Decay rates of spherical and deformed proton emitters

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Using Green's function techniques, we derive expressions for the width of a proton decaying state in spherical and deformed nuclei. We show that the proton decay widths calculated by the ''exact'' expressions of Maglione *et al.* are equivalent to the distorted wave expressions of Bugrov *et al.*, and that of Åberg *et al.* in the spherical case.

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I. INTRODUCTION

With the discovery of a number of new ground-state proton emitters, both spherical and deformed [1,2], the interpretation of proton radioactivity data has become of theoretical interest. Several authors have presented expressions for the proton decay width of spherical nuclei [3–5] and deformed nuclei [3,6,7]. These quasibound states are extremely narrow, having observable widths which do not exceed 10^{-10} eV. In the calculations described in this paper, the decaying states are treated as stationary states. The decay of the states is imposed by applying an outgoing wave Green's function to solve the Schrödinger equation.

Two apparently different methods have been used to calculate the decay width. In both cases one first determines the wave function for the relative motion of the proton and the daughter nucleus in the resonant state. We shall here illustrate how the two methods differ for a spherical proton emitter. The deformed case will be discussed in Sec. III.

In the first method [3], which we call the direct method (dir), the radial part of the wave function, $u_{lj}(r)/r$, is matched to an outgoing Coulomb wave:

$$u_{lj}(r)=N_{lj}^{\rm dir}O_l(kr)=N_{lj}^{\rm dir}[G_l(kr)+iF_l(kr)] \quad \text{at} \quad r=R, \tag{1}$$

where R is a large distance outside the range of the nuclear field. Here $F_l(kr)$ and $G_l(kr)$ are the regular and irregular Coulomb wave functions, respectively, calculated for the asymptotic relative kinetic energy $(\hbar k)^2/(2\mu)$ of the proton and the daughter nucleus, μ being the reduced mass. By calculating the radial probability flux through a sphere, using the radial wave function on the right-hand side of Eq. (1), one can then express the decay rate Γ_{lj} (or the mean lifetime τ) in terms of the matching amplitude $N_{lj}^{\rm dir}$ as in Ref. [3],

$$\Gamma_{lj}^{\rm dir}=\frac{\hbar}{\tau}=\frac{\hbar^2 k}{\mu}|N_{lj}^{\rm dir}|^2. \tag{2}$$

In the second method [4,5], which is based on a Gell-Mann-Goldberger transformation as we shall explain, one uses the following expression to calculate the decay width of a spherical proton emitter:

$$\Gamma_{lj}^{\rm DW}=\frac{4\mu}{\hbar^2 k}\left|\int_0^\infty F_l(kr)[V(r)-V_C^0(r)]u_{lj}(r)dr\right|^2. \tag{3}$$

Here $V(r)$ is the Coulomb plus nuclear interaction between the proton and the daughter nucleus, and $V_C^0(r)$ is the associated pointlike Coulomb interaction.

The second method is often believed to be approximate and it is sometimes referred to as the distorted wave Born approximation (DWBA). This is misleading because it does not make use of the Born approximation (BA). It does make use of distorted waves, namely, the Coulomb wave functions determined by the pointlike Coulomb interaction. We shall therefore refer to this method as the distorted wave (DW) method.

In this paper, we demonstrate that the distorted wave method, Eq. (3), is actually an exact method which gives the same result as the direct method, Eq. (2), provided that both methods are based on the same exact wave function of the resonance state. This is discussed in Sec. II for a spherical proton emitter. In Sec. III we show how this can be generalized to the case of a deformed nucleus. In Sec. IV we give a few numerical examples, and Sec. V contains our conclusions.

II. SPHERICAL NUCLEI

We consider proton emission from a system of A nucleons, leaving a residual daughter nucleus A-1. The wave function of the system at large distances will be a product of the daughter nucleus wave function and the proton-daughter relative motion wave function $\Psi(\mathbf{r})$, where $\mathbf{r}$ is the relative coordinate between the proton and the daughter nucleus. In the spherical case we consider only the decay to the ground state of the daughter nucleus, and treat the residual nucleus as an inert core. In this study we do not consider spectroscopic factors.

To calculate the outgoing proton wave function we use the exact Gell-Mann–Goldberger transformation and the distorted wave Green's function $G^{(+)}(\mathbf{r},\mathbf{r}')$ with outgoing Coulomb wave boundary conditions [8]. For the spherical case we generalize the asymptotic Green's function (i.e., for $r\rightarrow\infty$) by coupling in the proton spin:

$$G^{(+)}(\mathbf{r},\mathbf{r}')=-\frac{2\mu}{\hbar^2 k}\sum_{ljm}\frac{O_l(kr)}{r}|ljm\rangle\langle ljm|\frac{F_l(kr')}{r'}.$$

The wave function $|ljm\rangle$ is shorthand for

$$|ljm\rangle = \sum_{m_l m_s} \langle lm_l \tfrac{1}{2} m_s | jm\rangle Y_l^{m_l}(\hat{\mathbf{r}})\chi(m_s). \tag{4}$$

The outgoing wave function is

$$\Psi^{(+)}_{l_p j_p m_p}(\mathbf{r}) = -\frac{2\mu}{\hbar^2 k}\sum_{ljm}\frac{O_l(kr)}{r}|ljm\rangle \times\left\langle\left\langle ljm\left|\frac{F_l(kr')}{r'}\right|V(r')-V_C^0(r')\right|\Psi_{l_p j_p m_p}\right\rangle.$$

The initial state $|\Psi_{l_p j_p m_p}\rangle$ is a shell model state of a single particle plus inert core; in what follows we have omitted the core wave function since it acts only as a spectator:

$$|\Psi_{l_p j_p m_p}\rangle = \frac{u_{l_p j_p}(r')}{r'}|l_p j_p m_p\rangle.$$

Performing the angular integral in the matrix element yields $\delta_{ll_p}\delta_{jj_p}\delta_{mm_p}$, so

$$\Psi^{(+)}_{l_p j_p m_p}(\mathbf{r}) = -\frac{2\mu}{\hbar^2 k}\frac{O_{l_p}(kr)}{r}|l_p j_p m_p\rangle\int_0^\infty F_{l_p}(kr')[V(r') - V_C^0(r')]u_{l_p j_p}(r')dr'. \tag{5}$$

Comparing the radial part of Eq. (5) with Eq. (1), we obtain the normalization amplitude for the outgoing Coulomb wave

$$N^{\mathrm{DW}}_{l_p j_p} = -\frac{2\mu}{\hbar^2 k}\int_0^\infty F_{l_p}(kr)[V(r)-V_C^0(r)]u_{l_p j_p}(r)dr, \tag{6}$$

and thus the distorted wave expression (3) for the partial decay width follows directly from Eq. (2).

As mentioned in the introduction, Eq. (3) is identical to the expression obtained by Bugrov *et al.*, [4], who adapted their α-decay formalism to proton decay, and also to that used by Åberg *et al.* [5]. Since this expression has been derived using the exact Gell-Mann-Goldberger transformation, it must give exactly the same numerical result as the direct method, Eq. (2). This can, in fact, be demonstrated explicitly by replacing the interactions in the radial matrix element of Eq. (6) by the associated single-particle Hamiltonian minus the radial kinetic energy operators

$$V(r)-V_C^0(r)\rightarrow\left[H+\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2}\right]_{\mathrm{right}}-\left[H_0+\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2}\right]_{\mathrm{left}}. \tag{7}$$

The subscripts ''right'' and ''left'' indicate that the operators must act to the right and to the left, respectively, when inserted in the radial matrix element of Eq. (6).

Making the substitution (7) in Eq. (6), the contributions from the two single-particle Hamiltonians must cancel because the two wave functions have the same energy. The contribution from the radial kinetic energy operator can be written as

$$N^{\mathrm{DW}}_{l_p j_p} = -\frac{1}{k}\int_0^\infty\frac{d}{dr}\left[F_{l_p}(kr)\frac{du_{l_p j_p}(r)}{dr}-u_{l_p j_p}(r)\frac{dF_{l_p}(kr)}{dr}\right]dr = -\frac{1}{k}\left[F_{l_p}(kr)\frac{du_{l_p j_p}(r)}{dr}-u_{l_p j_p}(r)\frac{dF_{l_p}}{dr}\right]_0^\infty.$$

The contribution from the lower limit, $r=0$, vanishes because both radial wave functions are regular at the origin. The contribution from the upper limit, $r=\infty$, is the Wronskian of the two radial wave functions, and inserting the asymptotic form of the exact solution, Eq. (1), we obtain the desired result $N^{\mathrm{DW}}_{l_p j_p} = N^{\mathrm{dir}}_{l_p j_p}$. The same is true if the asymptotic wave function is purely real, i.e., $N^{\mathrm{dir}}_{l_p j_p} G_{l_p}(kr)$, as used in Ref. [5].

III. DEFORMED NUCLEI

We consider a deformed odd-A nucleus, consisting of a single particle strongly coupled to an axially symmetric even-even core. In analogy to the spherical case, the total outgoing wave function at large distances is

$$\Psi^{(+)}_{KIM}(\mathbf{r}) = -\frac{2\mu}{\hbar^2 k}\sum_{ljR}\frac{O_l(kr)}{r}|(ljR)IM\rangle \times\left\langle\left\langle (ljR)IM\left|\frac{F_l(kr')}{r'}\right|V(\mathbf{r}')-V_C^0(r')\right|\Psi_{KIM}\right\rangle. \tag{8}$$

Here $V(\mathbf{r}')$ is the total deformed potential acting between the proton and the core nucleus. The angular momentum part of the Green's function now includes the rotational states $|RM_R\rangle$ $(R=0,2,\ldots)$ of the core and the single-particle state $|ljm\rangle$, which are coupled to the total spin (IM):

$$|(ljR)IM\rangle = \sum_{mM_R}\langle jmRM_R|IM\rangle|ljm\rangle|RM_R\rangle. \tag{9}$$

It is noted that the total spin (IM) is preserved, as we shall see explicitly later on.

The total wave function of the initial state is of the form [9]

$$\Psi_{KIM} = \sqrt{\frac{\hat{I}}{16\pi^2}}[D^I_{MK}(\omega')\phi_K + (-1)^{I+K}D^I_{M-K}(\omega')\phi_{\bar{K}}], \tag{10}$$

where $\hat{I}=2I+1$ and all D functions, here and below, are functions of the orientation of the nucleus in the laboratory frame. The single-particle wave function ϕ_K is described (as in the Nilsson model) in terms of the intrinsic (body-fixed) coordinates of the daughter nucleus. It can be expanded in spherical components

$$\phi_K(\mathbf{r}')=\sum_{lj}\ \phi_{lj}^{(i)}(r')|ljK\rangle_0, \tag{11}$$

where the sum is over $j\geqslant|K|$ and the subscript ''0'' denotes a state in the intrinsic frame.

To specify the final state, we note that the rotational state wave function of the daughter is

$$|RM_R\rangle=\sqrt{\frac{\hat{R}}{8\pi^2}}D^R_{M_R0}(\omega'). \tag{12}$$

We pick out a particular outgoing channel, in which the proton carries off angular momentum l_pj_p, with projection m_p, leaving the daughter nucleus with angular momentum R and projection M_R. For the evaluation of the matrix element in Eq. (8), the final state $|(l_pj_pR)IM\rangle$ must be expressed in terms of the single-particle wave function in the intrinsic system

$$|l_pj_pm_p\rangle=\sum_{K'}\ D^{j_p}_{m_pK'}(\omega')|l_pj_pK'\rangle_0. \tag{13}$$

Inserting Eqs. (12) and (13) into Eq. (9) we obtain

$$\begin{aligned}|(l_pj_pR)IM\rangle&=\sqrt{\frac{\hat{R}}{8\pi^2}}\sum_{K'm_pM_R}\langle j_pm_pRM_R|IM\rangle D^R_{M_R0}(\omega')\\&\quad\times D^{j_p}_{m_pK'}(\omega')|l_pj_pK'\rangle_0\\&=\sqrt{\frac{\hat{R}}{8\pi^2}}\\&\quad\times\sum_{K'}\langle j_pK'R0|IK'\rangle D^I_{MK'}(\omega')|l_pj_pK'\rangle_0,\end{aligned} \tag{14}$$

where we have used a well-known relation involving a sum of D functions [10].

A. Evaluation of the matrix element

We can now calculate the matrix element found in Eq. (8). The integration over the orientation of the daughter nucleus ω' produces an expression that is diagonal in the quantum numbers IMK. This is evident from the first part of Eq. (10). The second term will select the value $-K$ from the sum (14) over final state K' values. In fact one can show that the two terms are of equal magnitude, so we obtain

$$\begin{aligned}&\left\langle\left\langle(l_pj_pR)IM\left|\frac{F_{l_p}(kr')}{r'}\right|V(\mathbf{r}')-V_C^0(r')\right|\Psi_{KIM}\right\rangle\\&\quad=\sqrt{\frac{2\hat{R}}{\hat{I}}}\langle j_pKR0|IK\rangle\mathcal{M}_{l_pj_pK},\end{aligned} \tag{15}$$

where

$$\mathcal{M}_{l_pj_pK}=\left\langle l_pj_pK\left|\frac{F_{l_p}(kr')}{r'}[V(\mathbf{r}')-V_C^0(r')]\right|\phi_K\right\rangle_0 \tag{16}$$

is evaluated in the intrinsic frame. It is noted that the matrix element (15) is independent of the M-quantum number.

B. Partial decay width

Having determined the matrix element we can now write the outgoing wave function (8) for a specific channel (l_pj_pRIK) as

$$\Psi^{(+)}_{l_pj_pR,KIM}(\mathbf{r})=N^{\mathrm{DW}}_{l_pj_pRIK}\frac{O_{l_p}(kr)}{r}|(l_pj_pR)IM\rangle,$$

where

$$N^{\mathrm{DW}}_{l_pj_pRIK}=-\frac{2\mu}{\hbar^2k}\sqrt{\frac{2\hat{R}}{\hat{I}}}\langle j_pKR0|IK\rangle\mathcal{M}_{l_pj_pK} \tag{17}$$

and $\mathcal{M}_{l_pj_pK}$ is given in Eq. (16). As in the spherical case we obtain the decay width from the outgoing flux, Eq. (2), as

$$\Gamma^{\mathrm{DW}}_{l_pj_pRIK}=\frac{4\mu}{\hbar^2k}\frac{2\hat{R}}{\hat{I}}\langle j_pKR0|IK\rangle^2|\mathcal{M}_{l_pj_pK}|^2.$$

Apart from a pairing term, this expression is identical to that obtained by Kadmensky and Bugrov [6,7]. This can be shown in detail by inserting the expansion (11) of the initial state into the matrix element (16) and using the form (4) of the single-particle states. Since the interaction does not change the proton spin, the single-particle matrix element will be diagonal in m_s. It is also diagonal in K. Thus one obtains

$$\begin{aligned}\mathcal{M}_{l_pj_pK}&=\sum_{ljm_s}\langle l_pm_{l_p}\tfrac{1}{2}m_s|j_pK\rangle\langle lm_l\tfrac{1}{2}m_s|jK\rangle\\&\quad\times\left\langle Y^{m_{l_p}}_{l_p}(\hat{\mathbf{r}}')\frac{F_{l_p}(kr')}{r'}\right|V(\mathbf{r}')\\&\quad-V_C^0(r')\left|\phi_{lj}^{(i)}(r')Y^{m_l}_l(\hat{\mathbf{r}}')\right\rangle.\end{aligned}$$

This expression has been simplified by noting that $m_{l_p}=m_l=K-m_s$, thus eliminating sums over those variables. The $Y^0_2(\hat{\mathbf{r}}')$ term in the deformed potential $V(\mathbf{r}')$ apparently allows for an angular momentum exchange at the nuclear surface between the outgoing proton and the daughter nucleus, leading to nondiagonal terms in the matrix element.

C. Direct method

We can also use the direct method to determine the decay width of a deformed proton emitter. To do this we expand the wave function $\Psi_{KIM}(\mathbf{r})$ of the initial state on the complete set of angular momentum basis states

TABLE I. Comparison of the proton half-lives of three spherical proton radioactivities calculated with the direct (dir) and distorted wave (DW) methods.

Nucleus	E_p(keV)	j_p	l_p	$t^{\rm dir}_{1/2,p}$	$t^{\rm DW}_{1/2,p}$
^{167}Ir^g	1064(6)	$\frac{1}{2}^+$	0	35.7687 ms	35.7671 ms
^{147}Tm^m	1119(5)	$\frac{3}{2}^+$	2	171.332 μs	171.327 μs
^{167}Ir^m	1238(7)	$\frac{11}{2}^-$	5	1.99360 s	1.99352 s

$$\Psi_{KIM}(\mathbf{r})=\sum_{ljR}|(ljR)IM\rangle\langle(ljR)IM|\Psi_{KIM}\rangle.$$

Using Eqs. (15) and (16) with $F_{l_p}(kr)/r[V(\mathbf{r})-V^0_C(r)]$ replaced by 1 we can immediately write the overlap matrix element as

$$\langle(ljR)IM|\Psi_{KIM}\rangle=\sqrt{\frac{2\hat{R}}{\hat{I}}}\langle jKR0|IK\rangle\langle ljK|\phi_K\rangle.$$

Inserting the expansion (11) for the outgoing channel l_pj_p we obtain

$$\langle l_pj_pK|\phi_K\rangle=\phi^{(i)}_{l_pj_p}(r)\rightarrow A_{l_pj_p}\frac{O_{l_p}(kr)}{r},\quad \text{for } r\rightarrow\infty, \tag{18}$$

assuming that the intrinsic states are matched to outgoing Coulomb waves as in Eq. (1). The outgoing wave is therefore

$$\Psi^{(+)}_{l_pj_pR,KIM}(\mathbf{r})=N^{\rm dir}_{l_pj_pRIK}\frac{O_{l_p}(kr)}{r}|(l_pj_pR)IM\rangle,$$

where

$$N^{\rm dir}_{l_pj_pRIK}=\sqrt{\frac{2\hat{R}}{\hat{I}}}\langle j_pKR0|IK\rangle A_{l_pj_p}. \tag{19}$$

This gives the decay width, according to Eq. (2), as

$$\Gamma^{\rm dir}_{l_pj_pRIK}=\frac{\hbar^2k}{\mu}\frac{2\hat{R}}{\hat{I}}\langle j_pKR0|IK\rangle^2|A_{l_pj_p}|^2.$$

This expression is consistent with the result given in Eq. (8) of Ref. [3]. In addition, it allows the calculation of partial decay widths to excited states of the daughter nucleus.

It can also be demonstrated explicitly that the distorted wave method and the direct method give identical results for the decay width of a deformed proton emitter. The proof is analogous to that presented in the spherical case [see Eq. (7)], but requires the use of the full three-dimensional kinetic energy operator $-\hbar^2\nabla^2/2\mu$. Thus the matrix element (16) can be expressed as

$$\mathcal{M}_{l_pj_pK}=\frac{\hbar^2}{2\mu}\left\langle l_pj_pK\left|\frac{F_{l_p}(kr)}{r}(\nabla^2_{\rm right}-\nabla^2_{\rm left})\right|\phi_K\right\rangle.$$

Using Green's theorem and inserting the expansion (11) for the initial state one obtains

$$\begin{aligned}\mathcal{M}_{l_pj_pK}&=\frac{\hbar^2}{2\mu}\left(F_{l_p}(kr)\frac{d[r\phi^{(i)}_{l_pj_p}(r)]}{dr}-[r\phi^{(i)}_{l_pj_p}(r)]\frac{dF_{l_p}(kr)}{dr}\right)_{r\rightarrow\infty}\\&=-\frac{\hbar^2k}{2\mu}A_{l_pj_p},\end{aligned}$$

where we again have used the asymptotic form (18) of $\phi^{(i)}_{l_pj_p}(r)$. Inserting this into the distorted wave amplitude (17) we see that it becomes identical to the direct amplitude, Eq. (19).

IV. NUMERICAL CALCULATIONS

A. Normalization

For the calculation of the decay widths for spherical and deformed nuclei, normalized wave functions are needed. In addition, the direct method includes a determination of the quantity N_{lj}. We will consider both of these normalization questions separately for the spherical and deformed cases.

1. Spherical normalization

Since the wave function $u_{lj}(r)$ decreases rapidly with radius outside the nucleus, typically by 10 orders of magnitude, it can be normalized by requiring that

$$\int_0^R[u_{lj}(r)]^2dr=1,$$

where $R<R_o$, the classical external turning point. In practice, integrating out to $R=25$ fm is adequate for the known spherical proton emitters. Once $u_{lj}(r)$ is suitably normalized, then N_{lj} can be determined from Eq. (1):

$$N_{lj}=\frac{u_{lj}(R)}{G_l(kR)+iF_l(kR)}. \tag{20}$$

We note that, strictly speaking, the matching condition (1) requires a complex energy and a complex wave function [3]. However, since the imaginary part of the resonance energy (i.e., the decay width) is usually extremely small, one could as well use energies and wave functions that are real [5]. In that case, one would replace Eq. (20) by

$$N_{lj}=\frac{u_{lj}(R)}{G_l(kR)}.$$

This condition will be quite close to Eq. (20) because it is usually possible to find a value for R such that $G_l(kR)\gg F_l(kR)$.

2. Deformed normalization

In the deformed case we make the substitution

$$\phi^{(i)}_{lj}(r)=C_{lj}(\beta_2)\frac{u_{lj}(r)}{r},$$

where β_2 is the quadrupole deformation parameter and the function $u_{lj}(r)$ is normalized as described for the spherical case. The C_{lj} coefficients can be calculated, for example, using the procedure described by Andersen, Back, and Bang [11]. With this substitution, the asymptotic amplitude A_{lj} defined in Eq. (18) is given by

$$A_{lj}=C_{lj}(\beta_2)\frac{u_{lj}(R)}{G_l(kR)},$$

where $R<R_o$. Under these conditions the deformed and spherical widths are related by a simple numerical factor

$$(\Gamma^{\rm dir}_{ljRIK})_{\rm def}=\frac{2\hat{R}}{\hat{I}}\langle jKR0|IK\rangle^2|C_{lj}(\beta_2)|^2(\Gamma^{\rm dir}_{lj})_{\rm sph}.$$

B. Comparison of half-life calculations

It is of interest to compare the numerical results for the half-lives [$t_{1/2}=\hbar\ \ln(2)/\Gamma$] obtained with the direct method (2) and the distorted wave method (3) for spherical nuclei. We show in Table I the calculated half-lives for three spherical decaying states having orbital angular momentum $l=0$, 2, and 5, respectively. The radial wave functions $u_{lj}(r)$ were calculated by integrating the radial Schrödinger equation, using for the proton-daughter nucleus potential the real part of the Becchetti-Greenlees optical model potential [12]. The potential depth was adjusted to match the energy eigenvalue to the proton decay Q value. It is seen that the calculated half-lives agree to better than 0.005% between the two methods. Since the two methods should give identical results, the difference must reflect the accuracy of the numerical techniques that have been used.

For the deformed case we show in Table II the calculated half-lives for three decaying states ($\beta_2=0.3$) and total angular momentum $j=3/2^+$, $5/2^+$, and $7/2^-$, respectively. Here the calculated half-lives only agree to within 20% between the two methods. The discrepancy is probably due to the truncation in the eigenfunction space, such that only the nearest spherical states were included. The initial state is therefore not the exact or complete solution to the deformed Hamiltonian, and the Gell-Mann-Goldberger transformation method will therefore not provide exactly the same result as the direct method. The comparison of the results of the two methods is therefore a test of how close the truncated solution comes to being correct. Further investigation in this area is needed. Perhaps a coupled-channels approach, such as that developed by Ref. [13], will offer closer agreement between the direct and distorted wave methods.

TABLE II. Comparison of the proton half-lives of deformed proton radioactivities calculated with the direct (dir) and distorted wave (DW) methods.

Nucleus	E_p(keV)	j_p	l_p	β_2	C_{lj}[a]	$t^{\rm dir}_{1/2,p}$	$t^{\rm DW}_{1/2,p}$
^{131}Eu	932(7)	$\frac{3}{2}^+$	2	0.3	−0.208	27.92 ms	24.09 ms
^{131}Eu	932(7)	$\frac{5}{2}^+$	2	0.3	−0.0999	176.2 ms	214.8 ms
^{141}Ho^g	1169(8)	$\frac{7}{2}^-$	3	0.3	0.240	4.087 ms	3.266 ms

[a]A.A. Sonzogni, private communication (1999).

C. Conclusions

We have shown that the distorted wave method and the direct method of calculating the width of spherical and deformed proton emitter are equivalent. In the spherical case numerical agreement is demonstrated to better than 0.005%, while for the deformed case the agreement is only within about 20%. Improved methods of calculating the wave functions should reduce this discrepancy. We recommend using either of these methods in place of the WKB method, which has certain problems related to the frequency factor (see Ref. [5] for a discussion of this point). For the cases where the radial wave function is known over the $0<r<25$ fm range, the direct method is preferred for its calculational simplicity. However, if the radial wave function is known reliably only in the region of the nuclear surface, the distorted wave method is to be preferred.

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