,\mathbf{q}_\alpha)[\mu_\alpha/\mu_\beta+2\epsilon_{\alpha\beta}\mathbf{v}_\beta\cdot\mathbf{k}_\alpha+k'^2_\beta v^2_\beta]^{1-i\hat{\eta}_\alpha}}, \tag{98}$$

with

$$D_1(\tilde{\mathbf{q}}'_\beta,\mathbf{q}_\alpha):=1+\frac{2\mu_\beta}{M_\beta}\mathbf{v}_\beta\cdot\tilde{\mathbf{q}}'_\beta+\frac{\mu_\beta}{M_\beta}k'^2_\beta v_\beta^2+2\epsilon_{\beta\alpha}(\mathbf{v}_\alpha+\lambda_{\alpha\gamma}k'_\beta\mathbf{v}_\beta)\cdot\hat{\mathbf{k}}'_\beta+(\mathbf{v}_\alpha+\lambda_{\alpha\gamma}k'_\beta\mathbf{v}_\beta)^2. \tag{99}$$

We mention that the derivation of this result requires the substitutions

$$\mathbf{\Delta}^0_\beta=\sigma_\beta(\tilde{q}_\beta;z)\mathbf{v}_\beta=k'^2_\beta\mathbf{v}_\beta \text{ and } \mathbf{\Delta}^0_\alpha=k'_\beta\mathbf{v}_\alpha. \tag{100}$$

Analogously, for $q_\alpha=\tilde{q}_\alpha$ but $q'_\beta\neq\tilde{q}_\beta$, in the limit $\sigma_\alpha(\tilde{q}_\alpha;z)=k^2_\alpha\to 0$, one finds

$$\tilde{\mathcal{V}}^{(2)(\alpha\beta)}_{\beta\alpha}(\mathbf{q}'_\beta,\tilde{\mathbf{q}}_\alpha;z)\overset{k_\alpha\to 0}{=}\frac{1}{k_\alpha^{1-2i\hat{\eta}_\beta}}J''(0;\mathbf{q}'_\beta,\tilde{\mathbf{q}}_\alpha;z), \tag{101}$$

where $J''(0;\mathbf{q}'_\beta,\tilde{\mathbf{q}}_\alpha;z)$ is given by an expression similar to Eq. (98). Finally, for $q'_\beta=\tilde{q}_\beta$ and $q_\alpha=\tilde{q}_\alpha$, using the substitutions $\mathbf{\Delta}^0_\beta=k'_\beta\mathbf{v}_\beta$ and $\mathbf{\Delta}^0_\alpha=k'_\beta\mathbf{v}_\alpha$ one obtains in the limit $\sigma_\beta(\tilde{q}_\beta;z)=k'^2_\beta=\mu_\beta\sigma_\alpha(\tilde{q}_\alpha;z)/\mu_\alpha=\mu_\beta k^2_\alpha/\mu_\alpha\to 0$

$$\tilde{\mathcal{V}}^{(2)(\alpha\beta)}_{\beta\alpha}(\tilde{\mathbf{q}}'_\beta,\tilde{\mathbf{q}}_\alpha;z)\overset{k'_\beta\to 0}{=}k'^0_\beta J'''(0;\mathbf{0},\mathbf{0};0) \text{ for } E=0. \tag{102}$$

Here,

$$J'''(0;\mathbf{0},\mathbf{0};0)=-2\mu_\beta\lim_{k'_\beta\to 0}\int\frac{d\mathbf{v}_\beta}{(2\pi)^3}\int\frac{d\mathbf{v}_\alpha}{(2\pi)^3}\frac{4\pi e_\beta e_\gamma}{v_\beta^2}\frac{4\pi e_\alpha e_\gamma}{v_\alpha^2}\frac{\tilde{\phi}^*_\beta(\epsilon_{\beta\alpha}k'_\beta\mathbf{v}_\alpha+\mathbf{k}'_\beta)\tilde{\phi}_\alpha(\epsilon_{\alpha\beta}k'_\beta\mathbf{v}_\beta+\mathbf{k}_\alpha)}{D_2(\mathbf{0},\mathbf{0})}, \tag{103}$$

with

$$D_2(\mathbf{0},\mathbf{0})=\mu_\beta v_\beta^2/M_\beta+(\mathbf{v}_\alpha+\lambda_{\alpha\gamma}\mathbf{v}_\beta+\epsilon_{\beta\alpha}\hat{\mathbf{k}}'_\beta)^2. \tag{104}$$

Note that in J', Eq. (98), $\mathbf{k}_\alpha$ and $\mathbf{k}'_\beta$ are considered expressed as linear combinations of $(\tilde{\mathbf{q}}'_\beta,\mathbf{q}_\alpha)$, in J'' as linear combinations of $(\mathbf{q}'_\beta,\tilde{\mathbf{q}}_\alpha)$.

It is apparent that the leading singularity, if the final and/or the initial momentum equals its on-shell value $\overline{\mathbf{q}'_\beta}$ and/or $\overline{\mathbf{q}}_\alpha$, respectively, in the limit $\overline{\sigma}_\beta\to 0$ can be obtained from the above off-shell results substituting $\eta^{(bs)}_\beta$ for $i\hat{\eta}_\beta$ and/or $\eta^{(bs)}_\alpha$ for $i\hat{\eta}_\alpha$.

This verifies the assertion that $\tilde{\mathcal{V}}^{(2)(\alpha\beta)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$ has the same leading singularity as $\mathcal{V}^{(a)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$ [cf. Eq. (65)], with the exception of the special points $q_\alpha=\tilde{q}_\alpha$ and/or $q'_\beta=\tilde{q}_\beta$.

4. Leading singularity of $\tilde{\mathcal{V}}^{(\alpha\beta)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$

Proof of Theorem 2. We are now ready to prove Theorem 2 by showing that the typical contribution $\tilde{\mathcal{V}}^{(\alpha\beta)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$ to the full effective potential, in spite of containing an infinite number of Coulombic rescatterings between all three particles in the intermediate state as represented by the three-body Coulomb resolvent, possesses the same leading singularity at the same position as the second-order contribution $\tilde{\mathcal{V}}^{(2)(\alpha\beta)}_{\beta\alpha}(\tilde{\mathbf{q}}'_\beta,\mathbf{q}_\alpha;z)$, or equivalently as $\mathcal{V}^{(a)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$, cf. Eq. (65), except for the special points $q'_\beta=\tilde{q}_\beta$ and/or $q_\alpha=\tilde{q}_\alpha$ where the character of the singularity differs. We start again by considering the case $q_\alpha\neq\tilde{q}_\alpha$ and $q'_\beta\neq\tilde{q}_\beta$. To this end we introduce in Eq. (81) the spectral representation (83) of the full three-body Coulomb Green's function:

$$\tilde{\mathcal{V}}_{\beta\alpha}^{(\alpha\beta)}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)=\int\frac{d\mathbf{k}''_\alpha}{(2\pi)^3}\int\frac{d\mathbf{q}''_\alpha}{(2\pi)^3}\int\frac{d\mathbf{k}''_\beta}{(2\pi)^3}\int\frac{d\mathbf{q}''_\beta}{(2\pi)^3}\int\frac{d\mathbf{k}^0_\alpha}{(2\pi)^3}\int\frac{d\mathbf{q}^0_\alpha}{(2\pi)^3}\frac{\tilde{\phi}^*_\beta[\epsilon_{\beta\alpha}(\mathbf{q}''_\alpha+\lambda_{\alpha\gamma}\mathbf{q}'_\beta)]}{[(\mathbf{q}''_\alpha+\lambda_{\alpha\gamma}\mathbf{q}'_\beta)^2-2\mu_\beta\hat{z}_\beta]^{1-i\hat{\eta}_\beta}}$$

$$\times\frac{4\pi e_\beta e_\gamma}{[\mathbf{k}''_\alpha-\epsilon_{\alpha\beta}(\mathbf{q}'_\beta+\lambda_{\beta\gamma}\mathbf{q}''_\alpha)]^2}\frac{\Psi^{C(+)}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{k}''_\alpha,\mathbf{q}''_\alpha)\Psi^{C(+)*}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{k}''_\beta,\mathbf{q}''_\beta)}{\left[z-\frac{q^{0^2}_\alpha}{2M_\alpha}-\frac{k^{0^2}_\alpha}{2\mu_\alpha}\right]}$$

$$\times\frac{4\pi e_\alpha e_\gamma}{[\mathbf{k}''_\beta-\epsilon_{\beta\alpha}(\mathbf{q}_\alpha+\lambda_{\alpha\gamma}\mathbf{q}''_\beta)]^2}\frac{\tilde{\phi}_\alpha[\epsilon_{\alpha\beta}(\mathbf{q}''_\beta+\lambda_{\beta\gamma}\mathbf{q}_\alpha)]}{[(\mathbf{q}''_\beta+\lambda_{\beta\gamma}\mathbf{q}_\alpha)^2-2\mu_\alpha\hat{z}_\alpha]^{1-i\hat{\eta}_\alpha}}.\tag{105}$$

As before, the leading singularity of $\tilde{\mathcal{V}}_{\beta\alpha}^{(\alpha\beta)}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$ emerges as the result of the coincidence of the zeros of all the denominators of the integrand in Eq. (105) and of the forward-scattering singularities of the three-body Coulomb scattering wave functions. To proceed any further requires knowledge of the explicit expression of the latter which, however, is unknown. But it turns out that in the region of integration which is relevant for generating the leading singularity, only the leading term of the asymptotic expansion of the three-body coordinate-space Coulomb scattering wave function enters which is known in analytic form. (As a side remark we mention that this situation is reminiscent of the nonperturbative derivation of the long-range behavior of the optical potential within the context of the three-charged particle theory in [21]. As is well known, the optical potential is likewise given as a certain matrix element of the three-body resolvent. However, when using the spectral representation for the latter, for the investigation of the large-distance behavior again only the asymptotic part of the three-body wave function was needed.)

To see this, let us rewrite expression (105) in the coordinate-space representation, yielding

$$\tilde{\mathcal{V}}_{\beta\alpha}^{(\alpha\beta)}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)=\int d\mathbf{r}'_\alpha\int d\boldsymbol{\rho}'_\alpha\int d\mathbf{r}_\beta\int d\boldsymbol{\rho}_\beta\int\frac{d\mathbf{k}^0_\alpha}{(2\pi)^3}\int\frac{d\mathbf{q}^0_\alpha}{(2\pi)^3}e^{-i\mathbf{q}'_\beta\cdot\boldsymbol{\rho}'_\beta}\psi^*_\beta(\mathbf{r}'_\beta;\hat{z}^*_\beta)V^C_\alpha(\mathbf{r}'_\alpha)\frac{\Psi^{C(+)}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{r}'_\alpha,\boldsymbol{\rho}'_\alpha)\Psi^{C(+)*}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{r}_\beta,\boldsymbol{\rho}_\beta)}{\left[z-\frac{k^{0^2}_\alpha}{2\mu_\alpha}-\frac{q^{0^2}_\alpha}{2M_\alpha}\right]}$$

$$\times V^C_\beta(\mathbf{r}_\beta)\psi_\alpha(\mathbf{r}_\alpha;\hat{z}_\alpha)e^{i\mathbf{q}_\alpha\cdot\boldsymbol{\rho}_\alpha}.\tag{106}$$

In this equation the Jacobian vector pair $\{\mathbf{r}'_\beta,\boldsymbol{\rho}'_\beta\}$ is considered expressed as linear combinations of the integration variables $\{\mathbf{r}'_\alpha,\boldsymbol{\rho}'_\alpha\}$, and similarly $\{\mathbf{r}_\alpha,\boldsymbol{\rho}_\alpha\}$ as linear combinations of $\{\mathbf{r}_\beta,\boldsymbol{\rho}_\beta\}$, according to Eq. (A3). Furthermore, the notation

$$\psi_\alpha(\mathbf{r}_\alpha;\hat{z}_\alpha):=\langle\mathbf{r}_\alpha|\hat{G}_0(\hat{z}_\alpha)|\phi_\alpha(\hat{z}_\alpha)\rangle$$

$$=\int d\mathbf{r}''_\alpha\hat{G}_0(\mathbf{r}_\alpha,\mathbf{r}''_\alpha;\hat{z}_\alpha)\phi_\alpha(\mathbf{r}''_\alpha;\hat{z}_\alpha),\tag{107}$$

has been introduced, with an analogous definition of $\psi_\beta(\mathbf{r}'_\beta;\hat{z}_\beta)$. The quantity $\hat{G}_0(\mathbf{r}_\alpha,\mathbf{r}''_\alpha;\hat{z}_\alpha)$ is the two-particle free Green's function in the coordinate representation.

As has been mentioned before, the singular behavior of $\tilde{\mathcal{V}}_{\beta\alpha}^{(\alpha\beta)}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$ at the point (38) results from the coincidence of the various singularities of the integrand in Eq. (105). Among these are the poles of the Fourier transforms $V^C_\alpha[\mathbf{k}''_\alpha-\epsilon_{\alpha\beta}(\mathbf{q}'_\beta+\lambda_{\beta\gamma}\mathbf{q}''_\alpha)]$ and $V^C_\beta[\mathbf{k}''_\beta-\epsilon_{\beta\alpha}(\mathbf{q}_\alpha+\lambda_{\alpha\gamma}\mathbf{q}''_\beta)]$ of the Coulomb potentials $V^C_\alpha(\mathbf{r}'_\alpha)$ and $V^C_\beta(\mathbf{r}_\beta)$, respectively. The singularity at $\mathbf{k}''_\alpha-\epsilon_{\alpha\beta}(\mathbf{q}'_\beta+\lambda_{\beta\gamma}\mathbf{q}''_\alpha)=0$ in Eq. (105) of the Fourier transform of $V^C_\alpha(\mathbf{r}'_\alpha)$ is generated by the divergence of the integral over $\mathbf{r}'_\alpha$ in Eq. (106), for $r'_\alpha\to\infty$. Similarly, the singularity of the Fourier transform of $V^C_\alpha(\mathbf{r}_\beta)$ at $\mathbf{k}''_\beta-\epsilon_{\beta\alpha}(\mathbf{q}_\alpha+\lambda_{\alpha\gamma}\mathbf{q}''_\beta)=0$ is generated by the divergence in Eq. (106) of the integral over $\mathbf{r}_\beta$ for $r_\beta\to\infty$. Thus, in order to extract the singular behavior of $\tilde{\mathcal{V}}_{\beta\alpha}^{(\alpha\beta)}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$ we must investigate the behavior of the integrand in Eq. (106) in the asymptotic region

$$\omega_{\alpha\beta}=\omega'_\alpha\cap\omega_\beta;\quad\omega'_\alpha:r'_\alpha\to\infty;\quad\omega_\beta:r_\beta\to\infty.\tag{108}$$

When $r_\beta\to\infty$, either one more coordinate r_α or r_γ, or both r_α and r_γ, have to approach infinity together with r_β. That is, we must distinguish the following three cases.

(1) $r_\beta,r_\alpha\to\infty$, i.e., $\rho_\gamma\to\infty$, and $r_\gamma/\rho_\gamma\to0$. (2) $r_\beta,r_\gamma\to\infty$, i.e., $\rho_\alpha\to\infty$, and $r_\alpha/\rho_\alpha\to0$. (3) $r_\beta\to\infty,r_\alpha\to\infty,r_\gamma\to\infty$.

Let us define the following four asymptotic regions:

$$\Omega_\nu:\quad\rho_\nu\to\infty,\;\; r_\nu/\rho_\nu\to0,\;\;\text{for}\;\;\nu=1,2,3,\tag{109}$$

$$\Omega_0:\quad r_1\to\infty,r_2\to\infty,r_3\to\infty.\tag{110}$$

With their help, ω_β can be expressed as

$$\omega_\beta=\Omega_\gamma\cup\Omega_\alpha\cup\Omega_0,\tag{111}$$

and ω'_α analogously in terms of the primed coordinates as

$$\omega'_\alpha=\Omega'_\gamma\cup\Omega'_\beta\cup\Omega'_0.\tag{112}$$

Thus, to extract the behavior of $\tilde{\mathcal{V}}_{\beta\alpha}^{(\alpha\beta)}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$ near the singularity at Eq. (38) it suffices to know the three-body Coulomb wave function $\Psi^{C(+)}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{r}_\beta,\boldsymbol{\rho}_\beta)$ in the asymptotic domain ω_β, and $\Psi^{C(+)}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{r}'_\alpha,\boldsymbol{\rho}'_\alpha)$ in the asymptotic domain ω'_α. Since, according to Eq. (111), ω_β is the union of three different asymptotic domains we have to consider the asymptotic behavior of $\Psi^{C(+)}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{r}_\beta,\boldsymbol{\rho}_\beta)$ in each of them.

The asymptotic form of the three-charged particle wave function in Ω_0 has been found in Refs. [22,23], and that valid in $\Omega_\nu,\nu=1,2,3$, in Ref. [11]. The corresponding expressions are collected in Appendix D. There it is argued that, when looking for the main singular part of $\tilde{\mathcal{V}}_{\beta\alpha}^{(\alpha\beta)}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$, the exact three-charged particle wave function may be approximated uniformly in all asymptotic regions $\Omega_\nu,\nu=0,\ldots,3$, i.e., in the whole domain ω_β, by [cf. Eq. (D17)]

$$\Psi^{C(+)}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{r}_\beta,\boldsymbol{\rho}_\beta)\overset{\omega_\beta}{\approx}\Psi^{C,as(+)'}_{P^0}(X)=e^{iP^0\cdot X}\prod_{\nu=1}^{3}e^{-i\mathbf{k}^0_\nu\cdot\mathbf{r}_\nu}\psi^{C(+)}_{\mathbf{k}^0_\nu}(\mathbf{r}_\nu). \quad (113)$$

Here, $P^0=\{\mathbf{k}^0_\nu,\mathbf{q}^0_\nu\}$ and $X=\{\mathbf{r}_\nu,\boldsymbol{\rho}_\nu\}$ are six-dimensional vectors with

$$P^0\cdot X=\mathbf{k}^0_\nu\cdot\mathbf{r}_\nu+\mathbf{q}^0_\nu\cdot\boldsymbol{\rho}_\nu,\qquad \nu=1,2,3. \quad (114)$$

Moreover, $\psi^{C(+)}_{\mathbf{k}^0_\alpha}(\mathbf{r}_\alpha)$ is the two-particle Coulomb scattering wave function for particles β and γ, belonging to the energy $k^{0\,2}_\alpha/2\mu_\alpha$. Analogously, in the domain ω'_α the wave function $\Psi^{C(+)}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{r}'_\alpha,\boldsymbol{\rho}'_\alpha)$ can be approximated by [cf. Eq. (D18)]

$$\Psi^{(+)}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{r}'_\alpha,\boldsymbol{\rho}'_\alpha)\overset{\omega'_\alpha}{\approx}\Psi^{C,as(+)'}_{P^0}(X')=e^{iP^0\cdot X'}\prod_{\nu=1}^{3}e^{-i\mathbf{k}^0_\nu\cdot\mathbf{r}'_\nu}\psi^{C(+)}_{\mathbf{k}^0_\nu}(\mathbf{r}'_\nu). \quad (115)$$

Thus, the leading singular part of $\tilde{\mathcal{V}}_{\beta\alpha}^{(\alpha\beta)}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$ in the limit $\sigma_\beta(q'_\beta;z)\to 0$, to be denoted by $\tilde{\mathcal{V}}_{\beta\alpha}^{(\alpha\beta)(s)}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$, can be extracted from expression (106), with $\Psi^{C(+)}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{r}'_\alpha,\boldsymbol{\rho}'_\alpha)$ replaced by $\Psi^{C,as(+)'}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{r}'_\alpha,\boldsymbol{\rho}'_\alpha)$, and $\Psi^{C(+)*}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{r}_\beta,\boldsymbol{\rho}_\beta)$ by $\Psi^{C,as(+)'*}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{r}_\beta,\boldsymbol{\rho}_\beta)$. Alternatively, rewriting the integrals in momentum space we have

$$\begin{aligned}\tilde{\mathcal{V}}_{\beta\alpha}^{(\alpha\beta)(s)}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)=&\int\frac{d\mathbf{k}''_\alpha}{(2\pi)^3}\int\frac{d\mathbf{q}''_\alpha}{(2\pi)^3}\int\frac{d\mathbf{k}''_\beta}{(2\pi)^3}\int\frac{d\mathbf{q}''_\beta}{(2\pi)^3}\int\frac{d\mathbf{k}^0_\alpha}{(2\pi)^3}\int\frac{d\mathbf{q}^0_\alpha}{(2\pi)^3}\frac{\tilde{\phi}^*_\beta[\boldsymbol{\epsilon}_{\beta\alpha}(\mathbf{q}''_\alpha+\lambda_{\alpha\gamma}\mathbf{q}'_\beta)]}{[(\mathbf{q}''_\alpha+\lambda_{\alpha\gamma}\mathbf{q}'_\beta)^2-2\mu_\beta\hat{z}_\beta]^{1-i\hat{\eta}_\beta}}\\&\times\frac{4\pi e_\beta e_\gamma}{[\mathbf{k}''_\alpha-\boldsymbol{\epsilon}_{\alpha\beta}(\mathbf{q}'_\beta+\lambda_{\beta\gamma}\mathbf{q}''_\alpha)]^2}\frac{\Psi^{C,as(+)'}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{k}''_\alpha,\mathbf{q}''_\alpha)\Psi^{C,as(+)'*}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{k}''_\beta,\mathbf{q}''_\beta)}{\left[z-\dfrac{q^{0^2}_\alpha}{2M_\alpha}-\dfrac{k^{0^2}_\alpha}{2\mu_\alpha}\right]}\\&\times\frac{4\pi e_\alpha e_\gamma}{[\mathbf{k}''_\beta-\boldsymbol{\epsilon}_{\beta\alpha}(\mathbf{q}_\alpha+\lambda_{\alpha\gamma}\mathbf{q}''_\beta)]^2}\frac{\tilde{\phi}_\alpha[\boldsymbol{\epsilon}_{\alpha\beta}(\mathbf{q}''_\beta+\lambda_{\beta\gamma}\mathbf{q}_\alpha)]}{[(\mathbf{q}''_\beta+\lambda_{\beta\gamma}\mathbf{q}_\alpha)^2-2\mu_\alpha\hat{z}_\alpha]^{1-i\hat{\eta}_\alpha}}.\end{aligned} \quad (116)$$

Given the explicit expressions for the three-body Coulomb wave functions in the asymptotic regions ω_β and ω'_α, we can now show that $\tilde{\mathcal{V}}_{\beta\alpha}^{(\alpha\beta)(s)}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$, and hence also $\tilde{\mathcal{V}}_{\beta\alpha}^{(\alpha\beta)}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$, has a branch point singularity of the type (40) at Eq. (38).

To begin with let us write down the Fourier transform of $\Psi^{as(+)'}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{r}_\alpha,\boldsymbol{\rho}_\alpha)$:

$$\Psi^{as(+)'}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{k}_\alpha,\mathbf{q}_\alpha)=\int\frac{d\mathbf{k}}{(2\pi)^3}\psi^{C(+)}_{\mathbf{k}^0_\beta}(\mathbf{k})\,\psi^{C(+)}_{\mathbf{k}^0_\gamma}[\mathbf{k}+\mathbf{k}^0_\gamma-\mathbf{k}^0_\beta+\boldsymbol{\epsilon}_{\alpha\beta}(\mathbf{q}_\alpha-\mathbf{q}^0_\alpha)]\,\psi^{C(+)}_{\mathbf{k}^0_\alpha}[\mathbf{k}+\mathbf{k}^0_\alpha-\mathbf{k}^0_\beta+\boldsymbol{\epsilon}_{\beta\alpha}(\mathbf{q}_\gamma-\mathbf{q}^0_\gamma)]. \quad (117)$$

This result follows from Eq. (113) in a straightforward manner by taking into account the linear dependence of the different Jacobian variables (cf. Appendix A) and $\Sigma_{\nu=1}^3\mathbf{q}_\nu=0$. The Fourier transform of the two-body Coulomb scattering wave function is defined as

$$\psi^{C(+)}_{\mathbf{k}^0_\alpha}(\mathbf{k}_\alpha)=\int d\mathbf{r}e^{-i\mathbf{r}_\alpha\cdot\mathbf{k}_\alpha}\psi^{C(+)}_{\mathbf{k}^0_\alpha}(\mathbf{r}_\alpha). \quad (118)$$

When inserting Eq. (117) into Eq. (116), one encounters an expression of the following type:

$$J_\alpha := \int \frac{d\mathbf{k}''_\beta}{(2\pi)^3}\int \frac{d\mathbf{q}''_\beta}{(2\pi)^3}\Psi^{C,as(+)'*}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{k}''_\beta,\mathbf{q}''_\beta)\frac{4\pi e_\alpha e_\gamma}{[\mathbf{k}''_\beta-\epsilon_{\beta\alpha}(\mathbf{q}_\alpha+\lambda_{\alpha\gamma}\mathbf{q}''_\beta)]^2}\frac{\tilde{\phi}_\alpha[\epsilon_{\alpha\beta}(\mathbf{q}''_\beta+\lambda_{\beta\gamma}\mathbf{q}_\alpha)]}{[(\mathbf{q}''_\beta+\lambda_{\beta\gamma}\mathbf{q}_\alpha)^2-2\mu_\alpha\hat{z}_\alpha]^{1-i\hat{\eta}_\alpha}}$$

$$=\int \frac{d\mathbf{q}''_\alpha}{(2\pi)^3}\int \frac{d\mathbf{q}''_\beta}{(2\pi)^3}\int \frac{d\mathbf{k}}{(2\pi)^3}\psi^{C(+)*}_{\mathbf{k}^0_\beta}(\mathbf{k})\psi^{C(+)*}_{\mathbf{k}^0_\gamma}[\mathbf{k}+\mathbf{k}^0_\gamma-\mathbf{k}^0_\beta+\epsilon_{\alpha\beta}(\mathbf{q}''_\alpha-\mathbf{q}^0_\alpha)]\psi^{C(+)*}_{\mathbf{k}^0_\alpha}[\mathbf{k}+\mathbf{k}^0_\alpha-\mathbf{k}^0_\beta-\epsilon_{\beta\alpha}(\mathbf{q}''_\alpha+\mathbf{q}''_\beta+\mathbf{q}^0_\gamma)]$$

$$\times\frac{4\pi e_\alpha e_\gamma}{(\mathbf{q}''_\alpha-\mathbf{q}_\alpha)^2}\frac{\tilde{\phi}_\alpha[\epsilon_{\alpha\beta}(\mathbf{q}''_\beta+\lambda_{\beta\gamma}\mathbf{q}_\alpha)]}{[(\mathbf{q}''_\beta+\lambda_{\beta\gamma}\mathbf{q}_\alpha)^2-2\mu_\alpha\hat{z}_\alpha]^{1-i\hat{\eta}_\alpha}}, \tag{119}$$

where to arrive at the second equality a change of the integration variable has been performed. We are looking for the behavior of J_α when the singularities of the integrand at

$$\mathbf{q}''_\alpha-\mathbf{q}_\alpha=0 \quad\text{and}\quad (\mathbf{q}''_\beta+\lambda_{\beta\gamma}\mathbf{q}_\alpha)^2-2\mu_\alpha\hat{z}_\alpha=0 \tag{120}$$

collide with the forward-scattering singularities of the wave function $\Psi^{C,as(+)'*}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{k}''_\beta,\mathbf{q}''_\beta)$. The latter occur at

$$\mathbf{q}''_\nu-\mathbf{q}^0_\nu=0, \quad \nu=1,2,3, \tag{121}$$

which can easily be verified by taking into account the fact that each factor $\psi^{C(+)}_{\mathbf{k}^0_\nu}(\mathbf{k}_\nu)$ has a singularity for $\mathbf{k}_\nu-\mathbf{k}^0_\nu=0$. The coincidence of these three forward-scattering singularities in Eq. (117) gives rise to the singularity conditions (121).

Since we presently assume $q_\alpha\neq\tilde{q}_\alpha$ and $q'_\beta\neq\tilde{q}_\beta$, J_α can, according to Eq. (E32), be written near the leading singularity in the form

$$J_\alpha=4\pi e_\alpha e_\gamma e^{-\pi(\eta^0_\alpha+\eta^0_\beta+\eta^0_\gamma)/2}\frac{\Gamma[1-i(\hat{\eta}_\alpha+\eta^0_\alpha)]\Gamma[1-i(\eta^0_\beta+\eta^0_\gamma)]}{\Gamma(1-i\hat{\eta}_\alpha)}[-2(\mathbf{k}^0_\alpha\cdot\mathbf{k}_\alpha+k^0_\alpha k_\alpha)]^{-i\eta^0_\alpha}[2\epsilon_{\alpha\beta}\mathbf{\Delta}^0_\alpha\cdot\mathbf{k}^0_\beta]^{-i\eta^0_\beta}$$

$$\times[-2\epsilon_{\alpha\beta}\mathbf{\Delta}^0_\alpha\cdot\mathbf{k}^0_\gamma]^{-i\eta^0_\gamma}\frac{1}{[2\epsilon_{\alpha\beta}\mathbf{\Delta}^0_\beta\cdot\mathbf{k}_\alpha+\sigma_\alpha(q_\alpha;z)]^{1-i(\hat{\eta}_\alpha+\eta^0_\alpha)}}\frac{\hat{J}_\alpha}{[(\Delta^0_\alpha)^2]^{1-i(\eta^0_\beta+\eta^0_\gamma)}}, \tag{122}$$

where $\hat{J}_\alpha$ remains finite at $\Delta^0_\alpha=0$, $\Delta^0_\beta=0$, and $\sigma_\alpha(q_\alpha;z)=0$. The vectors $\mathbf{\Delta}^0_\alpha$ and $\mathbf{\Delta}^0_\beta$ have been introduced in Eqs. (89) and (90), and $\mathbf{k}_\alpha$ in Eq. (27). Moreover,

$$\eta^0_\alpha\equiv\eta_\alpha(k^0_\alpha):=e_\beta e_\gamma\mu_\alpha/k^0_\alpha, \tag{123}$$

with analogous definitions for η^0_β and η^0_γ [cf. Eq. (41)]. $\Gamma(z)$ is the gamma function.

Similarly, the leading singular part of

$$J^*_\beta := \int \frac{d\mathbf{k}''_\alpha}{(2\pi)^3}\int \frac{d\mathbf{q}''_\alpha}{(2\pi)^3}\Psi^{C,as(+)'}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{k}''_\alpha,\mathbf{q}''_\alpha)\frac{\tilde{\phi}^*_\beta[\epsilon_{\beta\alpha}(\mathbf{q}''_\alpha+\lambda_{\alpha\gamma}\mathbf{q}'_\beta)]}{[(\mathbf{q}''_\alpha+\lambda_{\alpha\gamma}\mathbf{q}'_\beta)^2-2\mu_\beta\hat{z}_\beta]^{1-i\hat{\eta}_\beta}}\frac{4\pi e_\beta e_\gamma}{[\mathbf{k}''_\alpha-\epsilon_{\alpha\beta}(\mathbf{q}'_\beta+\lambda_{\beta\gamma}\mathbf{q}''_\alpha)]^2}$$

$$=\int \frac{d\mathbf{q}''_\beta}{(2\pi)^3}\int \frac{d\mathbf{q}''_\alpha}{(2\pi)^3}\int \frac{d\mathbf{k}}{(2\pi)^3}\psi^{C(+)}_{\mathbf{k}^0_\alpha}(\mathbf{k})\psi^{C(+)}_{\mathbf{k}^0_\beta}[\mathbf{k}-\mathbf{k}^0_\alpha+\mathbf{k}^0_\beta-\epsilon_{\alpha\beta}(\mathbf{q}''_\alpha+\mathbf{q}''_\beta+\mathbf{q}^0_\gamma)]\psi^{C(+)}_{\mathbf{k}^0_\gamma}[\mathbf{k}-\mathbf{k}^0_\alpha+\mathbf{k}^0_\gamma+\epsilon_{\beta\alpha}(\mathbf{q}''_\beta-\mathbf{q}^0_\beta)]$$

$$\times\frac{\tilde{\phi}^*_\beta[\epsilon_{\beta\alpha}(\mathbf{q}''_\alpha+\lambda_{\alpha\gamma}\mathbf{q}'_\beta)]}{[(\mathbf{q}''_\alpha+\lambda_{\alpha\gamma}\mathbf{q}'_\beta)^2-2\mu_\beta\hat{z}_\beta]^{1-i\hat{\eta}_\beta}}\frac{4\pi e_\beta e_\gamma}{(\mathbf{q}''_\beta-\mathbf{q}'_\beta)^2}, \tag{124}$$

when the zeros of the denominators at

$$\mathbf{q}''_\beta-\mathbf{q}'_\beta=0 \quad\text{and}\quad (\mathbf{q}''_\alpha+\lambda_{\alpha\gamma}\mathbf{q}'_\beta)^2-2\mu_\beta\hat{z}_\beta=0 \tag{125}$$

collide with the forward-scattering singularities of the wave function $\Psi^{C,as(+)'}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{k}''_\alpha,\mathbf{q}''_\alpha)$ which again occur at Eq. (121), is given by

$$J_{\beta}^{*}=4\pi e_{\beta}e_{\gamma}e^{-\pi(\eta_{\alpha}^{0}+\eta_{\beta}^{0}+\eta_{\gamma}^{0})/2}\frac{\Gamma[1-i(\hat{\eta}_{\beta}-\eta_{\beta}^{0})]\Gamma[1+i(\eta_{\alpha}^{0}+\eta_{\gamma}^{0})]}{\Gamma(1-i\hat{\eta}_{\beta})}[-2(\mathbf{k}_{\beta}^{0}\cdot\mathbf{k}_{\beta}'+k_{\beta}^{0}k_{\beta}')]^{i\eta_{\beta}^{0}}[2\epsilon_{\beta\alpha}\boldsymbol{\Delta}_{\beta}^{0}\cdot\mathbf{k}_{\alpha}^{0}]^{i\eta_{\alpha}^{0}}$$
$$\times[-2\epsilon_{\beta\alpha}\boldsymbol{\Delta}_{\beta}^{0}\cdot\mathbf{k}_{\gamma}^{0}]^{i\eta_{\gamma}^{0}}\frac{1}{[2\epsilon_{\beta\alpha}\boldsymbol{\Delta}_{\alpha}^{0}\cdot\mathbf{k}_{\beta}'+\sigma_{\beta}(q_{\beta}';z)]^{1-i(\hat{\eta}_{\beta}-\eta_{\beta}^{0})}}\frac{\hat{J}_{\beta}^{*}}{[(\Delta_{\beta}^{0})^{2}]^{1+i(\eta_{\alpha}^{0}+\eta_{\gamma}^{0})}}, \tag{126}$$

where $\hat{J}_{\beta}$ remains finite at $\Delta_{\alpha}^{0}=0=\Delta_{\beta}^{0}$ and $\sigma_{\beta}(q_{\beta}';z)=0=\sigma_{\alpha}(q_{\alpha};z)$.

Taking into account Eqs. (122) and (126), and repeating the steps that led from Eq. (85) to Eq. (91), we derive from Eq. (116) for the leading singular part of $\tilde{\mathcal{V}}_{\beta\alpha}^{(\alpha\beta)}(\mathbf{q}_{\beta}',\mathbf{q}_{\alpha};z)$:

$$\tilde{\mathcal{V}}_{\beta\alpha}^{(\alpha\beta)(s)}(\mathbf{q}_{\beta}',\mathbf{q}_{\alpha};z)=\int\frac{d\mathbf{k}_{\alpha}^{0}}{(2\pi)^{3}}\int\frac{d\mathbf{q}_{\alpha}^{0}}{(2\pi)^{3}}\frac{J_{\beta}^{*}J_{\alpha}}{\left[z-\dfrac{k_{\alpha}^{0^{2}}}{2\mu_{\alpha}}-\dfrac{q_{\alpha}^{0^{2}}}{2M_{\alpha}}\right]}\approx-\frac{16\pi^{2}e_{\alpha}e_{\beta}e_{\gamma}^{2}}{\Gamma(1-i\hat{\eta}_{\beta})\Gamma(1-i\hat{\eta}_{\alpha})}\int\frac{d\boldsymbol{\Delta}_{\beta}^{0}}{(2\pi)^{3}}\int\frac{d\boldsymbol{\Delta}_{\alpha}^{0}}{(2\pi)^{3}}e^{-\pi(\eta_{\alpha}^{0}+\eta_{\beta}^{0}+\eta_{\gamma}^{0})}$$
$$\times\Gamma[1-i(\hat{\eta}_{\beta}-\eta_{\beta}^{0})]\Gamma[1+i(\eta_{\alpha}^{0}+\eta_{\gamma}^{0})]\Gamma[1-i(\hat{\eta}_{\alpha}+\eta_{\alpha}^{0})]\Gamma[1-i(\eta_{\beta}^{0}+\eta_{\gamma}^{0})]$$
$$\times[-2(\mathbf{k}_{\beta}^{0}\cdot\mathbf{k}_{\beta}'+k_{\beta}^{0}k_{\beta}')]^{i\eta_{\beta}^{0}}[2\epsilon_{\beta\alpha}\boldsymbol{\Delta}_{\beta}^{0}\cdot\mathbf{k}_{\alpha}^{0}]^{i\eta_{\alpha}^{0}}[-2\epsilon_{\beta\alpha}\boldsymbol{\Delta}_{\beta}^{0}\cdot\mathbf{k}_{\gamma}^{0}]^{i\eta_{\gamma}^{0}}$$
$$\times\frac{1}{[2\epsilon_{\beta\alpha}\boldsymbol{\Delta}_{\alpha}^{0}\cdot\mathbf{k}_{\beta}'+\sigma_{\beta}(q_{\beta}';z)]^{1-i(\hat{\eta}_{\beta}-\eta_{\beta}^{0})}}\frac{1}{[(\Delta_{\beta}^{0})^{2}]^{1+i(\eta_{\alpha}^{0}+\eta_{\gamma}^{0})}}$$
$$\times\frac{\hat{J}_{\beta}^{*}\hat{J}_{\alpha}}{\left[\dfrac{1}{2\mu_{\alpha}}\sigma_{\alpha}(q_{\alpha};z)+\dfrac{1}{M_{\alpha}}\boldsymbol{\Delta}_{\alpha}^{0}\cdot\mathbf{q}_{\alpha}+\dfrac{\epsilon_{\alpha\beta}}{\mu_{\alpha}}(\boldsymbol{\Delta}_{\beta}^{0}+\lambda_{\beta\gamma}\boldsymbol{\Delta}_{\alpha}^{0})\cdot\mathbf{k}_{\alpha}\right]}\frac{1}{[(\Delta_{\alpha}^{0})^{2}]^{1-i(\eta_{\beta}^{0}+\eta_{\gamma}^{0})}}$$
$$\times\frac{1}{[2\epsilon_{\alpha\beta}\boldsymbol{\Delta}_{\beta}^{0}\cdot\mathbf{k}_{\alpha}+\sigma_{\alpha}(q_{\alpha};z)]^{1-i(\hat{\eta}_{\alpha}+\eta_{\alpha}^{0})}}[-2(\mathbf{k}_{\alpha}^{0}\cdot\mathbf{k}_{\alpha}+k_{\alpha}^{0}k_{\alpha})]^{-i\eta_{\alpha}^{0}}[2\epsilon_{\alpha\beta}\boldsymbol{\Delta}_{\alpha}^{0}\cdot\mathbf{k}_{\beta}^{0}]^{-i\eta_{\beta}^{0}}[-2\epsilon_{\alpha\beta}\boldsymbol{\Delta}_{\alpha}^{0}\cdot\mathbf{k}_{\gamma}^{0}]^{-i\eta_{\gamma}^{0}}. \tag{127}$$

Terms proportional to $(\boldsymbol{\Delta}_{\alpha}^{0})^{2}$ and to $(\boldsymbol{\Delta}_{\beta}^{0})^{2}$ have already been omitted. In order to extract the singular behavior of $\tilde{\mathcal{V}}_{\beta\alpha}^{(\alpha\beta)}(\mathbf{q}_{\beta}',\mathbf{q}_{\alpha};z)$ in the limit $\sigma_{\beta}(q_{\beta}';z)\to0$ we use the substitution (92), which motivates the neglect of the terms $\sim(\boldsymbol{\Delta}_{\alpha}^{0})^{2}=O(\sigma_{\beta}^{2}(q_{\beta}';z))$ and $\sim(\boldsymbol{\Delta}_{\beta}^{0})^{2}=O(\sigma_{\alpha}^{2}(q_{\alpha};z))$, and find

$$\tilde{\mathcal{V}}_{\beta\alpha}^{(\alpha\beta)}(\mathbf{q}_{\beta}',\mathbf{q}_{\alpha};z)\overset{\sigma_{\beta}(q_{\beta}';z)\to0}{=}\frac{A(0;\mathbf{q}_{\beta}',\mathbf{q}_{\alpha};z)}{[\sigma_{\beta}(q_{\beta}';z)]^{1-i(\hat{\eta}_{\alpha}+\hat{\eta}_{\beta})}}+o\left(\frac{1}{\sigma_{\beta}(q_{\beta}';z)}\right). \tag{128}$$

The function $A(0;\mathbf{q}_{\beta}',\mathbf{q}_{\alpha};z)$ is nonsingular at $\sigma_{\beta}(q_{\beta}';z)=0$. It is apparent that $\tilde{\mathcal{V}}_{\beta\alpha}^{(\alpha\beta)}(\mathbf{q}_{\beta}',\mathbf{q}_{\alpha};z)$ has the asserted singular branch point at $\sigma_{\beta}(q_{\beta}';z)=0$.

Let us comment on this result. First, it is easily seen that this singularity is integrable for any q_{β}' and q_{α} subject to the restrictions (39). For real $\hat{\eta}_{\beta}$ and $\hat{\eta}_{\alpha}$, i.e., for $\hat{E}_{\beta}$ and $\hat{E}_{\alpha}$ being positive, the singularity is integrable because

$$\frac{1}{|[\sigma_{\beta}(q_{\beta}';z)]^{1-i(\hat{\eta}_{\alpha}+\hat{\eta}_{\beta})}|}=\frac{1}{|\sigma_{\beta}(q_{\beta}';z)|}. \tag{129}$$

For $\hat{z}_{\alpha}=\hat{E}_{\alpha}<0$, as follows from Eq. (41a), $i\hat{\eta}_{\alpha}$ is positive real (recall that we are considering only particles with charges of the same sign), thus even weakening the singularity. In addition, taking into account Eqs. (32)–(34) one has

$$\sigma_{\beta}(q_{\beta}';z)=\frac{\mu_{\beta}}{\mu_{\alpha}}\sigma_{\alpha}(q_{\alpha};z)=\frac{\mu_{\beta}}{\mu_{\alpha}}[k_{\alpha}^{2}+2\mu_{\alpha}|\hat{E}_{\alpha}|]>0. \tag{130}$$

Hence, the singularity lies outside the region accessible for physical values of the momenta. An analogous situation prevails for $\hat{z}_{\beta}'=\hat{E}_{\beta}<0$ when $i\hat{\eta}_{\beta}$ is positive real [cf. Eq. (41b)] and again

$$\sigma_\beta(q'_\beta;z)=k'^2_\beta+2\mu_\beta|\hat{E}_\beta|>0. \tag{131}$$

As the next step we look for the leading singular part of $\tilde{\mathcal{V}}^{(\alpha\beta)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$ if $q'_\beta=\tilde{q}_\beta$ in which case $\hat{E}_\beta=0$ and $\sigma_\beta(\tilde{q}_\beta;E)=k'^2_\beta$, but let $q_\alpha\neq\tilde{q}_\alpha$. As before, near the leading singularity J_α is given by Eq. (122), while J^*_β for $q'_\beta=\tilde{q}_\beta$ behaves as the β-channel version of Eq. (E33),

$$J'^*_\beta=4\pi e_\beta e_\gamma e^{-\pi(\eta^0_\alpha+\eta^0_\beta+\eta^0_\gamma)/2}\Gamma(1+i\eta^0_\beta)\Gamma(1+i(\eta^0_\alpha+\eta^0_\gamma))[2\epsilon_{\beta\alpha}\mathbf{\Delta}^0_\beta\cdot\mathbf{k}^0_\alpha]^{i\eta^0_\alpha}[-2\epsilon_{\beta\alpha}\mathbf{\Delta}^0_\beta\cdot\mathbf{k}^0_\gamma]^{i\eta^0_\gamma}\frac{\hat{J}'^*_\beta}{[(\Delta^0_\beta)^2]^{1+i(\eta^0_\alpha+\eta^0_\gamma)}}, \tag{132}$$

with $\hat{J}'^*_\beta$ remaining regular.

Consequently, for $q'_\beta=\tilde{q}_\beta$ and $z=E+i0$ the leading singular part of $\tilde{\mathcal{V}}^{(\alpha\beta)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$ in the limit $\sigma_\beta(\tilde{q}_\beta;E)=k'^2_\beta\to 0$ takes, instead of Eq. (127), the form

$$\begin{aligned}\tilde{\mathcal{V}}^{(\alpha\beta)(s)}_{\beta\alpha}(\tilde{\mathbf{q}}'_\beta,\mathbf{q}_\alpha;E+i0)\approx&-\frac{16\pi^2 e_\alpha e_\beta e^2_\gamma}{\Gamma(1-i\hat{\eta}_\alpha)}\int\frac{d\mathbf{\Delta}^0_\beta}{(2\pi)^3}\int\frac{d\mathbf{\Delta}^0_\alpha}{(2\pi)^3}e^{-\pi(\eta^0_\alpha+\eta^0_\beta+\eta^0_\gamma)}\Gamma(1+i\eta^0_\beta)\Gamma(1+i(\eta^0_\alpha+\eta^0_\gamma))\Gamma(1-i(\hat{\eta}_\alpha\\&+\eta^0_\alpha))\Gamma(1-i(\eta^0_\beta+\eta^0_\gamma))[2\epsilon_{\beta\alpha}\mathbf{\Delta}^0_\beta\cdot\mathbf{k}^0_\alpha]^{i\eta^0_\alpha}[-2\epsilon_{\beta\alpha}\mathbf{\Delta}^0_\beta\cdot\mathbf{k}^0_\gamma]^{i\eta^0_\gamma}\frac{1}{[\Delta^{0\,2}_\beta]^{1+i(\eta^0_\alpha+\eta^0_\gamma)}}\\&\times\frac{\hat{J}'^*_\beta\hat{J}_\alpha}{\left[\frac{1}{2\mu_\beta}k'^2_\beta+\frac{1}{M_\beta}\mathbf{\Delta}^0_\beta\cdot\tilde{\mathbf{q}}'_\beta+\frac{1}{2M_\beta}(\Delta^0_\beta)^2+\frac{\epsilon_{\beta\alpha}}{\mu_\beta}(\mathbf{\Delta}^0_\alpha+\lambda_{\alpha\gamma}\mathbf{\Delta}^0_\beta)\cdot\mathbf{k}'_\beta+\frac{1}{2\mu_\beta}(\mathbf{\Delta}^0_\alpha+\lambda_{\alpha\gamma}\mathbf{\Delta}^0_\beta)^2\right]}\\&\times\frac{1}{[(\Delta^0_\alpha)^2]^{1-i(\eta^0_\beta+\eta^0_\gamma)}}\frac{1}{[(\mathbf{\Delta}^0_\beta)^2+2\epsilon_{\alpha\beta}\mathbf{\Delta}^0_\beta\cdot\mathbf{k}_\alpha+\mu_\alpha k'^2_\beta/\mu_\beta]^{1-i(\hat{\eta}_\alpha+\eta^0_\alpha)}}[-2(\mathbf{k}^0_\alpha\cdot\mathbf{k}_\alpha+k^0_\alpha k_\alpha)]^{-i\eta^0_\alpha}\\&\times[2\epsilon_{\alpha\beta}\mathbf{\Delta}^0_\alpha\cdot\mathbf{k}^0_\beta]^{-i\eta^0_\beta}[-2\epsilon_{\alpha\beta}\mathbf{\Delta}^0_\alpha\cdot\mathbf{k}^0_\gamma]^{-i\eta^0_\gamma}.\end{aligned} \tag{133}$$

To proceed further we make the substitutions

$$\mathbf{\Delta}^0_\alpha=k'_\beta\mathbf{u}\quad\text{and}\quad\mathbf{\Delta}^0_\beta=k'^2_\beta\mathbf{v}. \tag{134}$$

Expressing $\mathbf{k}^0_\beta:=\epsilon_{\beta\alpha}(\mathbf{q}^0_\alpha+\lambda_{\alpha\gamma}\mathbf{q}^0_\beta)$ in terms of $\mathbf{\Delta}^0_\alpha$, $\mathbf{\Delta}^0_\beta$, and $\mathbf{k}'_\beta$, using Eqs. (89), (90), and (27), we find

$$k^0_\beta=|\mathbf{\Delta}^0_\alpha+\lambda_{\alpha\gamma}\mathbf{\Delta}^0_\beta+\epsilon_{\beta\alpha}\mathbf{k}'_\beta|\overset{k'_\beta\to 0}{=}a_\beta k'_\beta+o(k'_\beta),\quad a_\beta\neq 0, \tag{135}$$

and similarly

$$k^0_\alpha\overset{k'_\beta\to 0}{\sim}k_\alpha+O(k'_\beta),\quad k^0_\gamma\overset{k'_\beta\to 0}{\sim}k_\gamma+O(k'_\beta). \tag{136}$$

For the factor

$$N=e^{-\pi(\eta^0_\alpha+\eta^0_\beta+\eta^0_\gamma)}\Gamma(1+i\eta^0_\beta)\Gamma(1+i(\eta^0_\alpha+\eta^0_\gamma))\Gamma(1-i(\hat{\eta}_\alpha+\eta^0_\alpha))\Gamma(1-i(\eta^0_\beta+\eta^0_\gamma)), \tag{137}$$

which occurs in the integrand of Eq. (133), this implies (recall $e_\alpha e_\beta>0$)

$$N\overset{k'_\beta\to 0}{\sim}\frac{1}{k'_\beta}\exp\left\{-\frac{2\pi e_\alpha e_\gamma\mu_\beta}{a_\beta k'_\beta}\right\}\quad(q'_\beta=\tilde{q}_\beta). \tag{138}$$

Thus we derive

$$\tilde{\mathcal{V}}^{(\alpha\beta)}_{\beta\alpha}(\tilde{\mathbf{q}}'_\beta,\mathbf{q}_\alpha;E+i0)\overset{k'_\beta\to 0}{=}\frac{1}{k'^{2-2i\hat{\eta}_\alpha+i\eta_\gamma}_\beta}C(k'_\beta;\tilde{\mathbf{q}}'_\beta,\mathbf{q}_\alpha;E+i0), \tag{139}$$

where $C(k'_\beta;\widetilde{\mathbf{q}}'_\beta,\mathbf{q}_\alpha;E)$ vanishes in the limit $k'_\beta\to 0$ because of the exponentially decreasing factor in N. Thus, for $q'_\beta=\widetilde{q}_\beta$, the leading singularity of $\widetilde{\mathcal{V}}^{(\alpha\beta)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;E+i0)$ is actually weaker than described in Eq. (139), and hence represents an integrable branch point at $\sigma_\beta(\widetilde{q}_\beta;E)=k'^2_\beta=0$. In the same way the result (45) for $q'_\beta\neq\widetilde{q}_\beta$ and $q_\alpha=\widetilde{q}_\alpha$ follows.

Finally, if $q'_\beta=\widetilde{q}_\beta$ and $q_\alpha=\widetilde{q}_\alpha$, which in the limit $k'^2_\beta\sim k^2_\alpha\to 0$ necessitates $E=0$ and, hence, also $\widetilde{q}_\beta=0=\widetilde{q}_\alpha$, use of Eq. (132) for J^*_β, and of Eq. (E33) for J_α, leads to the result that for $z=E+i0$, the leading singular part of $\widetilde{\mathcal{V}}^{(\alpha\beta)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$ takes in the limit $\sigma_\beta(\widetilde{q}_\beta;E)=k'^2_\beta=\mu_\beta\sigma_\alpha(\widetilde{q}_\alpha;E)/\mu_\alpha=\mu_\beta k^2_\alpha/\mu_\alpha\to 0$ the form

$$\widetilde{\mathcal{V}}^{(\alpha\beta)}_{\beta\alpha}(\widetilde{\mathbf{q}}'_\beta,\widetilde{\mathbf{q}}_\alpha;E+i0)\overset{k'_\beta\to 0}{=}\frac{1}{k'^2_\beta}D(k'_\beta;\mathbf{0},\mathbf{0};0)\quad\text{for}\quad E=0, \tag{140}$$

with $D(k'_\beta;\ldots)$ vanishing in the limit $k'_\beta\to 0$ for similar reasons. To arrive at this result the substitutions $\mathbf{\Delta}^0_\alpha=k'_\beta\mathbf{u}$ and $\mathbf{\Delta}^0_\beta=k'_\beta\mathbf{v}$ had to be made.

The above steps can be repeated in an analogous manner if the final and/or the initial momentum equals its on-shell value $\overline{\mathbf{q}}'_\beta$ and/or $\overline{\mathbf{q}}_\alpha$, respectively. The result is that the leading singularity in the limit $\overline{\sigma}_\beta\to 0$ is of the same form as shown in Eqs. (128) and (139), with $i\hat{\eta}_\beta$ appropriately substituted by $\eta^{(bs)}_\beta$ and/or $i\hat{\eta}_\alpha$ by $\eta^{(bs)}_\alpha$. Recall that the assumption of nonzero two-body binding energies $B_\nu>0$ entails $\overline{q}_\nu\neq\widetilde{q}_\nu$, $\nu=1,2,3$.

As an example consider the contribution $\widetilde{\mathcal{V}}^{(\alpha\beta)}_{\beta\alpha}(\overline{\mathbf{q}}'_\beta,\overline{\mathbf{q}}_\alpha;E+i0)$ to the fully on-shell effective potential. Using the appropriate forms (62a) and (64a) for the on-shell Coulomb-modified form factor, one obtains for the leading singular part

$$\begin{aligned}\widetilde{\mathcal{V}}^{(\alpha\beta)(s)}_{\beta\alpha}(\overline{\mathbf{q}}'_\beta,\overline{\mathbf{q}}_\alpha;z)=&\int\frac{d\mathbf{k}''_\alpha}{(2\pi)^3}\int\frac{d\mathbf{q}''_\alpha}{(2\pi)^3}\int\frac{d\mathbf{k}''_\beta}{(2\pi)^3}\int\frac{d\mathbf{q}''_\beta}{(2\pi)^3}\int\frac{d\mathbf{k}^0_\alpha}{(2\pi)^3}\int\frac{d\mathbf{q}^0_\alpha}{(2\pi)^3}\frac{\widetilde{\phi}^*_\beta[\epsilon_{\beta\alpha}(\mathbf{q}''_\alpha+\lambda_{\alpha\gamma}\overline{\mathbf{q}}'_\beta)]}{[(\mathbf{q}''_\alpha+\lambda_{\alpha\gamma}\overline{\mathbf{q}}'_\beta)^2+2\mu_\beta B_\beta]^{1-\eta^{(bs)}_\beta}}\\&\times\frac{4\pi e_\beta e_\gamma}{[\mathbf{k}''_\alpha-\epsilon_{\alpha\beta}(\overline{\mathbf{q}}'_\beta+\lambda_{\beta\gamma}\mathbf{q}''_\alpha)]^2}\frac{\Psi^{C,as(+)'}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{k}''_\alpha,\mathbf{q}''_\alpha)\Psi^{C,as(+)'*}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{k}''_\beta,\mathbf{q}''_\beta)}{\left[z-\frac{q^{0^2}_\alpha}{2M_\alpha}-\frac{k^{0^2}_\alpha}{2\mu_\alpha}\right]}\\&\times\frac{4\pi e_\alpha e_\gamma}{[\mathbf{k}''_\beta-\epsilon_{\beta\alpha}(\overline{\mathbf{q}}_\alpha+\lambda_{\alpha\gamma}\mathbf{q}''_\beta)]^2}\frac{\widetilde{\phi}_\alpha[\epsilon_{\alpha\beta}(\mathbf{q}''_\beta+\lambda_{\beta\gamma}\overline{\mathbf{q}}_\alpha)]}{[(\mathbf{q}''_\beta+\lambda_{\beta\gamma}\overline{\mathbf{q}}_\alpha)^2+2\mu_\alpha B_\alpha]^{1-\eta^{(bs)}_\alpha}}.\end{aligned} \tag{141}$$

Repeating the argumentation which led from Eq. (116) to Eq. (127) but now for expression (141), we arrive at

$$\begin{aligned}\widetilde{\mathcal{V}}^{(\alpha\beta)(s)}_{\beta\alpha}(\overline{\mathbf{q}}'_\beta,\overline{\mathbf{q}}_\alpha;E+i0)\approx&-\frac{16\pi^2e_\alpha e_\beta e^2_\gamma}{\Gamma(1-\eta^{(bs)}_\beta)\Gamma(1-\eta^{(bs)}_\alpha)}\int\frac{d\mathbf{\Delta}^0_\beta}{(2\pi)^3}\int\frac{d\mathbf{\Delta}^0_\alpha}{(2\pi)^3}e^{-\pi(\eta^0_\alpha+\eta^0_\beta+\eta^0_\gamma)}\Gamma(1-\eta^{(bs)}_\beta+i\eta^0_\beta)\Gamma(1+i(\eta^0_\alpha\\&+\eta^0_\gamma))\Gamma(1-(\eta^{(bs)}_\alpha-i\eta^0_\alpha))\Gamma(1-i(\eta^0_\beta+\eta^0_\gamma))[-2(\mathbf{k}^0_\beta\cdot\overline{\mathbf{k}}'_\beta+k^0_\beta\overline{k}'_\beta)]^{i\eta^0_\beta}[2\epsilon_{\beta\alpha}\mathbf{\Delta}^0_\beta\cdot\mathbf{k}^0_\alpha]^{i\eta^0_\alpha}\\&[-2\epsilon_{\beta\alpha}\mathbf{\Delta}^0_\beta\cdot\mathbf{k}^0_\gamma]^{i\eta^0_\gamma}\frac{1}{[2\epsilon_{\beta\alpha}\mathbf{\Delta}^0_\alpha\cdot\overline{\mathbf{k}}'_\beta+\overline{\sigma}_\beta]^{1-\eta^{(bs)}_\beta+i\eta^0_\beta}}\frac{1}{[(\Delta^0_\beta)^2]^{1+i(\eta^0_\alpha+\eta^0_\gamma)}}\\&\times\frac{\hat{J}^*_\beta\hat{J}_\alpha}{\left[\frac{1}{2\mu_\alpha}\overline{\sigma}_\alpha+\frac{1}{M_\alpha}\mathbf{\Delta}^0_\alpha\cdot\overline{\mathbf{q}}_\alpha+\frac{\epsilon_{\alpha\beta}}{\mu_\alpha}(\mathbf{\Delta}^0_\beta+\lambda_{\beta\gamma}\mathbf{\Delta}^0_\alpha)\cdot\overline{\mathbf{k}}_\alpha\right]}\frac{1}{[(\Delta^0_\alpha)^2]^{1-i(\eta^0_\beta+\eta^0_\gamma)}}\\&\times\frac{1}{[2\epsilon_{\alpha\beta}\mathbf{\Delta}^0_\beta\cdot\overline{\mathbf{k}}_\alpha+\overline{\sigma}_\alpha]^{1-\eta^{(bs)}_\alpha-i\eta^0_\alpha}}[-2(\mathbf{k}^0_\alpha\cdot\overline{\mathbf{k}}_\alpha+k^0_\alpha\overline{k}_\alpha)]^{-i\eta^0_\alpha}[2\epsilon_{\alpha\beta}\mathbf{\Delta}^0_\alpha\cdot\mathbf{k}^0_\beta]^{-i\eta^0_\beta}[-2\epsilon_{\alpha\beta}\mathbf{\Delta}^0_\alpha\cdot\mathbf{k}^0_\gamma]^{-i\eta^0_\gamma}\\&\overset{\overline{\sigma}_\beta\to 0}{=}\frac{\widetilde{N}(E)}{\overline{\sigma}_\beta^{\,1-\eta^{(bs)}_\alpha-\eta^{(bs)}_\beta}}+o\left(\frac{1}{\overline{\sigma}_\beta}\right),\end{aligned} \tag{142}$$

where $\tilde{N}(E)$ is a function which is finite at the singular point $\bar{\sigma}_\beta = \bar{k}_\beta^2 + 2\mu_\beta B_\beta = 0$. The momenta $\bar{\mathbf{k}}_\alpha$ and $\bar{\mathbf{k}}'_\beta$ are defined as in Eq. (27) but in terms of the on-shell momenta $\bar{\mathbf{q}}_\alpha$ and $\bar{\mathbf{q}}'_\beta$. Thus, the term $\tilde{\mathcal{V}}^{(\alpha\beta)}_{\beta\alpha}(\bar{\mathbf{q}}'_\beta, \bar{\mathbf{q}}_\alpha; E+i0)$ has the same singularity as $\mathcal{V}^{(a)}_{\beta\alpha}(\bar{\mathbf{q}}'_\beta, \bar{\mathbf{q}}_\alpha; E+i0)$. This shows that on the energy shell the complete nondiagonal effective potential part $\mathcal{V}^{(\alpha\beta)}_{\beta\alpha}(\bar{\mathbf{q}}'_\beta, \bar{\mathbf{q}}_\alpha; E+i0)$ has a branch point singularity at $\bar{\sigma}_\beta = 0$. That singularity is, however, located outside of the integration contour and, hence, is harmless.

Thus, we have proved that the replacement of the three-body free Green's function in Eq. (85) by the three-body Coulomb Green's function, leading to expression (81), influences neither position nor character (except for the special points $q'_\beta = \tilde{q}_\beta$ and $q_\alpha = \tilde{q}_\alpha$) of the leading singularity of the nondiagonal effective potential part $\tilde{\mathcal{V}}^{(\alpha\beta)}_{\beta\alpha}(\mathbf{q}'_\beta, \mathbf{q}_\alpha; z)$ in the momentum transfer variable. That is, the occurrence of the leading singularity of the type (128) to (142) is solely due to the Coulomb modifications of the form factors.

Though the proof has been concerned as yet only with the contribution $\tilde{\mathcal{V}}^{(\alpha\beta)}_{\beta\alpha}(\mathbf{q}'_\beta, \mathbf{q}_\alpha; z)$, it is obvious that it can be repeated in absolutely the same way for either term of the sum (79). Consequently, the total nondiagonal effective potential $\mathcal{V}_{\beta\alpha}(\mathbf{q}'_\beta, \mathbf{q}_\alpha; z), \alpha \neq \beta$, possesses the branch point singularities as asserted in the Theorem.

Let us add the following comments.

(i) The actual proof of the Theorem had relied on the simplicity of the explicit form (24) of the effective potential which on its part had resulted from the assumption that the two-body short-range interactions are given as separable potentials of rank one, cf. Eq. (7). It is now easy to see that this assumption was only technically convenient but does not limit the generality of the results obtained. Namely, if we allow for an arbitrary form of the two-body short-range interactions V^S_ν, $\nu = 1,2,3$, instead of the resolvent (12) of the three-body Coulomb Hamiltonian H^C, the resolvent $G'(z) = (z - H')^{-1}$ of the Hamiltonian $H' = H^C + \Sigma^3_{\nu=1} V'_\nu$, with $V'_\nu \equiv V^S_\nu - |\chi_\nu\rangle \Lambda_\nu \langle \chi_\nu|$, would occur in the definition of the effective potential [16]. Clearly, with V^S_ν also V'_ν is of short range. But for the extraction of the leading singularity of $\mathcal{V}_{\beta\alpha}(\mathbf{q}'_\beta, \mathbf{q}_\alpha; z) = \langle \mathbf{q}'_\beta, \chi_\beta | G'(z) | \chi_\alpha, \mathbf{q}_\alpha \rangle$ $(\beta \neq \alpha)$, which represents the appropriate generalization of expression (24), only knowledge of the asymptotic parts of the eigenfunctions of H' are needed in the spectral resolution of $G'(z)$. The latter are, of course, not influenced by any short-range modifications of the potentials, and thus could again be approximated by Eqs. (113) and (115).

(ii) Also the assumption that each subsystem supports only one bound state, is easily seen to have been of technical nature only. For, if in subsystem ν an arbitrary but finite number N_ν bound states exist, this could be accounted for by splitting off the potential V^S_ν a separable potential of rank N_ν (see Refs. [16,24]). As a result only the dimension of the effective potential matrix $\mathcal{V}$ would be blown up from 3×3 to $(N_\alpha + N_\beta + N_\gamma) \times (N_\alpha + N_\beta + N_\gamma)$, without altering anything else as compared to case (i).

E. Singular behavior of the kernel $\mathcal{K}_{\beta\alpha}(\mathbf{q}'_\beta, \mathbf{q}_\alpha; z)$

Given the leading singularity of the nondiagonal effective potential $\mathcal{V}_{\beta\alpha}(\mathbf{q}'_\beta, \mathbf{q}_\alpha; z)$, with $z = E + i0$, the singularity structure of the kernel $\mathcal{K}_{\beta\alpha}(\mathbf{q}'_\beta, \mathbf{q}_\alpha; z)$, Eq. (10), follows immediately. The corresponding AGS equations are given in Eq. (9) where the integration is over the right-hand variable which is presently denoted by $\mathbf{q}_\alpha$. $\mathbf{q}'_\beta$ is a free vector-valued parameter. Consider first the case $q'_\beta \neq \tilde{q}_\beta$. The leading singularities of the kernel are (i) the one that originates from $\mathcal{V}_{\beta\alpha}(\mathbf{q}'_\beta, \mathbf{q}_\alpha; z)$ and is located at $(\mathbf{q}_\alpha + \lambda_{\alpha\gamma}\mathbf{q}'_\beta)^2/2\mu_\beta + q'^2_\beta/2M_\beta - z = 0 = (\mathbf{q}'_\beta + \lambda_{\beta\gamma}\mathbf{q}_\alpha)^2/2\mu_\alpha + q^2_\alpha/2M_\alpha - z$ $(\gamma \neq \alpha \neq \beta)$ for $q_\alpha \neq \tilde{q}_\alpha$ [cf. Eq. (38)] and at $(\mathbf{q}'_\beta + \lambda_{\beta\gamma}\tilde{\mathbf{q}}_\alpha)^2 = 0$ for $q_\alpha = \tilde{q}_\alpha$, and (ii) the pole located at $z - q^2_\alpha/2M_\alpha + B_\alpha = 0$ which is due to the effective propagator $\mathcal{G}_\alpha(\mathbf{q}_\alpha; z)$, Eq. (19). It is obvious that these two singularities can never coincide for physical values of the momenta, i.e., for momenta lying on the integration contour (here the assumption $B_\nu > 0$ for $\nu = 1,2,3$, enters). Furthermore, according to Appendix B no dangerous singularity arises from $S_\alpha(z - q^2_\alpha/2M_\alpha)$.

V. SUMMARY

Let us summarize the results obtained.

(i) We have shown that in the presence of additional, purely repulsive Coulomb interactions between two or all three particles, the leading singular behavior of the nondiagonal kernels of the effective-two-body AGS equations, although being changed into a branch point as compared to the simple pole for (separable) short-range interactions, remains integrable. Hence, all solution methods developed for the nondiagonal kernels for short-range potentials are applicable also to the nondiagonal kernels for short-range plus (unscreened) repulsive Coulomb interactions.

(ii) Even in the presence of only repulsive Coulomb interactions there exists a—from the practical point of view possibly unpleasant—complication. For, in order to solve the effective-two-body AGS equations one needs to know the expressions for the effective potentials which contain the three-body Coulomb Green's function, cf. Eq. (11). (In this respect the problem resembles somewhat that encountered in the Noble-Bencze approach [7,8].) At least in principle, for the calculation of the latter perturbative methods can be employed (''quasi-Born expansion'' of the effective potentials obtained by using Eq. (16b) in Eq. (11b) [16]). But, according to Theorems 1 and 2, each term in this quasi-Born series has the same branch point singularity (except for the special points $q'_\beta = \tilde{q}_\beta$ and/or $q_\alpha = \tilde{q}_\alpha$). This could imply that, in principle, all terms should be taken into account unless, of course, their contribution to the singularity strength is found to decrease with increase of the number of intermediate-state Coulomb rescatterings, i.e., with the order of iteration. Clearly, the question of practical convergence of such an expansion requires further investigations. Note that this remark does not apply when only two of the three particles are charged because in that case the quasi-Born series rigorously collapses to the first two terms [16,15] shown in Eq. (18).

(iii) As already mentioned, the singularity caused by Eq. (38) is the leading dynamical one, i.e., it is the (energy-dependent) singularity which is strongest and closest to the integration region. In addition, each term of the quasi-Born series has its own singularities which are, however, more distant than this singularity and are, therefore, less dangerous.

At the end we mention that a following paper deals with the singularity structure of the diagonal kernels. There, it will be shown that, if the charges of all three particles are of the same sign, nonintegrable singularities appear only on the energy shell, and coincide with those investigated by Veselova [12] below the breakup threshold. These singularities can, however, be explicitly singled out and inverted as it has been done by Alt and Sandhas [15]. The off-the-energy-shell singularities of the diagonal kernel turn out to be integrable. Taken together these results imply that, after a few iterations, the appropriately modified effective-two-body AGS equations become integral equations with compact kernels.

APPENDIX A: JACOBI VARIABLES

For the convenience of the reader we collect here a few formulas relating different sets of Jacobi variables since these relations are frequently used in the present paper.

We always work in the total center-of-mass system. It is advantageous to introduce the antisymmetric symbol $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$, with $\epsilon_{\alpha\beta} = +1$ if (α,β) is a cyclic ordering of (1,2,3). Moreover, let $\alpha \neq \beta \neq \gamma \neq \alpha$. Then

$$\begin{pmatrix} \mathbf{q}_\alpha \\ \mathbf{k}_\alpha \end{pmatrix} = \begin{pmatrix} -\dfrac{\mu_\beta}{m_\gamma} & \epsilon_{\beta\alpha} \\ \epsilon_{\alpha\beta}\dfrac{\mu_\beta}{M_\alpha} & -\dfrac{\mu_\alpha}{m_\gamma} \end{pmatrix} \begin{pmatrix} \mathbf{q}_\beta \\ \mathbf{k}_\beta \end{pmatrix}, \tag{A1}$$

$$\begin{pmatrix} \mathbf{k}_\alpha \\ \mathbf{k}_\beta \end{pmatrix} = \begin{pmatrix} \epsilon_{\alpha\beta}\dfrac{\mu_\alpha}{m_\gamma} & \epsilon_{\alpha\beta} \\ \epsilon_{\beta\alpha} & \epsilon_{\beta\alpha}\dfrac{\mu_\beta}{m_\gamma} \end{pmatrix} \begin{pmatrix} \mathbf{q}_\alpha \\ \mathbf{q}_\beta \end{pmatrix}. \tag{A2}$$

Of the various relations between the different sets of coordinates we only need

$$\begin{pmatrix} \boldsymbol{\rho}_\alpha \\ \mathbf{r}_\alpha \end{pmatrix} = \begin{pmatrix} -\dfrac{\mu_\alpha}{m_\gamma} & \epsilon_{\beta\alpha}\dfrac{\mu_\beta}{M_\alpha} \\ \epsilon_{\alpha\beta} & -\dfrac{\mu_\beta}{m_\gamma} \end{pmatrix} \begin{pmatrix} \boldsymbol{\rho}_\beta \\ \mathbf{r}_\beta \end{pmatrix}. \tag{A3}$$

APPENDIX B: SINGULARITIES OF $S_\alpha(\hat{E}_\alpha + i0)$

In this appendix we investigate the singular behavior of the numerator function $S_\alpha(\hat{E}_\alpha + i0)$ of the effective free propagator, Eq. (19), where $\hat{E}_\alpha$ is the relative kinetic energy of the particles of the pair $(\beta+\gamma)$. Using the spectral decomposition of the two-particle Coulomb resolvent

$$\hat{G}^C_\alpha(\hat{E}_\alpha + i0) = \int \frac{d\mathbf{k}^0_\alpha}{(2\pi)^3} \frac{|\psi^{C(+)}_{\mathbf{k}^0_\alpha}\rangle\langle\psi^{C(+)}_{\mathbf{k}^0_\alpha}|}{[\hat{E}_\alpha + i0 - k^{0^2}_\alpha/2\mu_\alpha]}, \tag{B1}$$

and the orthogonality of two-particle Coulomb scattering wave functions, $\langle\psi^{C(+)}_{\mathbf{k}^0_\alpha}|\psi^{C(+)}_{\mathbf{k}^{0\prime}_\alpha}\rangle = \delta(\mathbf{k}^0_\alpha - \mathbf{k}^{0\prime}_\alpha)$, we find for $S^{-1}_\alpha(\hat{E}_\alpha)$ [cf. Eq. (20)]

$$S^{-1}_\alpha(\hat{E}_\alpha + i0) = -\int \frac{d\mathbf{k}^0_\alpha}{(2\pi)^3} \frac{|\langle\chi_\alpha|\psi^{C(+)}_{\mathbf{k}^0_\alpha}\rangle|^2}{[B_\alpha + k^{0^2}_\alpha/2\mu_\alpha][\hat{E}_\alpha + i0 - k^{0^2}_\alpha/2\mu_\alpha]}. \tag{B2}$$

Choosing for simplicity a form factor of the form (37) we can take over the result (C10) for the overlap $\langle\chi_\alpha|\psi^{C(+)}_{\mathbf{k}^0_\alpha}\rangle$ and obtain

$$S^{-1}_\alpha(\hat{E}_\alpha + i0) = -\int \frac{d\mathbf{k}^0_\alpha}{(2\pi)^3} \frac{2\pi\eta^0_\alpha}{[e^{2\pi\eta^0_\alpha} - 1]} \frac{e^{4\eta^0_\alpha \arctan(p_\alpha/\beta_\alpha)}}{(k^{0^2}_\alpha + \beta^2_\alpha)^2[B_\alpha + k^{0^2}_\alpha/2\mu_\alpha][\hat{E}_\alpha + i0 - k^{0^2}_\alpha/2\mu_\alpha]}, \tag{B3}$$

where η^0_α is defined in Eq. (123). Since $B_\alpha > 0$, the only singularity of the integrand in the integration region is the pole at

$$k^{0^2}_\alpha/2\mu_\alpha = \hat{E}_\alpha. \tag{B4}$$

The coincidence of this singularity with the lower limit of integration, $k^0_\alpha = 0$, will in general generate a singularity of $S_\alpha(\hat{E}_\alpha + i0)$ at the subsystem threshold energy

$$\hat{E}_\alpha = 0. \tag{B5}$$

However, for the case considered presently, namely that the charges have the same sign ($e_\beta e_\gamma > 0$, i.e., $\eta^0_\alpha > 0$), the pole at threshold is compensated by the Coulomb penetration factor which vanishes exponentially there,

$$2\pi\eta_\alpha^0 \exp\{-2\pi\eta_\alpha^0\} \overset{k_\alpha^0\to 0}{\to} 0. \tag{B6}$$

Hence, $S_\alpha^{-1}(\hat{E}_\alpha+i0)$ remains nonsingular and in fact positive definite at the subsystem threshold energy (B5). Consequently, $S_\alpha(\hat{E}_\alpha+i0)$ is regular also at $\hat{E}_\alpha=0$. For oppositely charged particles, however, i.e., for $e_\beta e_\gamma<0$, the integral (B3) goes to infinity if $\hat{E}_\alpha\to 0$, due to the divergence of the integrand at the lower limit of integration. In fact, it is easy to see that

$$S_\alpha(\hat{E}_\alpha+i0) \overset{\hat{E}_\alpha\to 0}{\sim} \frac{1}{\ln|\hat{E}_\alpha|} \quad (e_\beta e_\gamma<0), \tag{B7}$$

for form factors $\chi_\alpha(\mathbf{k}_\alpha)$ which remain regular at the origin.

APPENDIX C: LEADING SINGULARITY OF THE COULOMB-MODIFIED FORM FACTOR

Consider the ‘‘off-shell Coulomb-modified form factor’’ [cf. Eq. (54)]

$$\phi_\alpha(\mathbf{k}_\alpha;\hat{z}_\alpha):=\langle \mathbf{k}_\alpha|\hat{G}_0^{-1}(\hat{z}_\alpha)\hat{G}_\alpha^C(\hat{z}_\alpha)|\chi_\alpha\rangle, \tag{C1}$$

with $\hat{z}_\alpha\equiv\hat{E}_\alpha+i0$, $\hat{E}_\alpha$ being the energy parameter in subsystem α, for $\hat{E}_\alpha\neq k_\alpha^2/2\mu_\alpha$. As the notation indicates, all operators act in the two-body space. We prove the following.

Auxiliary Theorem. (i) For $\hat{E}_\alpha>0$, $\phi_\alpha(\mathbf{k}_\alpha;\hat{z}_\alpha)$ behaves in the (α-subsystem) on-shell limit $k_\alpha^2-2\mu_\alpha\hat{z}_\alpha\to 0$ as

$$\phi_\alpha(\mathbf{k}_\alpha;\hat{z}_\alpha) \overset{k_\alpha^2-2\mu_\alpha\hat{z}_\alpha\to 0}{\approx} [k_\alpha^2-2\mu_\alpha\hat{z}_\alpha]^{i\hat{\eta}_\alpha}\tilde{\phi}_\alpha(\mathbf{k}_\alpha;\hat{z}_\alpha), \tag{C2}$$

with $\tilde{\phi}_\alpha(\mathbf{k}_\alpha;\hat{z}_\alpha)$ remaining finite at $k_\alpha^2=2\mu_\alpha\hat{E}_\alpha$. Here, $\hat{\eta}_\alpha$ is defined in Eq. (41a).

(ii) For $\hat{z}_\alpha=\hat{E}_\alpha<0$, the same result (C2) holds but with $\hat{\eta}_\alpha=-ie_\beta e_\gamma\mu_\alpha/\sqrt{2\mu_\alpha|\hat{E}_\alpha|}$. The important special case that $|\hat{E}_\alpha|$ equals the binding energy, i.e., $\hat{E}_\alpha=-B_\alpha$, deserves extra mention. The ‘‘off-shell Coulomb-modified bound state form factor’’ behaves in the limit $k_\alpha^2+2\mu_\alpha B_\alpha\to 0$ as

$$\phi_\alpha(\mathbf{k}_\alpha;-B_\alpha) \overset{k_\alpha^2+2\mu_\alpha B_\alpha\to 0}{\approx} [k_\alpha^2+2\mu_\alpha B_\alpha]^{\eta_\alpha^{(bs)}}\tilde{\phi}_\alpha(\mathbf{k}_\alpha;-B_\alpha), \tag{C3}$$

with $\tilde{\phi}_\alpha(\mathbf{k}_\alpha;-B_\alpha)$ being regular at $k_\alpha^2+2\mu_\alpha B_\alpha=0$. The bound state Coulomb parameter $\eta_\alpha^{(bs)}$ is defined in Eq. (49).

(iii) The ‘‘off-shell Coulomb-modified zero-energy form factor’’ $\phi_\alpha(\mathbf{k}_\alpha;0)$ is nonsingular in the on-shell limit $k_\alpha\to 0$,

$$\phi_\alpha(\mathbf{k}_\alpha;0) \overset{k_\alpha\to 0}{\approx} k_\alpha^2\tilde{\phi}_\alpha(0), \tag{C4}$$

with $\tilde{\phi}_\alpha(0)$ being finite.

Comment. This theorem is valid for both repulsive and attractive Coulomb interaction.

Proofs. (i). We start from the definition (C1), with $k_\alpha\neq p_\alpha:=\sqrt{2\mu_\alpha\hat{E}_\alpha}>0$. With the help of the so-called stationary off-shell Coulomb scattering states $|\psi^{C(\pm)}_{p_\alpha,\mathbf{k}_\alpha}\rangle:=\hat{G}_\alpha^C(p_\alpha^2/2\mu_\alpha\pm i0)\hat{G}_0^{-1}(p_\alpha^2/2\mu_\alpha\pm i0)|\mathbf{k}_\alpha\rangle$, this can be rewritten as

$$\phi_\alpha(\mathbf{k}_\alpha;\hat{z}_\alpha)=\langle\psi^{C(-)}_{p_\alpha,\mathbf{k}_\alpha}|\chi_\alpha\rangle. \tag{C5}$$

The limit relation between off-shell and on-shell ($k_\alpha=p_\alpha,\mathbf{p}_\alpha\equiv p_\alpha\hat{\mathbf{k}}_\alpha$) Coulomb scattering states is [25]

$$\lim_{k_\alpha\to p_\alpha}\langle\psi^{C(-)}_{p_\alpha,\mathbf{k}_\alpha}|\Omega(k_\alpha,p_\alpha)=\langle\psi^{C(-)}_{\mathbf{p}_\alpha}|, \tag{C6}$$

where

$$\Omega(k_\alpha,p_\alpha)=[e^{-\pi\hat{\eta}_\alpha/2}\Gamma(1-i\hat{\eta}_\alpha)(4\hat{z}_\alpha)^{-i\hat{\eta}_\alpha}\times[k_\alpha^2-2\mu_\alpha\hat{z}_\alpha]^{i\hat{\eta}_\alpha}]^{-1}. \tag{C7}$$

This gives for the limit relation between the off- and the on-shell Coulomb-modified form factor

$$\phi_\alpha(\mathbf{k}_\alpha;\hat{z}_\alpha) \overset{k_\alpha^2-2\mu_\alpha\hat{z}_\alpha\to 0}{=} [\Omega(k_\alpha,p_\alpha)]^{-1}\langle\psi^{C(-)}_{\mathbf{p}_\alpha}|\chi_\alpha\rangle. \tag{C8}$$

The on-shell Coulomb-modified form factor $\langle\psi^{C(-)}_{\mathbf{p}_\alpha}|\chi_\alpha\rangle$ is well-behaved, for reasonable functions $\chi_\alpha(\mathbf{k}_\alpha)$. For instance, for a form factor of the type (37), it can be calculated analytically using the formula [26]

$$\int\frac{d\mathbf{k}_\nu}{(2\pi)^3}\frac{\psi^{C(+)}_{\mathbf{p}_\nu}(\mathbf{k}_\nu)}{[(\mathbf{k}_\nu-\mathbf{a})^2+b^2]}=e^{-\pi\eta_\nu/2}\Gamma(1+i\eta_\nu)\times\frac{[a^2-(p_\nu+ib)^2]^{i\eta_\nu}}{[(\mathbf{p}_\nu-\mathbf{a})^2+b^2]^{1+i\eta_\nu}}, \tag{C9}$$

with η_ν defined as in Eq. (123) but with k_ν^0 replaced by p_ν, yielding

$$\langle\psi^{C(-)}_{\mathbf{p}_\alpha}|\chi_\alpha\rangle=e^{-\pi\hat{\eta}_\alpha/2}\Gamma(1-i\hat{\eta}_\alpha)\frac{e^{2\hat{\eta}_\alpha \arctan p_\alpha/\beta}}{p_\alpha^2+\beta^2}. \tag{C10}$$

This function has only a pole at $p_\alpha^2=-\beta^2<0$. Hence, using Eqs. (C7) and (C8), the desired result (C2) follows.

(ii) This case has already been described in Ref. [27] (see also [28]). It is apparent that its singularity is in general branch point at $k_\alpha^2+2\mu_\alpha B_\alpha=0$, although for the case that the charges of particles β and γ are of opposite signs it may be a pole (viz., if $\eta_\alpha^{(bs)}=-n$, $n\in\mathcal{N}$ being a positive integer, as it happens for hydrogenic bound state form factors).

(iii) This assertion will be verified for the Yukawa-type *S*-wave form factor $\chi(p^2)$ introduced in Eq. (37). The explicit expression for the corresponding off-shell Coulomb-modified form factor is known for $\hat{E}_\alpha \equiv p_\alpha^2/2\mu_\alpha > 0$ (see, e.g., Ref. [25]),

$$\phi_\alpha(\mathbf{k}_\alpha;\hat{E}_\alpha+i0)=\chi(k_\alpha^2)-\chi(p_\alpha^2)\frac{p_\alpha}{k_\alpha}[F_{i\hat{\eta}_\alpha}(Ba)-F_{i\hat{\eta}_\alpha}(B/a)], \tag{C11}$$

with

$$B=\frac{\beta+ip_\alpha}{\beta-ip_\alpha},\quad a=\frac{k_\alpha-p_\alpha}{k_\alpha+p_\alpha}. \tag{C12}$$

Here, $F_{i\hat{\eta}_\alpha}(z)$ is a short notation for the hypergeometric function ${}_2F_1(1,i\hat{\eta}_\alpha;1+i\hat{\eta}_\alpha;z)$. We first need the zero-energy limit of $\phi_\alpha(\mathbf{k}_\alpha;\hat{E}_\alpha+i0)$, and hence the limit $p_\alpha \to 0$ of the hypergeometric functions, for $k_\alpha \neq 0$. This limit is nontrivial since not only $Ba=1+O(p_\alpha)$ and the same behavior for B/a [recall that $F_{i\hat{\eta}_\alpha}(z)$ has a branch point at $z=1$] but also two of the parameters go to infinity at the same time since $\hat{\eta}_\alpha \sim 1/p_\alpha$. This problem can be solved in the following way. First we note that, because the variable in either hypergeometric function approaches the value 1 in this limit, it is advantageous to represent $F_{i\hat{\eta}_\alpha}(z)$ as a function of the variable $1-z$. Such a representation is well known [29]:

$$\frac{1}{i\hat{\eta}_\alpha}F_{i\hat{\eta}_\alpha}(z)=\sum_{n=0}^{\infty}\frac{(i\hat{\eta}_\alpha)_n(1-z)^n}{n!}[\psi(n+1)-\psi(n+i\hat{\eta}_\alpha)-\ln(1-z)]. \tag{C13}$$

Here, $\psi(z)$ is the psi function and $(a)_n$ the Pochhammer symbol. By expanding all functions on the right-hand side of Eq. (C13) in powers of $1/i\hat{\eta}_\alpha$ and keeping in mind that $(1-Ba)\hat{\eta}_\alpha=O(1)$, a calculation similar to that reported in [30] yields

$$F_{i\hat{\eta}_\alpha}(Ba)=i\hat{\eta}_\alpha\Gamma(0,i\hat{\eta}_\alpha(1-Ba))e^{i\hat{\eta}_\alpha(1-Ba)}[1+O(p_\alpha)], \tag{C14}$$

with an analogous expression for $F_{i\hat{\eta}_\alpha}(B/a)$. Here, $\Gamma(0,z)$ is the incomplete gamma function. With this result the limit to zero energy can be performed in Eq. (C11). Introducing $s_\alpha := p_\alpha\hat{\eta}_\alpha = e_\beta e_\gamma \mu_\alpha$ one finds

$$\phi_\alpha(\mathbf{k}_\alpha;0):=\lim_{p_\alpha\to 0}\phi_\alpha(\mathbf{k}_\alpha;\hat{E}_\alpha+i0)=\chi(k_\alpha^2)+\chi(0)\frac{2s_\alpha}{k_\alpha}\mathrm{Im}[\Gamma(0,2s_\alpha(\beta^{-1}+ik_\alpha^{-1}))e^{2s_\alpha(\beta^{-1}+ik_\alpha^{-1})}]. \tag{C15}$$

The on-shell limit $k_\alpha \to 0$ of the off-shell Coulomb-modified zero-energy form factor (C15) is now easily found. Use of [29]

$$\Gamma(0,z)z \underset{=}{\to}{}^{\!\infty} \frac{e^{-z}}{z}\left[1-\frac{1}{z}+O\left(\frac{1}{z^2}\right)\right],\quad |\arg z|<3\pi/2, \tag{C16}$$

leads to

$$\phi_\alpha(\mathbf{k}_\alpha;0)\overset{k_\alpha\to 0}{=}\chi(0)\left\{1+k_\alpha^2\frac{d\ln\chi(k_\alpha^2)}{dk_\alpha^2}+\cdots+\frac{2s_\alpha}{k_\alpha}\mathrm{Im}\left[\frac{\beta k_\alpha}{2s_\alpha(k_\alpha+i\beta)}-\frac{(\beta k_\alpha)^2}{4s_\alpha^2(k_\alpha+i\beta)^2}+\cdots\right]\right\}=:k_\alpha^2\tilde{\phi}_\alpha(0)+o(k_\alpha^2), \tag{C17}$$

with $\tilde{\phi}_\alpha(0)=\chi(0)/\beta s_\alpha$ being nonsingular.

Note that, although the proof has relied on the explicit form (37) of $\chi(p^2)$, the result is nevertheless valid for arbitrary, at the origin nonsingular (*S*-wave) form factors since any such form factor can be represented as linear combination of functions of the type (37).

This completes the proof of the Auxiliary Theorem.

APPENDIX D: ASYMPTOTIC FORM OF THE THREE-CHARGED PARTICLE WAVE FUNCTION

The asymptotic behavior of the three-charged particle wave function in the region Ω_0, which is valid outside of the so-called singular directions characterized by $k_\nu^0 r_\nu - \mathbf{k}_\nu^0\cdot\mathbf{r}_\nu = 0$ for $\nu=1,2,3$, has been given in Refs. [22,23]:

$$\Psi^{C(+)}_{\mathbf{k}_\alpha^0,\mathbf{q}_\alpha^0}(\mathbf{r}_\alpha,\boldsymbol{\rho}_\alpha)\equiv\Psi^{C(+)}_{P^0}(X)\overset{\Omega_0}{\to}\Psi^{C,as(+)}_{P^0}(X)=e^{iP^0\cdot X}\prod_{\nu=1}^{3}e^{i\eta_\nu^0\ln(k_\nu^0 r_\nu-\mathbf{k}_\nu^0\cdot\mathbf{r}_\nu)}+O\left(\frac{1}{r_1},\frac{1}{r_2},\frac{1}{r_3}\right). \tag{D1}$$

Here, $P^0=\{\mathbf{k}_\nu^0,\mathbf{q}_\nu^0\}$ and $X=\{\mathbf{r}_\nu,\boldsymbol{\rho}_\nu\}$ are vectors in six-dimensional space, and

$$P^0\cdot X=\mathbf{k}_1^0\cdot\mathbf{r}_1+\mathbf{q}_1^0\cdot\boldsymbol{\rho}_1\equiv\mathbf{k}_2^0\cdot\mathbf{r}_2+\mathbf{q}_2^0\cdot\boldsymbol{\rho}_2\equiv\mathbf{k}_3^0\cdot\mathbf{r}_3+\mathbf{q}_3^0\cdot\boldsymbol{\rho}_3. \tag{D2}$$

Clearly, either set of Jacobi coordinates $\{\mathbf{r}_\nu,\boldsymbol{\rho}_\nu\}$ and conjugate momenta $\{\mathbf{k}_\nu^0,\mathbf{q}_\nu^0\}$, $\nu=1,2,$ or 3, can be used as the variables in $\Psi^{C(+)}_{P^0}(X)$ and $\Psi^{C,as(+)}_{P^0}(X)$. Moreover, $\eta_\nu^0=\eta_\nu(k_\nu^0)$ are the appropriate Coulomb parameters [cf. the definition (123)]. The leading asymptotic term on the right-hand side of Eq. (D1) is conventionally called three-particle Coulomb-distorted plane wave. Note that it is equivalent, in the sense of being asymptotic solution of the Schrödinger equation in Ω_0, to [31,32]

$$\Psi^{C,as(+)'}_{P^0}(X) := e^{iP^0\cdot X}\prod_{\nu=1}^{3} e^{-i\mathbf{k}^0_\nu\cdot\mathbf{r}_\nu}\psi^{C(+)}_{\mathbf{k}^0_\nu}(\mathbf{r}_\nu), \tag{D3}$$

with

$$\psi^{C(+)}_{\mathbf{k}^0_\alpha}(\mathbf{r}_\alpha) \equiv e^{i\mathbf{k}^0_\alpha\cdot\mathbf{r}_\alpha}N^0_\alpha F(-i\eta^0_\alpha,1;i(k^0_\alpha r_\alpha-\mathbf{k}^0_\alpha\cdot\mathbf{r}_\alpha)) \tag{D4}$$

being the continuum solution of the two-particle Schrödinger equation (Coulomb scattering wave function) for the particles β and γ, interacting via the Coulomb potential $V^C_\alpha(\mathbf{r}_\alpha)$ and moving asymptotically with the relative momentum $\mathbf{k}^0_\alpha$. Here, $N^0_\alpha = e^{-\pi\eta^0_\alpha/2}\Gamma(1+i\eta^0_\alpha)$, $F(a,b;x)$ is the confluent hypergeometric function, and $\Gamma(x)$ the Gamma function. That the leading asymptotic term of the wave function $\Psi^{C,as(+)'}_{P^0}(X)$ in Ω_0 is nothing but $\Psi^{C,as(+)}_{P^0}(X)$ becomes evident by recalling that the asymptotic behavior of $\psi^{C(+)}_{\mathbf{k}^0_\alpha}(\mathbf{r}_\alpha)$ in the nonsingular direction (i.e., for $k^0_\alpha r_\alpha - \mathbf{k}^0_\alpha\cdot\mathbf{r}_\alpha\neq 0$) is given by

$$\psi^{C(+)}_{\mathbf{k}^0_\alpha}(\mathbf{r}_\alpha) \overset{r_\alpha\to\infty}{=} e^{i\mathbf{k}^0_\alpha\cdot\mathbf{r}_\alpha}e^{i\eta^0_\alpha\ln(k^0_\alpha r_\alpha-\mathbf{k}^0_\alpha\cdot\mathbf{r}_\alpha)} + O\left(\frac{1}{k^0_\alpha r_\alpha-\mathbf{k}^0_\alpha\cdot\mathbf{r}_\alpha}\right). \tag{D5}$$

The asymptotic solution of the Schrödinger equation in Ω_α has been found in [11,33]. Here, however, we only need its leading term which has the form

$$\Psi^{C,as(+)}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{r}_\alpha,\boldsymbol{\rho}_\alpha) = \psi^{C(+)}_{\mathbf{k}^0_\alpha(\boldsymbol{\rho}_\alpha)}(\mathbf{r}_\alpha)e^{i\mathbf{q}^0_\alpha\cdot\boldsymbol{\rho}_\alpha}\prod_{\nu\neq\alpha} e^{i\eta^0_\nu\ln(k^0_\nu r_\nu-\mathbf{k}^0_\nu\cdot\mathbf{r}_\nu)}. \tag{D6}$$

The wave function $\psi^{C(+)}_{\mathbf{k}^0_\alpha(\boldsymbol{\rho}_\alpha)}(\mathbf{r}_\alpha)$ is the continuum solution of the two-body-like Schrödinger equation with the Coulomb potential $V^C_\alpha(\mathbf{r}_\alpha)$,

$$\left\{\frac{k^{0\,2}_\alpha(\boldsymbol{\rho}_\alpha)}{2\mu_\alpha}+\frac{\Delta_{\mathbf{r}_\alpha}}{2\mu_\alpha}-V^C_\alpha(\mathbf{r}_\alpha)\right\}\psi^{C(+)}_{\mathbf{k}^0_\alpha(\boldsymbol{\rho}_\alpha)}(\mathbf{r}_\alpha)=0, \tag{D7}$$

to the *local* energy $k^{0\,2}_\alpha(\boldsymbol{\rho}_\alpha)/2\mu_\alpha$, where the *local* momentum $\mathbf{k}^0_\alpha(\boldsymbol{\rho}_\alpha)$ is defined as

$$\mathbf{k}^0_\alpha(\boldsymbol{\rho}_\alpha) = \mathbf{k}^0_\alpha + \frac{\mathbf{a}_\alpha(\hat{\boldsymbol{\rho}}_\alpha)}{\rho_\alpha}, \tag{D8}$$

$$\mathbf{a}_\alpha(\hat{\boldsymbol{\rho}}_\alpha) = -\sum_{\nu=\beta,\gamma}\bar{\delta}_{\nu\alpha}\eta^0_\nu\lambda_{\nu\gamma}\frac{\epsilon_{\alpha\nu}\hat{\boldsymbol{\rho}}_\alpha-\hat{\mathbf{k}}^0_\nu}{1-\epsilon_{\alpha\nu}\hat{\boldsymbol{\rho}}_\alpha\cdot\hat{\mathbf{k}}^0_\nu}. \tag{D9}$$

The solution of Eq. (D7) is precisely of the form (D4) but with momentum $\mathbf{k}^0_\alpha(\boldsymbol{\rho}_\alpha)$ instead of $\mathbf{k}^0_\alpha$. We remark that the asymptotic form (D6) is valid only outside the singular directions characterized by $k^0_\nu r_\nu - \mathbf{k}^0_\nu\cdot\mathbf{r}_\nu = 0$ and $1-\epsilon_{\alpha\nu}\hat{\boldsymbol{\rho}}_\alpha\cdot\hat{\mathbf{k}}^0_\nu = 0$, for $\nu=\beta,\gamma$.

Taking into account (D5) it is apparent that in Ω_α the representation (D6) is equivalent to (cf. Ref. [11])

$$\Psi^{C,as(+)'}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{r}_\alpha,\boldsymbol{\rho}_\alpha) = e^{iP^0\cdot X}e^{-i\mathbf{k}^0_\alpha\cdot\mathbf{r}_\alpha}\psi^{C(+)}_{\mathbf{k}^0_\alpha(\boldsymbol{\rho}_\alpha)}(\mathbf{r}_\alpha) \times\prod_{\nu\neq\alpha} e^{-i\mathbf{k}^0_\nu\cdot\mathbf{r}_\nu}\psi^{C(+)}_{\mathbf{k}^0_\nu}(\mathbf{r}_\nu). \tag{D10}$$

Again, equivalence means that $\Psi^{C,as(+)}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{r}_\alpha,\boldsymbol{\rho}_\alpha)$ is the leading asymptotic term of $\Psi^{C,as(+)'}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{r}_\alpha,\boldsymbol{\rho}_\alpha)$ for $\rho_\alpha\to\infty$ and $r_\alpha/\rho_\alpha\to 0$.

Since the local momentum $\mathbf{k}^0_\alpha(\boldsymbol{\rho}_\alpha)$ coincides with the asymptotic momentum $\mathbf{k}^0_\alpha$ up to terms of the order $O(1/\rho_\alpha)$ [recall its definition (D8)], $\psi^{C(+)}_{\mathbf{k}^0_\alpha(\boldsymbol{\rho}_\alpha)}(\mathbf{r}_\alpha)$ can be written in Ω_α as

$$\psi^{C(+)}_{\mathbf{k}^0_\alpha(\boldsymbol{\rho}_\alpha)}(\mathbf{r}_\alpha) = \psi^{C(+)}_{\mathbf{k}^0_\alpha}(\mathbf{r}_\alpha) + O\left(\frac{1}{\rho_\alpha}\right). \tag{D11}$$

Consequently, in Ω_α we can write for $\Psi^{C,as(+)}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{r}_\alpha,\boldsymbol{\rho}_\alpha)$:

$$\Psi^{C,as(+)}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{r}_\alpha,\boldsymbol{\rho}_\alpha) = \psi^{C(+)}_{\mathbf{k}^0_\alpha}(\mathbf{r}_\alpha)e^{i\mathbf{q}^0_\alpha\cdot\boldsymbol{\rho}_\alpha} \times\prod_{\nu\neq\alpha} e^{i\eta^0_\nu\ln(k^0_\nu r_\nu-\mathbf{k}^0_\nu\cdot\mathbf{r}_\nu)} + O\left(\frac{1}{\rho_\alpha}\right), \tag{D12}$$

and for $\Psi^{C,as(+)'}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{r}_\alpha,\boldsymbol{\rho}_\alpha)$:

$$\Psi^{C,as(+)'}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{r}_\alpha,\boldsymbol{\rho}_\alpha) = \Psi^{C,as(+)'}_{P^0}(X) + O\left(\frac{1}{\rho_\alpha}\right). \tag{D13}$$

Note that the term $O(1/\rho_\alpha)$ is decisive for $\Psi^{C,as(+)}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{r}_\alpha,\boldsymbol{\rho}_\alpha)$ to satisfy the Schrödinger equation in the asymptotic domain Ω_α up to terms $O(1/\rho^2_\alpha)$ [11]. However, when looking for the leading singular part of $\tilde{\mathcal{V}}^{(\alpha\beta)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$ it plays no role and can therefore be neglected. To see this, substitute in the integrand of Eq. (106) for the exact three-body wave function $\Psi^{C(+)*}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{r}_\beta,\boldsymbol{\rho}_\beta) \equiv \Psi^{C(+)*}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{r}_\alpha,\boldsymbol{\rho}_\alpha)$, where on the right-hand side the variables $\{\mathbf{r}_\alpha,\boldsymbol{\rho}_\alpha\}$ are considered expressed by the set $\{\mathbf{r}_\beta,\boldsymbol{\rho}_\beta\}$, its asymptotic expression $\Psi^{C,as(+)*}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{r}_\alpha,\boldsymbol{\rho}_\alpha)$. In this context it is helpful to keep in mind that any Jacobian vector pair $\{\mathbf{k}_\nu,\mathbf{q}_\nu\}, \nu=1,2,$ or 3, can be chosen as the variables in the three-body wave function. The same remark applies also to the integration variables $\{\mathbf{k}^0_\nu,\mathbf{q}^0_\nu\}$ in the spectral decomposition (82). Since the singularity we are looking for, is generated by the divergence of the integral over $\mathbf{r}_\beta$ for $r_\beta\to\infty$, we investigate the behavior of the integrand for large r_β. Consider first the asymptotic region Ω_α where $r_\beta\to\infty$ implies $\rho_\alpha\approx r_\beta$. Use of Eq. (D12) shows that

$$\Psi^{C(+)*}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{r}_\alpha,\boldsymbol{\rho}_\alpha)V^C_\beta(\mathbf{r}_\beta)\overset{r_\beta\to\infty}{=}O\left(\frac{1}{r_\beta}\right)+O\left(\frac{1}{r^2_\beta}\right)+o\left(\frac{1}{r^2_\beta}\right), \tag{D14}$$

where the terms $\sim O(1/\rho_\alpha)$ in the asymptotic wave function, when multiplied with $V^C_\beta(\mathbf{r}_\beta)$, contribute to the $O(1/r^2_\beta)$. It is evident that in the integration over $\mathbf{r}_\beta$, because of its faster asymptotic decrease the latter contribution gives rise to a term which is less singular than that arising from $O(1/r_\beta)$; hence it can be discarded when looking for the leading singular part of $\tilde{\mathcal{V}}^{(\alpha\beta)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$. Since $\Psi^{C,as(+)*}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{r}_\alpha,\boldsymbol{\rho}_\alpha)$ and $\Psi^{C,as(+)*'}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{r}_\alpha,\boldsymbol{\rho}_\alpha)$ are equivalent in Ω_α, also in Eq. (D13) the term $\sim O(1/\rho_\alpha)$ can be omitted. Thus, when acting on the Coulomb potential $V^C_\beta(\mathbf{r}_\beta)$ the wave function $\Psi^{C(+)}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{r}_\beta,\boldsymbol{\rho}_\beta)$ may be approximated in Ω_α by

$$\Psi^{C(+)}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{r}_\beta,\boldsymbol{\rho}_\beta)\overset{\Omega_\alpha}{\approx}\Psi^{C,as(+)'}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{r}_\alpha,\boldsymbol{\rho}_\alpha)\overset{\Omega_\alpha}{\approx}\Psi^{C,as(+)'}_{P^0}(X). \tag{D15}$$

Similarly, in the asymptotic domain Ω_γ we find

$$\Psi^{C(+)}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{r}_\beta,\boldsymbol{\rho}_\beta)\overset{\Omega_\gamma}{\approx}\Psi^{C,as(+)'}_{\mathbf{k}^0_\gamma,\mathbf{q}^0_\gamma}(\mathbf{r}_\gamma,\boldsymbol{\rho}_\gamma)\overset{\Omega_\gamma}{\approx}\Psi^{C,as(+)'}_{P^0}(X), \tag{D16}$$

where in the first approximate equality the γ-channel coordinates $\{\mathbf{r}_\gamma,\boldsymbol{\rho}_\gamma\}$ are considered expressed by $\{\mathbf{r}_\beta,\boldsymbol{\rho}_\beta\}$, and the momenta $\{\mathbf{k}^0_\gamma,\mathbf{q}^0_\gamma\}$ by $\{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha\}$.

Thus we have derived the very important result: in all three asymptotic domains Ω_0, Ω_α, and Ω_γ, and hence also in all of ω_β [cf. Eq. (111)], when looking for the main singular part of the amplitude $\tilde{\mathcal{V}}^{(\alpha\beta)}_{\beta\alpha}(\mathbf{q}'_\beta,\mathbf{q}_\alpha;z)$ the wave function $\Psi^{C(+)}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{r}_\beta,\boldsymbol{\rho}_\beta)$ may be approximated by

$$\Psi^{C(+)}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{r}_\beta,\boldsymbol{\rho}_\beta)\overset{\omega_\beta}{\approx}\Psi^{C,as(+)'}_{P^0}(X). \tag{D17}$$

An analogous consideration shows that in the asymptotic domain ω'_α the wave function $\Psi^{C(+)}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{r}'_\alpha,\boldsymbol{\rho}'_\alpha)$ may be approximated by

$$\Psi^{C(+)}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{r}'_\alpha,\boldsymbol{\rho}'_\alpha)\overset{\omega'_\alpha}{\approx}\Psi^{C,as(+)'}_{P^0}(X'). \tag{D18}$$

APPENDIX E: LEADING SINGULAR PART OF THE INTEGRAL J_α

In this appendix we derive the leading singular part of J_α, Eq. (119), near the singularity caused by the coincidence of the singularities of its integrand given in Eq. (120) with the forward-scattering singularities of the wave function $\Psi^{C,as(+)'*}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{k}''_\beta,\mathbf{q}''_\beta)$ which occur at Eq. (121).

To begin with we introduce the following notation. We are only interested in the leading singularity of a given quantity F. Hence, throughout this section we denote the leading singular part of F by $F^{(s)}$ so that

$$F=F^{(s)}+\text{less singular}+\text{nonsingular terms}. \tag{E1}$$

(a) We start by investigating the leading singular part of J_α for $q_\alpha\neq\tilde{q}_\alpha$ when the singular behavior of the Coulomb-modified form factor is as shown in Eq. (62a). First of all we recall that the ''reduced Coulomb-modified form factor'' $\tilde{\phi}_\alpha[\epsilon_{\alpha\beta}(\mathbf{q}''_\beta+\lambda_{\beta\gamma}\mathbf{q}_\alpha)]$ is regular near the forward-scattering singularities of $\Psi^{C,as(+)'*}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{k}''_\beta,\mathbf{q}''_\beta)$ and the singularities (120). Hence it can be taken out from under the integral signs in Eq. (119) at the point $\mathbf{q}''_\beta=\mathbf{q}^0_\beta$ which corresponds to the forward-scattering singularity of $\Psi^{C,as(+)'*}_{\mathbf{k}^0_\alpha,\mathbf{q}^0_\alpha}(\mathbf{k}''_\beta,\mathbf{q}''_\beta)$. Next consider the following integral over $\mathbf{q}''_\beta$:

$$L_1:=\int\frac{d\mathbf{q}''_\beta}{(2\pi)^3}\frac{\psi^{C(+)*}_{\mathbf{k}^0_\alpha}[\mathbf{k}+\mathbf{k}^0_\alpha-\mathbf{k}^0_\beta-\epsilon_{\beta\alpha}(\mathbf{q}''_\alpha+\mathbf{q}''_\beta+\mathbf{q}^0_\gamma)]}{[(\mathbf{q}''_\beta+\lambda_{\beta\gamma}\mathbf{q}_\alpha)^2-2\mu_\alpha\hat{z}_\alpha]^{1-i\hat{\eta}_\alpha}}. \tag{E2}$$

The Coulomb parameter $\hat{\eta}_\alpha$ is defined in Eq. (41a). In the following we need the basic integral (for $0\leqslant\arg u<2\pi,\arg u\neq\pi$)

$$\int_0^\infty\frac{\mathrm{d}y}{y^{1-i\lambda}}\frac{1}{(u+y)^{1-i\mu}}=\frac{1}{u^{1-i(\lambda+\mu)}}\frac{\Gamma(i\lambda)\Gamma(1-i(\lambda+\mu))}{\Gamma(1-i\mu)},\quad 0<\mathrm{Re}(i\lambda)<\mathrm{Re}(1-i\mu), \tag{E3}$$

which represents a slight generalization of Eq. 3.194. of Ref. [34], in several variations. We first use it as integral representation for $1/[(\mathbf{q}''_\beta+\lambda_{\beta\gamma}\mathbf{q}_\alpha)^2-2\mu_\alpha\hat{z}_\alpha]^{1-i\hat{\eta}_\alpha}$ to rewrite the right-hand side of Eq. (E2) as (recall that $\hat{z}_\alpha=\hat{E}_\alpha+i\epsilon$ with $\epsilon>0$ for $\hat{E}_\alpha>0$, and that we consider only particles with charges of the same sign)

$$L_1=\frac{1}{\Gamma(i\hat{\eta}_\alpha)\Gamma(1-i\hat{\eta}_\alpha)}\int_0^\infty\frac{\mathrm{d}y}{y^{1-i\hat{\eta}_\alpha}}\int\frac{d\mathbf{q}''_\beta}{(2\pi)^3}\frac{\psi^{C(+)*}_{\mathbf{k}^0_\alpha}[\mathbf{k}+\mathbf{k}^0_\alpha-\mathbf{k}^0_\beta-\epsilon_{\beta\alpha}(\mathbf{q}''_\alpha+\mathbf{q}''_\beta+\mathbf{q}^0_\gamma)]}{[(\mathbf{q}''_\beta+\lambda_{\beta\gamma}\mathbf{q}_\alpha)^2-2\mu_\alpha\hat{z}_\alpha+y]}. \tag{E4}$$

The integral over $\mathbf{q}''_\beta$ can now be done explicitly with the help of Eq. (C9), yielding

$$L_1=\frac{e^{-\pi\eta_\alpha^0/2}\Gamma(1-i\eta_\alpha^0)}{\Gamma(i\hat{\eta}_\alpha)\Gamma(1-i\hat{\eta}_\alpha)}\int_0^\infty\frac{\mathrm{d}y}{y^{1-i\hat{\eta}_\alpha}}\frac{[(\mathbf{k}-\mathbf{K}(\mathbf{q}_\alpha'')+\mathbf{k}_\alpha^0)^2-(k_\alpha^0-i\sqrt{y-2\mu_\alpha\hat{z}_\alpha^*})^2]^{-i\eta_\alpha^0}}{[(\mathbf{k}-\mathbf{K}(\mathbf{q}_\alpha''))^2+y-2\mu_\alpha\hat{z}_\alpha^*]^{1-i\eta_\alpha^0}},\tag{E5}$$

with the abbreviation

$$\mathbf{K}(\mathbf{q}_\alpha''):=\mathbf{k}_\beta^0+\epsilon_{\beta\alpha}(\mathbf{q}_\alpha''+\mathbf{q}_\gamma^0-\lambda_{\beta\gamma}\mathbf{q}_\alpha);\tag{E6}$$

η_α^0 is defined in Eq. (123). Introducing the new variable v via

$$y=[(\mathbf{k}-\mathbf{K}(\mathbf{q}_\alpha''))^2-2\mu_\alpha\hat{z}_\alpha^*]v\tag{E7}$$

and retaining the leading singular parts only we derive, using again the integral (E3),

$$L_1^{(s)}=\frac{e^{-\pi\eta_\alpha^0/2}\Gamma(1-i(\hat{\eta}_\alpha+\eta_\alpha^0))}{\Gamma(1-i\hat{\eta}_\alpha)}\frac{[(\mathbf{k}-\mathbf{K}(\mathbf{q}_\alpha'')+\mathbf{k}_\alpha^0)^2-(k_\alpha^0-i\sqrt{-2\mu_\alpha\hat{z}_\alpha^*})^2]^{-i\eta_\alpha^0}}{[(\mathbf{k}-\mathbf{K}(\mathbf{q}_\alpha''))^2-2\mu_\alpha\hat{z}_\alpha^*]^{1-i(\hat{\eta}_\alpha+\eta_\alpha^0)}}.\tag{E8}$$

This result when inserted into expression (119) for J_α yields

$$J_\alpha^{(s)}=\frac{e^{-\pi\eta_\alpha^0/2}\Gamma(1-i(\hat{\eta}_\alpha+\eta_\alpha^0))}{\Gamma(1-i\hat{\eta}_\alpha)}\tilde{\phi}_\alpha[\epsilon_{\alpha\beta}(\mathbf{q}_\beta^0+\lambda_{\beta\gamma}\mathbf{q}_\alpha)]\int\frac{d\mathbf{q}_\alpha''}{(2\pi)^3}\int\frac{d\mathbf{k}}{(2\pi)^3}\psi_{\mathbf{k}_\beta^0}^{C(+)*}(\mathbf{k})\psi_{\mathbf{k}_\gamma^0}^{C(+)*}$$
$$\times(\mathbf{k}+\mathbf{k}_\gamma^0-\mathbf{k}_\beta^0+\epsilon_{\alpha\beta}(\mathbf{q}_\alpha''-\mathbf{q}_\alpha^0))\frac{4\pi e_\alpha e_\gamma}{(\mathbf{q}_\alpha''-\mathbf{q}_\alpha)^2}\frac{[(\mathbf{k}-\mathbf{K}(\mathbf{q}_\alpha'')+\mathbf{k}_\alpha^0)^2-(k_\alpha^0-i\sqrt{-2\mu_\alpha\hat{z}_\alpha^*})^2]^{-i\eta_\alpha^0}}{[(\mathbf{k}-\mathbf{K}(\mathbf{q}_\alpha''))^2-2\mu_\alpha\hat{z}_\alpha^*]^{1-i(\hat{\eta}_\alpha+\eta_\alpha^0)}}.\tag{E9}$$

When looking for the leading singular part of Eq. (E9), only those functions of the integrand which contain the leading singularities have to be retained under the integral sign. This implies that the term $[(\mathbf{k}-\mathbf{K}(\mathbf{q}_\alpha'')+\mathbf{k}_\alpha^0)^2-(k_\alpha^0-i\sqrt{-2\mu_\alpha\hat{z}_\alpha^*})^2]^{-i\eta_\alpha^0}$ can be taken out from under the integral sign at the forward-scattering singularities $\mathbf{q}_\alpha''=\mathbf{q}_\alpha^0$ and $\mathbf{k}=\mathbf{k}_\beta^0$ of the Coulomb wave functions. To proceed further it proves convenient to introduce a new variable

$$\mathbf{p}=\mathbf{k}+\mathbf{k}_\gamma^0-\mathbf{k}_\beta^0+\epsilon_{\alpha\beta}(\mathbf{q}_\alpha''-\mathbf{q}_\alpha^0).\tag{E10}$$

Then Eq. (E9) takes the form

$$J_\alpha^{(s)}=4\pi e_\alpha e_\gamma e^{-\pi\eta_\alpha^0/2}\frac{\Gamma(1-i(\hat{\eta}_\alpha+\eta_\alpha^0))}{\Gamma(1-i\hat{\eta}_\alpha)}\tilde{\phi}_\alpha(\epsilon_{\alpha\beta}\mathbf{\Delta}_\beta^0+\mathbf{k}_\alpha)[(\mathbf{k}_\alpha^0-\epsilon_{\alpha\beta}\mathbf{\Delta}_\beta^0-\mathbf{k}_\alpha)^2-(k_\alpha^0$$
$$-i\sqrt{-2\mu_\alpha\hat{z}_\alpha^*})^2]^{-i\eta_\alpha^0}\int\frac{d\mathbf{p}}{(2\pi)^3}\frac{\psi_{\mathbf{k}_\gamma^0}^{C(+)*}(\mathbf{p})}{[(\mathbf{p}-\mathbf{k}_\gamma^0-\epsilon_{\alpha\beta}\mathbf{\Delta}_\beta^0-\mathbf{k}_\alpha)^2-2\mu_\alpha\hat{z}_\alpha^*]^{1-i(\hat{\eta}_\alpha+\eta_\alpha^0)}}\int\frac{d\mathbf{k}\;\psi_{\mathbf{k}_\beta^0}^{C(+)*}(\mathbf{k})}{(2\pi)^3[\mathbf{p}-\mathbf{k}-\mathbf{k}_\gamma^0+\mathbf{k}_\beta^0+\epsilon_{\alpha\beta}\mathbf{\Delta}_\alpha^0]^2}.\tag{E11}$$

Here we have used the notations (89) and (90). Integration over $\mathbf{k}$ with the help of Eq. (C9) yields

$$J_\alpha^{(s)}=4\pi e_\alpha e_\gamma e^{-\pi(\eta_\alpha^0+\eta_\beta^0)/2}\frac{\Gamma(1-i\eta_\beta^0)\Gamma(1-i(\hat{\eta}_\alpha+\eta_\alpha^0))}{\Gamma(1-i\hat{\eta}_\alpha)}\tilde{\phi}_\alpha(\epsilon_{\alpha\beta}\mathbf{\Delta}_\beta^0+\mathbf{k}_\alpha)$$
$$\times[(\mathbf{k}_\alpha^0-\epsilon_{\alpha\beta}\mathbf{\Delta}_\beta^0-\mathbf{k}_\alpha)^2-(k_\alpha^0-i\sqrt{-2\mu_\alpha\hat{z}_\alpha^*})^2]^{-i\eta_\alpha^0}$$
$$\times\lim_{\varepsilon\to+0}\int\frac{d\mathbf{p}}{(2\pi)^3}\frac{[(\mathbf{p}-\mathbf{k}_\gamma^0+\mathbf{k}_\beta^0+\epsilon_{\alpha\beta}\mathbf{\Delta}_\alpha^0)^2-(k_\beta^0-i\varepsilon)^2]^{-i\eta_\beta^0}}{[(\mathbf{p}-\mathbf{k}_\gamma^0+\epsilon_{\alpha\beta}\mathbf{\Delta}_\alpha^0)^2+\varepsilon^2]^{1-i\eta_\beta^0}}\frac{\psi_{\mathbf{k}_\gamma^0}^{C(+)*}(\mathbf{p})}{[(\mathbf{p}-\mathbf{k}_\gamma^0-\epsilon_{\alpha\beta}\mathbf{\Delta}_\beta^0-\mathbf{k}_\alpha)^2-2\mu_\alpha\hat{z}_\alpha^*]^{1-i(\hat{\eta}_\alpha+\eta_\alpha^0)}}.\tag{E12}$$

Again only those terms of the integrand need to be left under the integral sign which contain the leading singularities; consequently, the square-braketed numerator function can be taken out at $\mathbf{p}=\mathbf{k}_\gamma^0$. Then in the main order one obtains for $J_\alpha^{(s)}$

$$J_{\alpha}^{(s)}=4\pi e_{\alpha}e_{\gamma}e^{-\pi(\eta_{\alpha}^{0}+\eta_{\beta}^{0})/2}\frac{\Gamma(1-i\eta_{\beta}^{0})\Gamma(1-i(\hat{\eta}_{\alpha}+\eta_{\alpha}^{0}))}{\Gamma(1-i\hat{\eta}_{\alpha})}\tilde{\phi}_{\alpha}(\epsilon_{\alpha\beta}\mathbf{\Delta}_{\beta}^{0}+\mathbf{k}_{\alpha})[(\mathbf{k}_{\alpha}^{0}-\epsilon_{\alpha\beta}\mathbf{\Delta}_{\beta}^{0}-\mathbf{k}_{\alpha})^{2}-(k_{\alpha}^{0}-i\sqrt{-2\mu_{\alpha}\hat{z}_{\alpha}^{*}})^{2}]^{-i\eta_{\alpha}^{0}}$$
$$\times[(\mathbf{k}_{\beta}^{0}+\epsilon_{\alpha\beta}\mathbf{\Delta}_{\alpha}^{0})^{2}-k_{\beta}^{0\,2}]^{-i\eta_{\beta}^{0}}I_{\alpha}, \tag{E13}$$

with

$$I_{\alpha}:=\int\frac{d\mathbf{p}}{(2\pi)^{3}}\frac{\psi_{\mathbf{k}_{\gamma}^{0}}^{C(+)*}(\mathbf{p})}{[(\mathbf{p}-\mathbf{k}_{\gamma}^{0}+\epsilon_{\alpha\beta}\mathbf{\Delta}_{\alpha}^{0})^{2}]^{1-i\eta_{\beta}^{0}}}\frac{1}{[(\mathbf{p}-\mathbf{k}_{\gamma}^{0}-\epsilon_{\alpha\beta}\mathbf{\Delta}_{\beta}^{0}-\mathbf{k}_{\alpha})^{2}-2\mu_{\alpha}\hat{z}_{\alpha}^{*}]^{1-i(\hat{\eta}_{\alpha}+\eta_{\alpha}^{0})}}. \tag{E14}$$

To evaluate this latter integral we use Eq. (E3) twice as integral transform of the denominator functions, once with $u\equiv\omega_{1}(\mathbf{p}):=(\mathbf{p}-\mathbf{k}_{\gamma}^{0}+\epsilon_{\alpha\beta}\mathbf{\Delta}_{\alpha}^{0})^{2}$, and once with $u\equiv\omega_{2}(\mathbf{p}):=(\mathbf{p}-\mathbf{k}_{\gamma}^{0}-\epsilon_{\alpha\beta}\mathbf{\Delta}_{\beta}^{0}-\mathbf{k}_{\alpha})^{2}-2\mu_{\alpha}\hat{z}_{\alpha}^{*}$. This gives

$$I_{\alpha}=-\frac{1}{\pi^{2}}\sinh[\pi(\hat{\eta}_{\alpha}+\eta_{\alpha}^{0})]\sinh(\pi\eta_{\beta}^{0})\int_{0}^{\infty}\mathrm{d}x x^{-1+i(\hat{\eta}_{\alpha}+\eta_{\alpha}^{0})}\int_{0}^{\infty}\mathrm{d}y y^{-1+i\eta_{\beta}^{0}}\int\frac{d\mathbf{p}}{(2\pi)^{3}}\frac{\psi_{\mathbf{k}_{\gamma}^{0}}^{C(+)*}(\mathbf{p})}{(\omega_{1}(\mathbf{p})+y)(\omega_{2}(\mathbf{p})+x)}. \tag{E15}$$

The integral over $\mathbf{p}$ is first rewritten by means of the Feynman integral

$$\frac{1}{ab}=\int_{0}^{1}\frac{\mathrm{d}t}{[ta+(1-t)b]^{2}}=-\lim_{\varepsilon_{1}\to 0}\frac{d}{d\varepsilon_{1}}\int_{0}^{1}\frac{\mathrm{d}t}{[ta+(1-t)b+\varepsilon_{1}]}. \tag{E16}$$

Thus

$$I_{\alpha}=-\frac{1}{\pi^{2}}\sinh[\pi(\hat{\eta}_{\alpha}+\eta_{\alpha}^{0})]\sinh(\pi\eta_{\beta}^{0})\int_{0}^{\infty}\mathrm{d}x x^{-1+i(\hat{\eta}_{\alpha}+\eta_{\alpha}^{0})}\int_{0}^{\infty}\mathrm{d}y y^{-1+i\eta_{\beta}^{0}}\int_{0}^{1}\mathrm{d}t L_{2}(t), \tag{E17}$$

with

$$L_{2}(t):=-\lim_{\varepsilon_{1}\to+0}\frac{d}{d\varepsilon_{1}}\int\frac{d\mathbf{p}}{(2\pi)^{3}}\frac{\psi_{\mathbf{k}_{\gamma}^{0}}^{C(+)*}(\mathbf{p})}{[t(\omega_{1}(\mathbf{p})+y)+(1-t)(\omega_{2}(\mathbf{p})+x)+\varepsilon_{1}]}. \tag{E18}$$

Since the denominator in the integrand of Eq. (E18) can be represented in the form $[\mathbf{p}-\mathbf{c}(t)]^{2}+\chi^{2}(t)+\varepsilon_{1}$, with

$$\mathbf{c}(t)=\mathbf{k}_{\gamma}^{0}-\epsilon_{\alpha\beta}\mathbf{\Delta}_{\alpha}^{0}t+(1-t)[\epsilon_{\alpha\beta}\mathbf{\Delta}_{\beta}^{0}+\mathbf{k}_{\alpha}] \tag{E19}$$

and

$$\chi^{2}(t)=t(1-t)(\mathbf{\Delta}_{\alpha}^{0}+\mathbf{\Delta}_{\beta}^{0}+\epsilon_{\alpha\beta}\mathbf{k}_{\alpha})^{2}+ty+(1-t)(x-2\mu_{\alpha}\hat{z}_{\alpha}^{*}), \tag{E20}$$

we can easily integrate over $\mathbf{p}$ using Eq. (C9) and obtain

$$L_{2}(t)=,-e^{-\pi\eta_{\gamma}^{0}/2}\Gamma(1-i\eta_{\gamma}^{0})\lim_{\varepsilon_{1}\to+0}\frac{d}{d\varepsilon_{1}}\frac{[c^{2}(t)-(k_{\gamma}^{0}-i\sqrt{\chi^{2}(t)+\varepsilon_{1}})^{2}]^{-i\eta_{\gamma}^{0}}}{[t((\mathbf{\Delta}_{\alpha}^{0})^{2}+y)+(1-t)((\Delta_{\beta}^{0})^{2}+2\epsilon_{\alpha\beta}\mathbf{\Delta}_{\beta}^{0}\cdot\mathbf{k}_{\alpha}+\sigma_{\alpha}(q_{\alpha};z)+x)+\varepsilon_{1}]^{1-i\eta_{\gamma}^{0}}}. \tag{E21}$$

Here we took into account that

$$(\mathbf{\Delta}_{\beta}^{0}+\epsilon_{\alpha\beta}\mathbf{k}_{\alpha})^{2}-2\mu_{\alpha}\hat{z}_{\alpha}=(\Delta_{\beta}^{0})^{2}+2\epsilon_{\alpha\beta}\mathbf{\Delta}_{\beta}^{0}\cdot\mathbf{k}_{\alpha}+\sigma_{\alpha}(q_{\alpha};z). \tag{E22}$$

Differentiation of the denominator with respect to ε_{1} gives the leading singular part $L_{2}^{(s)}$ of L_{2} in the limit $\sigma_{\alpha}(q_{\alpha};z)\to 0$ as

$$L_{2}^{(s)}(t)=\frac{e^{-\pi\eta_{\gamma}^{0}/2}\Gamma(2-i\eta_{\gamma}^{0})[c^{2}(t)-(k_{\gamma}^{0}-i\chi(t))^{2}]^{-i\eta_{\gamma}^{0}}}{[t((\Delta_{\alpha}^{0})^{2}+y)+(1-t)((\Delta_{\beta}^{0})^{2}+2\epsilon_{\alpha\beta}\mathbf{\Delta}_{\beta}^{0}\cdot\mathbf{k}_{\alpha}+\sigma_{\alpha}(q_{\alpha};z)+x)]^{2-i\eta_{\gamma}^{0}}}. \tag{E23}$$

Let us now insert this result for L_{2} in Eq. (E17), and consider the integrals over y and x. The leading singular part $I_{\alpha}^{(s)}$ of I_{α} is generated by the singularities of the denominator at the lower limits $y=0$ and $x=0$ of the integrals over y and x, respectively. Consequently, the numerator can be taken out under from the integrals over y and x at $x=y=0$. The integration over y and x is now straightforward. Introducing the variable v as

$$y=\left((\Delta_\alpha^0)^2+\frac{1-t}{t}[(\Delta_\beta^0)^2+2\epsilon_{\alpha\beta}\mathbf{\Delta}_\beta^0\cdot\mathbf{k}_\alpha+\sigma_\alpha(q_\alpha;z)+x]\right)v, \tag{E24}$$

we derive with the help of Eq. (E3) for $I_\alpha^{(s)}$ in the leading order

$$I_\alpha^{(s)}=-\frac{1}{\pi^2}\sinh[\pi(\hat{\eta}_\alpha+\eta_\alpha^0)]\sinh(\pi\eta_\beta^0)e^{-\pi\eta_\gamma^0/2}\Gamma(i\eta_\beta^0)\Gamma(2-i(\eta_\beta^0+\eta_\gamma^0))\int_0^1 \mathrm{d}t\, t^{-i\eta_\beta^0}[c^2(t)-(k_\gamma^0-i\chi_0(t))^2]^{-i\eta_\gamma^0}$$
$$\times\int_0^\infty \mathrm{d}x\frac{x^{-1+i(\hat{\eta}_\alpha+\eta_\alpha^0)}}{[t(\Delta_\alpha^0)^2+(1-t)((\Delta_\beta^0)^2+2\epsilon_{\alpha\beta}\mathbf{\Delta}_\beta^0\cdot\mathbf{k}_\alpha+\sigma_\alpha(q_\alpha;z)+x)]^{2-i(\eta_\beta^0+\eta_\gamma^0)}}. \tag{E25}$$

Here, we have used the short-hand notation $\chi_0^2(t):=\lim_{x,y\to 0}\chi^2(t)$. Similarly, defining a variable u via

$$x=\left(\frac{t}{1-t}(\Delta_\alpha^0)^2+(\Delta_\beta^0)^2+2\epsilon_{\alpha\beta}\mathbf{\Delta}_\beta^0\cdot\mathbf{k}_\alpha+\sigma_\alpha(q_\alpha;z)\right)u, \tag{E26}$$

an analogous integration over x yields

$$I_\alpha^{(s)}=e^{-\pi\eta_\gamma^0/2}\frac{\Gamma\left(2-i\left(\hat{\eta}_\alpha+\sum_\nu\eta_\nu^0\right)\right)}{\Gamma(1-i\eta_\beta^0)\Gamma(1-i(\hat{\eta}_\alpha+\eta_\alpha^0))}\int_0^1\mathrm{d}t\frac{t^{-i\eta_\beta^0}(1-t)^{-i(\hat{\eta}_\alpha+\eta_\alpha^0)}[c^2(t)-(k_\gamma^0-i\chi_0(t))^2]^{-i\eta_\gamma^0}}{[t\Delta_\alpha{}^2+(1-t)((\Delta_\beta^0)^2+2\epsilon_{\alpha\beta}\mathbf{\Delta}_\beta^0\cdot\mathbf{k}_\alpha+\sigma_\alpha(q_\alpha;z))]^{2-i(\hat{\eta}_\alpha+\sum_\nu\eta_\nu^0)}}. \tag{E27}$$

Recall Eq. (92) which shows that $\Delta_\alpha^0\sim\Delta_\beta^0\sim\sigma_\alpha$. Thus, the leading singularity of $I_\alpha^{(s)}$ in the limit $\sigma_\alpha(q_\alpha;z)\to 0$ is due to the zero of the denominator in Eq. (E27) at the upper limit of integration. For, at $t=1$ the denominator is proportional to $(\Delta_\alpha^0)^2\sim\sigma_\beta^2$ while at $t=0$ the leading term of the denominator is proportional to σ_α. Consequently, the square-bracketed term in the numerator in Eq. (E27), being a less singular function at $t=1$ than the denominator can be taken out of the integral at $t=1$. Using Eqs. (E19) and (E20) one finds in the leading order

$$\lim_{t\to 1}[c^2(t)-(k_\gamma^0-i\sqrt{\chi_0(t)})^2]\approx -2\epsilon_{\alpha\beta}\mathbf{k}_\gamma^0\cdot\mathbf{\Delta}_\alpha^0. \tag{E28}$$

The remaining integral over t can be evaluated explicitly [see [34], Eq. (3.197.3)],

$$I_\alpha^{(s)}=e^{-\pi\eta_\gamma^0/2}\frac{\Gamma\left(2-i\left(\hat{\eta}_\alpha+\sum_\nu\eta_\nu^0\right)\right)}{\Gamma(2-i(\hat{\eta}_\alpha+\eta_\alpha^0+\eta_\beta^0))}\frac{[-2\epsilon_{\alpha\beta}\mathbf{k}_\gamma^0\cdot\mathbf{\Delta}_\alpha^0]^{-i\eta_\gamma^0}}{[(\Delta_\beta^0)^2+2\epsilon_{\alpha\beta}\mathbf{\Delta}_\beta^0\cdot\mathbf{k}_\alpha+\sigma_\alpha(q_\alpha;z)]^{2-i(\hat{\eta}_\alpha+\sum_\nu\eta_\nu^0)}}{}_2F_1\left(2-i\left(\hat{\eta}_\alpha+\sum_\nu\eta_\nu^0\right),1-i\eta_\beta^0;2\right.$$
$$\left.-i(\hat{\eta}_\alpha+\eta_\alpha^0+\eta_\beta^0);1-u\right), \tag{E29}$$

with

$$u:=\frac{(\Delta_\alpha^0)^2}{(\Delta_\beta^0)^2+2\epsilon_{\alpha\beta}\mathbf{\Delta}_\beta^0\cdot\mathbf{k}_\alpha+\sigma_\alpha(q_\alpha;z)}. \tag{E30}$$

Here, ${}_2F_1(a,b;c;z)$ is the hypergeometric function. By using, e.g., the transformation formula (15.3.6) of [29] for ${}_2F_1(a,b;c;1-u)$ one easily extracts the leading term for $u\to 0$ [recall Eq. (92)]

$$I_\alpha^{(s)}=e^{-\pi\eta_\gamma^0/2}\frac{\Gamma(1-i(\eta_\beta^0+\eta_\gamma^0))}{\Gamma(1-i\eta_\beta^0)}[-2\epsilon_{\alpha\beta}\mathbf{k}_\gamma^0\cdot\mathbf{\Delta}_\alpha^0]^{-i\eta_\gamma^0}\frac{1}{[2\epsilon_{\alpha\beta}\mathbf{\Delta}_\beta^0\cdot\mathbf{k}_\alpha+\sigma_\alpha(q_\alpha;z)]^{1-i(\hat{\eta}_\alpha+\eta_\alpha^0)}}\frac{1}{[(\Delta_\alpha^0)^2]^{1-i(\eta_\beta^0+\eta_\gamma^0)}}. \tag{E31}$$

Correspondingly the leading singular part of J_α takes the form

$$J_\alpha^{(s)}=4\pi e_\alpha e_\gamma e^{-\pi(\eta_\alpha^0+\eta_\beta^0+\eta_\gamma^0)/2}\frac{\Gamma(1-i(\hat{\eta}_\alpha+\eta_\alpha^0))\Gamma(1-i(\eta_\beta^0+\eta_\gamma^0))}{\Gamma(1-i\hat{\eta}_\alpha)}\tilde{\phi}_\alpha(\epsilon_{\alpha\beta}\mathbf{\Delta}_\beta^0+\mathbf{k}_\alpha)[-2(\mathbf{k}_\alpha^0\cdot\mathbf{k}_\alpha+k_\alpha^0k_\alpha)]^{-i\eta_\alpha^0}$$
$$\times[2\epsilon_{\alpha\beta}\mathbf{\Delta}_\alpha^0\cdot\mathbf{k}_\beta^0]^{-i\eta_\beta^0}[-2\epsilon_{\alpha\beta}\mathbf{\Delta}_\alpha^0\cdot\mathbf{k}_\gamma^0]^{-i\eta_\gamma^0}\frac{1}{[2\epsilon_{\alpha\beta}\mathbf{\Delta}_\beta^0\cdot\mathbf{k}_\alpha+\sigma_\alpha(q_\alpha;z)]^{1-i(\hat{\eta}_\alpha+\eta_\alpha^0)}}\frac{1}{[(\Delta_\alpha^0)^2]^{1-i(\eta_\beta^0+\eta_\gamma^0)}}. \tag{E32}$$

(b) We also need the leading singular part of J_α for $q_\alpha=\tilde{q}_\alpha$, in the limit $\sigma_\alpha(\tilde{q}_\alpha;z)\to 0$. The appropriate singular behavior of the off-shell Coulomb wave function is given in Eq. (62b). Repeating the above steps one finds with a similar albeit simpler calculation

$$J_\alpha^{(s)}=4\pi e_\alpha e_\gamma e^{-\pi(\eta_\alpha^0+\eta_\beta^0+\eta_\gamma^0)/2}\Gamma(1-i\eta_\alpha^0)\Gamma(1-i(\eta_\beta^0+\eta_\gamma^0))[2\epsilon_{\alpha\beta}\mathbf{\Delta}_\alpha^0\cdot\mathbf{k}_\beta^0]^{-i\eta_\beta^0}[-2\epsilon_{\alpha\beta}\mathbf{\Delta}_\alpha^0\cdot\mathbf{k}_\gamma^0]^{-i\eta_\gamma^0}\frac{\tilde{L}_1}{[(\Delta_\alpha^0)^2]^{1-i(\eta_\beta^0+\eta_\gamma^0)}}. \tag{E33}$$

Here, $\tilde{L}_1$ is the nonsingular quantity

$$\tilde{L}_1:=\frac{e^{\pi\eta_\alpha^0/2}}{\Gamma(1-i\eta_\alpha^0)}\int\frac{d\mathbf{q}_\beta''}{(2\pi)^3}\psi_{\mathbf{k}_\alpha^0}^{C(+)*}[\mathbf{k}+\mathbf{k}_\alpha^0-\mathbf{k}_\beta^0-\epsilon_{\beta\alpha}(\mathbf{q}_\alpha''+\mathbf{q}_\beta''+\mathbf{q}_\gamma^0)]\tilde{\phi}_\alpha(\epsilon_{\alpha\beta}(\mathbf{q}_\beta''+\lambda_{\beta\gamma}\mathbf{q}_\alpha)). \tag{E34}$$

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