State dependent effective interaction for the hyperspherical formalism

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(Received 21 October 1999; published 30 March 2000)

The effective interaction method, traditionally used in the framework of a harmonic oscillator basis, is applied to the hyperspherical formalism of few-body nuclei ($A=3-6$). The separation of the hyperradial part leads to a state dependent effective potential. Undesirable features of the harmonic oscillator approach associated with the introduction of a spurious confining potential are avoided. It is shown that with the present method one obtains an enormous improvement of the convergence of the hyperspherical harmonics series in calculating ground state properties, excitation energies, and transitions to continuum states.

PACS number(s): 21.45.+v, 21.30.Fe, 31.15.Ja

I. INTRODUCTION

Shell-model calculations in complex nuclei make use of the single particle harmonic oscillator (HO) basis and are carried out in a truncated model space. In the last few years impressive progress has been made in the application of these shell-model methods to the study of light nuclei [1–5]. In the so called no-core shell-model calculations one keeps all the nucleons active and, instead of the single particle HO basis, one introduces HO basis functions that depend on the Jacobi coordinates, thus removing the spurious center of mass motion from the beginning [4,5].

Since the HO series has a slow convergence rate one generally has to replace the bare nucleon-nucleon (*NN*) interaction by an effective interaction tailored to the truncated model space. Theoretically for a given model space one can find an effective interaction such that the spectrum of the effective *A*-body Hamiltonian coincides with a subset of the spectrum of the full-space bare Hamiltonian. In practice, however, finding such an effective interaction is as difficult as solving the full *A*-body problem. Therefore one resorts to an approximate effective interaction, usually obtained from the solution of a two-body Hamiltonian. These two-body effective interactions no longer lead to the exact result in the truncated space, but, if constructed properly, they retain two important properties: (i) they converge to the bare Hamiltonian if the model space is enlarged up to the full Hilbert space; (ii) the energy levels of the effective Hamiltonian converge to the exact values faster than those of the bare Hamiltonian.

The HO basis functions resulting from a confining Hamiltonian do not possess the correct asymptotic behavior of the nuclear *A*-body Hamiltonian. As a result the use of the HO basis may lead to a rather slow convergence for energy levels as well as other observables. This limitation can be circumvented by using hyperspherical harmonic (HH) basis functions instead of a HO basis. In the HH formalism, which was successfully applied to the nuclear few-body problem [6–9], the Jacobi coordinates are replaced by a single length coordinate, the hyperradius, and a set of $3A-4$ hyperangles. The HH are the *A*-body generalization of the two-body spherical harmonics, and likewise depend only on the hyperangular (angular) coordinates in the hyperspherical (spherical) decomposition of the *A*-body (two-body) system. In general, the wave function can be expanded in a series consisting of products of HH basis functions and hyperradial basis functions. Very often the use of correlation functions will accelerate the slow convergence rate of the HH basis [6,8–10]. It is the aim of the present work to investigate an alternative way of improving the convergence. To this end we reformulate the effective interaction method for the HH expansion.

The HO and the HH expansion become equivalent for a particular choice for the hyperradial basis functions. Therefore a trivial way to achieve a reformulation would be to make the HH expansion equivalent to the HO expansion. However, by doing so one would lose the extra flexibility the HH basis has in comparison to the HO basis and, moreover, impose an incorrect asymptotic behavior on the wave function. Therefore we reject such an equivalent formulation.

There is a further advantage of the HH basis, which is due to the presence of the collective hyperradial coordinate. Eventually it will allow the introduction of a state dependent effective interaction similar to the recently formulated HO multivalued effective interaction [2,11]. At first sight, however, it appears that the hyperradius leads to two problems. First, its collective feature seems to make it difficult to single out a two-body Hamiltonian in a natural way. Secondly, by using a general form for the hyperradial basis functions one may find it difficult to identify a model space in accordance with a two-body effective interaction. In this work we propose to solve these problems by defining a model space that consists of a complete hyperradial set and the set of HH functions with generalized angular momentum quantum number $K \leqslant K_{\max}$. The effective interaction is then deduced from a hyperangular Hamiltonian associated with the two-body problem.

In Sec. II we review the method of hyperspherical coordinates and of the HH expansion. In Sec. III we derive the effective interaction for the HH expansion. Numerical results are then given in Sec. IV and conclusions are drawn in Sec. V.

II. THE HYPERSPHERICAL HARMONIC FUNCTIONS

To introduce the hyperspherical coordinates we start from the center-of-mass coordinate $\vec{R}=(1/A)\Sigma_{i=1}^{A}\vec{r}_i$ and the normalized reversed order $N=A-1$ Jacobi coordinates

$$\vec{\eta}_1=\sqrt{\frac{A-1}{A}}\left(\vec{r}_1-\frac{1}{A-1}(\vec{r}_2+\vec{r}_3+\cdots+\vec{r}_A)\right), \quad (1)$$

$$\vec{\eta}_2=\sqrt{\frac{A-2}{A-1}}\left(\vec{r}_2-\frac{1}{A-2}(\vec{r}_3+\vec{r}_4+\cdots+\vec{r}_A)\right),$$

$$\cdots,$$

$$\vec{\eta}_N=\sqrt{\frac{1}{2}}(\vec{r}_{A-1}-\vec{r}_A),$$

where the jth particle is specified relative to the center of mass of particles $j+1$ to A. The Jacobi coordinate $\vec{\eta}_j$ consists of a radial coordinate η_j and a pair of angular coordinates $\hat{\eta}_j\equiv(\theta_j,\phi_j)$.

These coordinates are then transformed into the hyperangular coordinates $\alpha_2,\ldots,\alpha_N$ through the relation

$$\sin\alpha_n=\eta_n/\rho_n, \quad (2)$$

where

$$\rho_n^2=\rho_{n-1}^2+\eta_n^2=\sum_{j=1}^{n}\eta_j^2. \quad (3)$$

For $n=N$ we also find the relation

$$\rho^2\equiv\rho_N^2=\frac{1}{A}\sum_{i<j}^{A}(\vec{r}_i-\vec{r}_j)^2. \quad (4)$$

Therefore, the hyperradial coordinate ρ is symmetric with respect to permutations of the underlying single particle coordinates.

The $3N=3(A-1)$ internal coordinates for the A-particle system consist of the hyperradial coordinate $\rho\equiv\rho_N$, and the $3N-1$ ''hyperangular'' coordinates $\Omega_N\equiv\{\hat{\eta}_1,\hat{\eta}_2,\ldots,\hat{\eta}_N,\alpha_2,\alpha_3,\ldots,\alpha_N\}$. These coordinates depend on the set of Jacobi coordinates specified in Eq. (1).

By using hyperspherical coordinates one can write the Laplace operator for n Jacobi coordinates $n=1\ldots N$, as a sum of two terms

$$\Delta_n=\frac{1}{\rho_n^{3n-1}}\frac{\partial}{\partial\rho_n}\rho_n^{3n-1}\frac{\partial}{\partial\rho_n}-\frac{1}{\rho_n^2}\hat{K}_n^2. \quad (5)$$

The hyperspherical, or grand angular momentum operator $\hat{K}_n^2$ of the n Jacobi coordinates can be expressed in terms of $\hat{K}_{n-1}^2$ and $\hat{l}_n^2$ as follows [13]:

$$\hat{K}_n^2=-\frac{\partial^2}{\partial\alpha_n^2}+\frac{3n-6-(3n-2)\cos(2\alpha_n)}{\sin(2\alpha_n)}\frac{\partial}{\partial\alpha_n}+\frac{1}{\cos^2\alpha_n}\hat{K}_{n-1}^2+\frac{1}{\sin^2\alpha_n}\hat{l}_n^2, \quad (6)$$

where we define $\hat{K}_1^2\equiv\hat{l}_1^2$. The angular momentum operator associated with these n coordinates is $\vec{\hat{L}}_n=\vec{\hat{L}}_{n-1}+\vec{\hat{l}}_n$. The operators $\hat{K}_n^2$, $\hat{l}_n^2$, $\hat{K}_{n-1}^2$, $\hat{L}_n^2$, and $\hat{L}_{n_z}$ commute with each other. The hyperspherical harmonic functions $\mathcal{Y}_{[K_n]}$ are the eigenfunctions of this hyperangular operator. The explicit expression for the HH functions of the first n Jacobi coordinates is given by [12]

$$\mathcal{Y}_{[K_n]}=\left[\sum_{m_1,\ldots,m_n}\langle l_1m_1l_2m_2|L_2M_2\rangle\langle L_2M_2l_3m_3|L_3M_3\rangle\times\cdots\langle L_{n-1}M_{n-1}l_nm_n|L_nM_n\rangle\prod_{j=1}^{n}Y_{l_j,m_j}(\hat{\eta}_j)\right]$$

$$\times\left[\prod_{j=2}^{n}\mathcal{N}_j(K_j;l_jK_{j-1})(\sin\alpha_j)^{l_j}(\cos\alpha_j)^{K_{j-1}}P_{\mu_j}^{[l_j+1/2,K_{j-1}+(3j-5)/2]}(\cos(2\alpha_j))\right], \quad (7)$$

where $Y_{l,m}$ are the spherical harmonic functions, $P_\mu^{(a,b)}$ are the Jacobi polynomials, and $\mathcal{N}_j(K_j;l_jK_{j-1})$ are normalization constants given by [13]

$$\mathcal{N}_j(K_j;l_jK_{j-1})=\left[\frac{(2K_j+3j-2)\mu_j!\Gamma[\mu_j+K_{j-1}+l_j+(3j-2)/2]}{\Gamma\left(\mu_j+l_j+\frac{3}{2}\right)\Gamma[\mu_j+K_{j-1}+(3j-3)/2]}\right]^{1/2}. \quad (8)$$

The symbol $[K_n]$ stands for the set of quantum numbers $l_1,\ldots,l_n$, $L_2,\ldots,L_n$, $\mu_2,\ldots,\mu_n$, and M_n. The quantum numbers K_j are given by

$$K_j=2\mu_j+K_{j-1}+l_j;\quad \mu_1\equiv 0, \tag{9}$$

and the μ_j are non-negative integers. By construction, $\rho_n^{K_n}\mathcal{Y}_{[K_n]}$ is a harmonic polynomial of degree K_n. The HH function $\mathcal{Y}_{[K_n]}$ is an eigenfunction of $\hat{K}_n^2$ with eigenvalues

$$K_n(K_n+3n-2). \tag{10}$$

It is evident that the HH functions (7) do not possess any special properties under particle permutation. Therefore the first step in applying the HH expansion to the A-body problem is the symmetrization of the HH basis. In the current work we employ two powerful algorithms [14,15] recently developed for the construction of a HH basis with well defined permutational symmetry. This enabled us to check our results, since they could be obtained in two independent ways.

In view of Eqs. (3) and (4) it is evident that the HO Hamiltonian, written in the form

$$\sum_{j=1}^{N}\left(-\frac{\Delta_j}{2}+\omega^2\eta_j^2\right)=\frac{1}{2}\left(\frac{\partial^2}{\partial\rho^2}+\frac{3N+4}{\rho}\frac{\partial}{\partial\rho}-\frac{\hat{K}^2}{\rho^2}+\omega^2\rho\right), \tag{11}$$

has eigenvectors of the form

$$\Psi_{HO}=R_{n_\rho}(\rho)\mathcal{Y}_{[K]}, \tag{12}$$

with eigenvalues

$$E_n=\hbar\omega\left(\frac{3(A-1)}{2}+n\right)=\hbar\omega\left(\frac{3(A-1)}{2}+2n_\rho+K\right). \tag{13}$$

Therefore the HH K-quantum number can be associated with the quanta of excitations of the HO wave function.

III. THE EFFECTIVE INTERACTION

In general we would like to use the HH basis functions to solve the A-body Hamiltonian

$$H=\sum_{i=1}^{A}\frac{\vec{p}_i^{\,2}}{2m}+\sum_{i<j}^{A}V_{ij}, \tag{14}$$

where m is the nucleon mass and V_{ij} is the NN interaction. In practice, looking for the eigenvectors of H in terms of the HH expansion turns out to be a notoriously difficult task. Therefore, one usually has to introduce correlation functions in order to accelerate the convergence of the calculation [6,8–10]. In this work, however, we shall explore another possibility and instead of using correlation functions we shall use the method of effective interactions [16]. This approach is largely used in shell-model calculations (see, e.g., Ref. [17]), where the harmonic oscillator basis is used in a truncated model space. Instead of the bare NN interaction one uses effective interactions inside the model space. Defining P as the projection operator onto the model space and $Q=1-P$ as the projection onto the complementary space, the model space Hamiltonian can be written as

$$H_P=P\left[\sum_{i=1}^{A}\frac{\vec{p}_i^{\,2}}{2m}\right]P+P\left[\sum_{i<j}^{A}V_{ij}\right]_{\rm eff}P. \tag{15}$$

In general the effective interaction appearing in Eq. (15) is an A-body interaction. If it is determined without any approximation, then the model-space Hamiltonian provides a set of eigenvalues which coincide with a subset of the eigenvalues of the original full-space Hamiltonian, Eq. (14). However, calculation of the exact A-body effective interaction is as difficult as finding the full-space solution.

In the HH formalism the model space can be defined as a product of the hyperradial subspace and the complete set of HH basis functions with generalized angular momentum quantum number $K\leqslant K_{\max}$. Instead of calculating the exact effective interaction we shall look for an approximate effective interaction with the following properties: $V_{\rm eff}\rightarrow V$ as $K_{\max}\rightarrow\infty$; the eigenvalues, $E_i(K_{\max})$, and eigenvectors of the effective A-body Hamiltonian converge to their limiting values faster than the eigenvalues and eigenvectors of the bare Hamiltonian.

Let us now turn to the problem of constructing the effective interaction. It is customary to approximate $V_{\rm eff}$ by a sum of two-body effective interactions determined from a two-body problem. As the nuclear two-body system contains only one bound state one is forced to introduce a confining potential into the two-body problem in order to ensure large overlaps between the model space states and the eigenvectors of the two-body problem. It will be shown that in the present approach one does not need such an additional confining potential.

Using the symmetrized HH basis, one can deduce the matrix elements of the effective interaction from the matrix elements of the ''last'' pair,

$$\left\langle\sum_{i<j}^{A}V_{2\ {\rm eff}}(\vec{r}_{ij})\right\rangle=\frac{A(A-1)}{2}\langle V_{2\ {\rm eff}}(\vec{r}_{A,A-1})\rangle. \tag{16}$$

The relevant hyperspherical degrees of freedom associated with $V_{2\ {\rm eff}}(\vec{r}_{A,A-1})$ are $\hat{\eta}_N$ and the hyperangle,

$$\sin\alpha_N=\frac{r_{A,A-1}}{\sqrt{2}\rho}. \tag{17}$$

A natural choice for the corresponding hyperspherical ''two-body'' Hamiltonian is

$$H_2(\rho)=\frac{1}{2m}\frac{\hat{K}_N^2}{\rho^2}+V(\sqrt{2}\rho\sin\alpha_N\cdot\hat{\eta}_N), \tag{18}$$

since $\hat{K}_N^2$ contains the canonical kinetic energy associated with the two-body variables α_N and $\hat{\eta}_N$ [see Eq. (6)]. Such

an H_2 is in fact an A-body effective interaction as it contains the hyperspherical part of the A-body kinetic energy operator and it is a function of the collective coordinate ρ. The hyperradial kinetic energy operator has not been included in H_2. The reason is that we can use a complete basis set for the ρ space and therefore we do not need to define an effective interaction for the hyperradial part.

Due to the collective coordinate, ρ, in H_2 one has automatically a confinement of the two-body system: for moderate values of ρ the relation $0 \leqslant r_{A,A-1} \leqslant \sqrt{2}\rho$ ensures localization of the two-body wave function and for large values of ρ the effective Hamiltonian coincides with the bare one, since the NN interaction vanishes. Therefore large overlaps between the model space states and the eigenvectors of the two-body problem are ensured.

As opposed to the HO approach we do not calculate an effective interaction for a ''free'' two-body system but for a ''bound'' one. Therefore we can avoid the introduction in H_2 of the additional confining potential which is necessary for the HO effective interaction and which leads to undesirable features.

The matrix elements of H_2 between the A-body HH functions, Eq. (7), are given by

$$\langle [K_N]|H_2(\rho)|[K'_N]\rangle = \delta_{[K_N][K'_N]}\frac{1}{2m}\frac{K_N(K_N+3N-2)}{\rho^2} + \delta_{[K_{N-1}][K'_{N-1}]}V^{K_{N-1}L_{N-1}}_{K_N L_N l_N, K'_N L'_N l'_N}(\rho), \tag{19}$$

where

$$V^{K_{N-1}L_{N-1}}_{K_N L_N l_N, K'_N L'_N l'_N}(\rho) = \int d\Omega_N \mathcal{Y}^*_{[K_N]} V(\sqrt{2}\rho \sin\alpha_N \cdot \hat{\eta}_N)\mathcal{Y}_{[K'_N]}. \tag{20}$$

We see that H_2 is diagonal in the quantum numbers $[K_{N-1}]$ and also, for central potentials, in L_N, l_N. Due to the hyperangular integration H_2 explicitly depends on quantum numbers of the residual system, i.e., K_{N-1} and, for noncentral forces, L_{N-1}. The effective two-body Hamiltonian is independent of the other quantum numbers in $[K_{N-1}]$ [see Eq. (7)]. As a result the HH effective interaction depends on the state of the residual $A-2$ particle subsystem. Such a ''medium correction'' of the two-body force is of course a great advantage of our approach and is similar to the HO multivalued effective interaction [2,3]. On the other hand one has to pay for it with greater numerical effort, since the effective interaction has to be calculated for all the various states and because it also depends on the specific A-body system considered.

We solve the hyperradial equation on a grid, where H_2 is diagonalized for each grid point ρ_i and for all the possible values of K_{N-1} in our model space. In general one should reach a K_{MAX} value for the Q space of about 60 for the ground state of the s-shell nuclei and $K_{\mathrm{MAX}} \sim 200$ for p-shell nuclei and excited states. From this point we can follow the same procedure as Barrett and Navrátil [4,5]. That is, by employing the Lee-Suzuki [16] similarity transformation method, we can use the eigenvectors, $\{|i\rangle\}$, and eigenvalues, $\{\epsilon_i\}$, of $H_2^{K_{N-1},L_{N-1}}(\rho)$ to construct the effective interaction. Let us denote by $|\alpha\rangle$ the HH functions that belong to our model space, i.e., the HH function $|[K_N]\rangle$ such that $K_N \leqslant K_{\max}$, and by $|\beta\rangle$ the states that belong to the Q space, $Q=\{|[K_N]\rangle;\ K_N > K_{\max}\}$. The Lee-Suzuki effective interaction then takes the form

$$P\tilde{H}_2 P = PH_2P + PH_2Q\omega P, \tag{21}$$

where the transformation operator $\omega = Q\omega P$ is given by the equation

$$\langle\beta|i\rangle = \sum_\alpha \langle\beta|\omega|\alpha\rangle\langle\alpha|i\rangle. \tag{22}$$

If n_P is the number of model-space HH basis functions that belong to the subspace K_{N-1}, we may solve Eq. (22) for ω by choosing a set, $\mathcal{A}$, of n_P eigenvectors with the lowest eigenvalues $|i\rangle$ and inverting the matrix $\langle\alpha|i\rangle$. The resulting effective two-body Hamiltonian

$$\langle\alpha|\tilde{H}_2(K_{N-1},\rho)|\alpha'\rangle = \sum_i^{n_P}\left[\langle\alpha|i\rangle\epsilon_i\langle i|\alpha'\rangle + \sum_\beta \langle\alpha|i\rangle\epsilon_i\langle i|\beta\rangle\langle\beta|\omega|\alpha\rangle\right], \tag{23}$$

will have the property that $P|i\rangle$, $|i\rangle \in \mathcal{A}$, is a right eigenvector of $\tilde{H}_2$ with eigenvalue ϵ_i. The effective interaction is in general a non-Hermitian operator, however it can be Hermitized, using the transformation [18]

$$H_{2\ \mathrm{eff}} = [P(1+\omega^\dagger\omega)P]^{1/2}\tilde{H}_2[P(1+\omega^\dagger\omega)P]^{-1/2}. \tag{24}$$

The effective interaction can be now deduced from $H_{2\ \mathrm{eff}}$, by subtracting the kinetic energy term,

$$V_{\mathrm{eff}} = H_{2\ \mathrm{eff}} - \frac{1}{2m}\frac{\hat{K}_N^2}{\rho^2}. \tag{25}$$

It can be seen that as $K_{\max}\to\infty$ $V_{2\ \mathrm{eff}}$ indeed reproduces the bare NN interaction in contrast to the HO effective interaction, where only the total Hamiltonian converges to the correct result. Another interesting feature of the current formulation is that, in contrast to the HO effective interaction, the HH effective interaction vanishes at large distances as it should for a system of noninteracting particles. In addition we would like to mention that as in the HO approach [4,19] the present formalism can be extended beyond two-body effective interactions to also incorporate three- and more-body effective interactions.

IV. NUMERICAL RESULTS

In order to check the proposed formulation of the hyperspherical effective interaction, we have applied the formalism to few-body nuclei in the mass range $A=3-6$. The

TABLE I. List of the parameters of the *NN* potentials used in this work. The potential is written as a sum of a few terms; each is expressed as $V_i f(\mu_i,r)(W_i+B_iP_\sigma-H_iP_\tau+M_iP_r)$, where P_σ, P_τ, P_r are spin-, isospin-, and space-exchange operators. $f(\mu_i,r)=\exp(-\mu r^2)$ for Gauss-type potential; $f(\mu_i,r)=\exp(-\mu r)/r$ for Yukawa-type potential. The potential strengths V_i are in (MeV). The range μ is in (fm^{-2}) for Gauss-type or (fm^{-1}) for Yukawa type.

Potential	Type	i	V_i	μ_i	W_i	M_i	B_i	H_i
MTV [20]	Yukawa	1	1458.05	3.11	1.0	0.0	0.0	0.0
		2	− 578.09	1.55	1.0	0.0	0.0	0.0
MTI-III [21]	Yukawa	1	1458.27	3.11	0.5	0.5	0.0	0.0
		2	−578.178	1.555	0.5	0.5	0.0495	0.0495
MN [22]	Gauss	1	200.0	1.487	0.5	0.5	0.0	0.0
		2	−178.0	0.639	0.25	0.25	0.25	0.25
		3	−91.85	0.465	0.25	0.25	−0.25	−0.25

following simple *NN* interactions are used: Malfliet-Tjon potentials MTV [20] and MTI-III [21] as well as the Minnesota potential MN [22] (see Table I). The MTV interaction is considered in all partial waves (not only in *s* waves) and in the case of MTI-III the interaction is taken into account in all even partial waves. Of course we are aware that realistic potential models have already been used even for nuclei with $A>4$ [23], but the principal aim of the present work is the introduction of the HH effective interaction. To this end we investigate the rate of convergence of the HH series with the effective interaction for ground state energies, radii, and excitation energies. In addition we study transitions to continuum states via the method of the Lorentz integral transform [24].

By expanding the effective wave functions into HH basis functions we transform the effective Hamiltonian [Eq. (15)] into a set of coupled differential equations in the hyperradius ρ. These equations are then solved by expanding the solution in terms of generalized Laguerre polynomials.

Our results for ground state energies and radii as well as first excitation energies are summarized in Table II. In cases where results from other authors were available, they are also given in Table II. In general one observes very good agreement between our results and those of the various other methods. In the following we discuss in detail the quality of the convergence for the calculated observables.

In Fig. 1 we illustrate the convergence patterns with bare and effective interactions for binding energy and radius of the $A=3$ system with the MN potential. It is readily seen that the effective interaction improves the convergence drastically. Already with $K=2$ one finds for the energy (radius) a deviation of only 0.6% (1.3%) from the converged value, while with the bare interaction one needs $K=10(8)$ for a similarly good result. For $K=10$ one obtains sufficiently converged results with the effective interaction, whereas for the bare interaction one has to go up to $K=18$ or higher to reach a similar precision.

In Fig. 2 we show the corresponding results for ^{4}He with the MTV potential. In this case the difference between the convergence of effective and bare interactions is even stronger. For the effective interaction one nearly obtains the correct values for energy and radius with a rather low K of 4. The convergence with the bare interaction is considerably worse since even with $K=20$ one does not have completely converged results. In Fig. 3 we compare our results to those of Navrátil and Barrett with the HO effective interaction [5,25]. One also obtains a very nice convergence in the HO case, although it depends on the chosen harmonic oscillator frequency Ω. On the other hand it is evident that the parameter-free HH effective interaction leads to a considerable improvement.

In Figs. 4–6 we give an overview of our results for the bound state properties with the various potential models. For $A=4$ one finds very good convergence for both the binding energy and radius. For the systems with $A>4$, which can be considered approximately as an α core plus remaining nucleons at larger distances, there is generally good convergence, except for the case of the radii with the MTV potential. We do not reach the same extremely good precision as in the cases with $A\leqslant 4$, however, since we have to restrict our calculation to a somewhat lower K. This is due to the fact that an increasing number of nucleons leads to a much higher number of HH functions for the same K. In general our calculations are limited to about 400 HH basis functions.

The first excitation energies of the $A=6$ systems are illustrated in Fig. 7. One sees that the convergence patterns are quite similar to the ground state patterns. Figure 7 also shows that also non-ground-state observables can be calculated with sufficient precision with the HH effective interaction.

One may ask what happens at even higher energies, e.g., in reactions where states in the continuum are involved. In order to address this question we consider response functions describing transitions from the ground state to the continuum due to an external probe. Such response functions can be calculated with the method of the Lorentz integral transform [24]. In this formalism, one needs, in addition to the ground state, an additional ''Lorentz-state'' $\tilde{\Psi}$, which is localized and which carries all the information about the excitation of the system and of the final state interactions. The conver-

TABLE II. Comparison of binding energies (E_B) in (MeV) and root mean square radii ($\langle r^2\rangle^{1/2}$) in (fm) obtained with the present effective interaction method in the HH formalism (EIHH) with results of other methods. In the case of EIHH the calculations with MTI-III and MN potentials include the Coulomb interaction. For EIHH the number in parenthesis indicates the variance with respect to the result obtained with $K=K_{\max}-1$. The quality of the convergence can be inferred from Figs. 1–6.

		MN		MTI–III		MTV	
Nucleus	Method [Ref.]	E	$\langle r^2\rangle^{1/2}$	E	$\langle r^2\rangle^{1/2}$	E	$\langle r^2\rangle^{1/2}$
^{3}H	EIHH	−8.3856(5)	1.7036(1)	−8.718(9)	1.7064(2)	−8.244(8)	1.6798(3)
	EIHO [25]					−8.235(5)	
	SVM [26]	−8.380	1.698			−8.2527	1.682
	Faddeev [27]					−8.25273	
	ATMS [28]					−8.26(1)	1.682
	CHH1 [29]					−8.240	
	CHH2 [30]			−8.716			
	GFMC [31]					−8.26(1)	1.682
	VMC [26]					−8.27(3)	1.68
^{3}H*	EIHH	−0.421(9)	4.757(5)	−1.01(1)	7.2(1)	−0.073(6)	6.973(8)
^{4}He	EIHH	−29.96(1)	1.4106(1)	−30.71(2)	1.4222(2)	−31.358(9)	1.40851(3)
	SVM [26]	−29.937	1.41			−31.360	1.4087
	FY [32]					−31.36	
	ATMS [28]					−31.36	1.40
	CHH2			−30.69	1.421		
	CRCG [33]					−31.357	
	GFMC [31]					−31.3(2)	1.36
	VMC [26]					−31.30(5)	1.39
^{4}He*	EIHH	−7.848(4)	3.405(7)	−8.48(1)	3.59(2)		
^{5}He	EIHH	−28.1(2)	2.17(6)	−29.4(1)	2.13(5)	−43.7(3)	1.499(5)
	SVM [26]					−43.48	1.51
	VMC [26]					−43.0(2)	1.51
^{5}He*	EIHH	−20.8(9)	3.33(1)	−22.(1)	3.312(2)		
^{6}He	EIHH	−30.6(5)	2.323(2)	−32.5(3)	2.263(9)	−68.5(2)	1.512(4)
	SVM [26]	−30.07	2.44			−66.30	1.52
	VMC [26]					−66.3(3)	1.50
^{6}He*	EIHH	−22.5(3)	3.55(9)	−23.5(2)	3.54(3)		
^{6}Li	EIHH	−35.2(4)	2.16(2)	−36.6(3)	2.15(2)	−68.5(2)	1.512(4)
	SVM [26]	−34.59	2.22			−66.30	1.52
	CHH3 [9]					−64.55	
^{6}Li*	EIHH	−25.7(4)		−26.6(5)	3.35(3)		

gence of the Lorentz integral transform depends on the K_0, the maximum hyperangular quantum number in the ground state, and on K_T, the maximum hyperangular quantum number in $\tilde{\Psi}$. Both effects are illustrated in Fig. 8, where we show the Lorentz integral transform of the ^{4}He total photoabsorption cross section in the electric dipole approximation with the MTI-III potential. One sees that for a convergent result of the transform a rather low K_T for the HH expansion

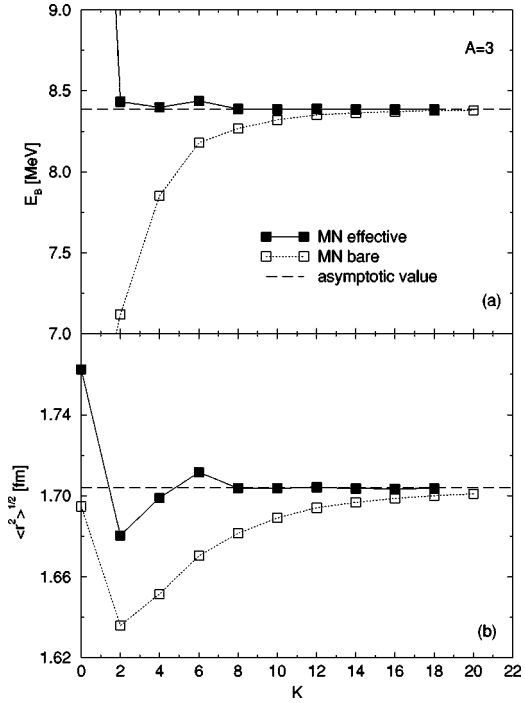

FIG. 1. Binding energy (a) and root mean square radius (b) of the $A=3$ system for the Minnesota potential [22] as a function of the hyperangular quantum number K. The asymptotic value has been indicated by a dashed line.

of $\tilde{\Psi}$ is sufficient, while one has to include somewhat higher K_0's in the HH expansion of the ground state. Our result for the Lorentz transform is a bit lower than that of Ref. [34] since there, as opposed to the Lorentz state, the ground state

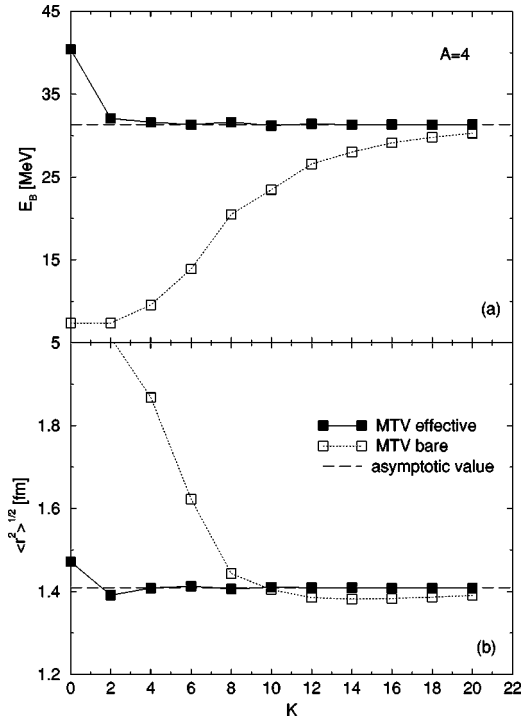

FIG. 2. Same as Fig. 1 for $A=4$ and the MTV potential [20].

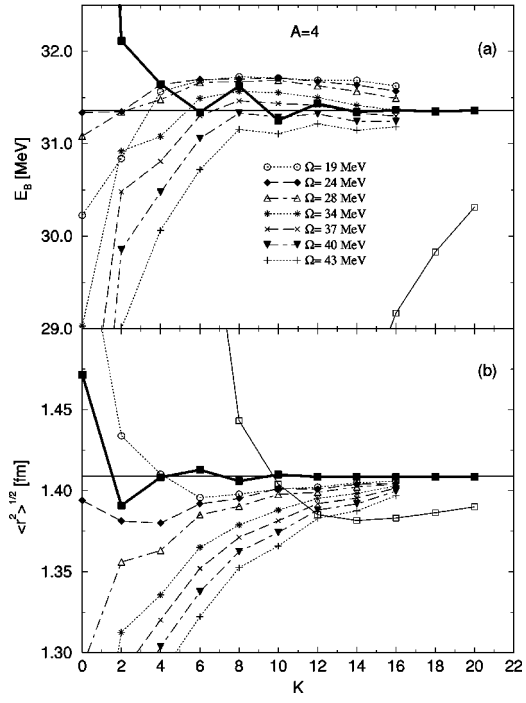

FIG. 3. Comparison between results of the present method (full squares: effective interaction; open squares: bare interaction) and that of Refs. [5,25] obtained with different values of the HO parameter $\Omega(\hbar=1)$. Binding energy (a) and root mean square radius (b) of the $A=4$ system for the MTV potential [20] as a function of the hyperangular quantum number K or of the HO quantum number N.

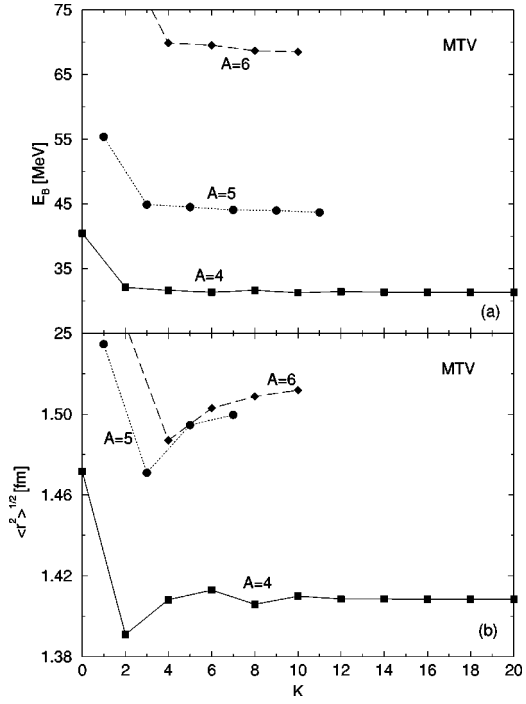

FIG. 4. Binding energies (a) and root mean square radii (b) of different A-body systems for the MTV potential [20] as a function of the hyperangular quantum number K.

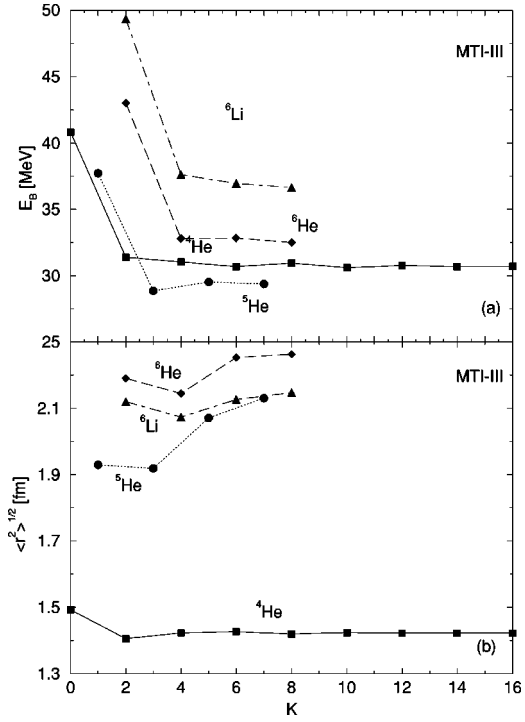

FIG. 5. Same as Fig. 3 for MTI-III potential [21].

was calculated without correlations and was not fully convergent. However the conclusions of Ref. [34] remain unchanged.

V. CONCLUSION

In this work we have introduced a hyperspherical effective interaction. To this end we have defined a model space consisting of a complete hyperradial basis and a set of HH functions with generalized angular momentum quantum number $K \leqslant K_{\max}$. The effective interaction has been derived from a hyperangular Hamiltonian related to the two-body

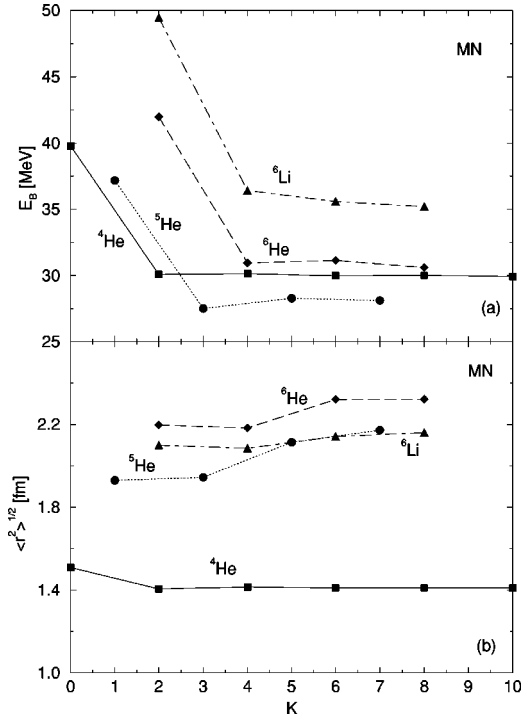

FIG. 6. Same as Fig. 3 for MN potential [22].

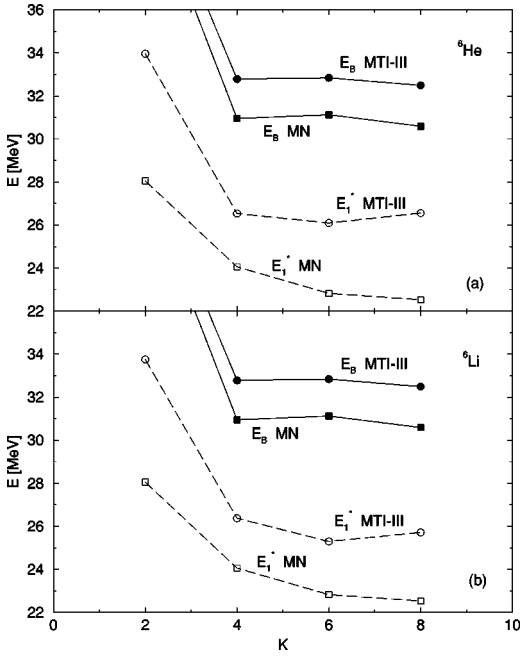

FIG. 7. Binding energies E_B and first excitation energies $E_1^\star$ of ^{6}He (a) and ^{6}Li (b) for the MTI-III [21] and MN potentials [22].

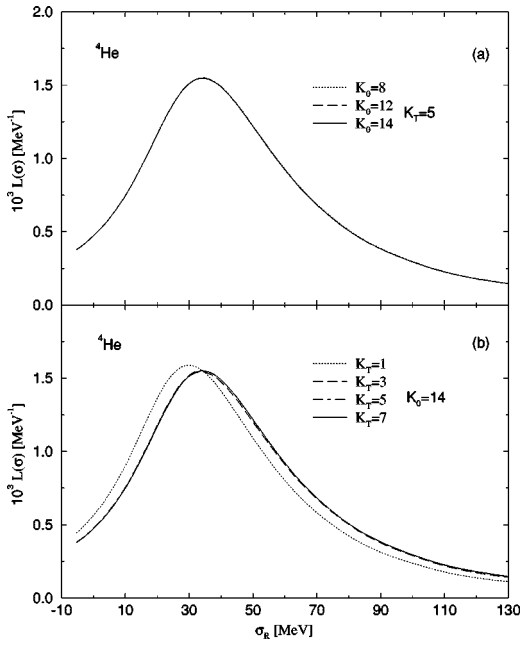

FIG. 8. Lorentz integral transform of the ^{4}He total photoabsorption cross section as a function of σ_R ($\sigma_I = 20$ MeV). In (a) the convergence in the ground state hyperangular quantum number K_0 is shown for a fixed value of the hyperangular quantum number K_T of the ''Lorentz state'' $\tilde{\Psi}$; in (b) the convergence in the hyperangular quantum number K_T of the ''Lorentz state'' $\tilde{\Psi}$ is shown for a fixed value of the ground state hyperangular quantum number K_0.

problem. The Hamiltonian also includes the hyperangular kinetic energy, which is proportional to $1/\rho^2$, where ρ is the collective hyperradial coordinate. Because of this additional A-body piece the $A-2$ residual system cannot be considered as a pure spectator. We are led to an effective interaction depending explicitly on the state of the residual system, similar to the HO multivalued effective interaction.

We should mention that the present approach can be extended in a straightforward way to derive an HH three- or more-body effective interaction.

We have applied the formalism developed here to few-body systems in the mass range $A=3-6$. For these systems we have calculated binding/excitation energies, radii and, via the method of the Lorentz integral transform, reactions at energies far into the continuum. For the larger A's these calculations have become feasible due to a powerful algorithm for constructing symmetrized basis states. In general we obtain nicely converging results showing that the HH effective interaction is a very powerful tool. Particularly interesting is the fact that the rate of convergence is always very good and does not depend much on the observable under consideration. We believe that the inherent confinement of the effective ''two-body'' Hamiltonian is largely responsible for this fact.

We have compared our results to other calculations available in the literature. In general a very good agreement is obtained.

Finally we would like to point out that the present approach has various advantages compared to the harmonic oscillator formalism: (i) one does not need to introduce an additional confining potential and, contrary to the HO approach, one obtains an effective interaction which is essentially parameter free depending only upon the size of the model space. Therefore one does not have a problem that is typical for the HO formalism, namely, that the convergence of different observables, e.g., binding energies and radii, lead to rather different optimal choices for such a parameter; (ii) the HH effective interaction is automatically state dependent; and (iii) the HH basis functions are more appropriate for describing the asymptotic part of the wave function than are the HO basis functions. On the other hand, it will require additional effort to elevate the present approach to the same level of sophistication as the HO formalism when incorporating modern realistic interactions and a larger number of particles.

ACKNOWLEDGMENTS

The authors would like to thank Petr Navrátil for triggering this work by presenting his results at an ECT* workshop, for stimulating discussions and for providing us with the ^{4}He results for the MTV potential. We would also like to acknowledge Bruce Barrett and Kalman Varga for useful discussions.

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