Radiative corrections to elastic electron-proton scattering for polarized electrons

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We analyze the radiative correction to high energy elastic electron-proton scattering of polarized electrons. We show that if the approximations inherent in the calculations developed by Tsai and given in the work of Mo and Tsai which have been used in the analysis of almost all experimental data pertaining to medium and high energy elastic electron scattering for the past three decades are maintained, then the same radiative correction applies both in the case of initially polarized and unpolarized electrons.

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I. INTRODUCTION

The radiative correction to elastic electron-proton scattering is well known from the work of Tsai [1] and Mo and Tsai [2], and the expressions given in [2] have been used in the analysis of almost all experimental data pertaining to medium and high energy elastic electron scattering for the past three decades. Very recently, experiments using polarized electron beams have been carried out at Jefferson Lab [3]; specifically, longitudinally polarized electrons were scattered from unpolarized protons ($\vec{e}p\to e\vec{p}$) and the transverse and longitudinal polarizations of the recoil protons were measured in order to obtain the ratio of the proton's elastic electromagnetic form factors G_{E_p}/G_{M_p}. Given that radiative corrections to elastic electron-proton scattering are generally of the order of 20–30 % for four-momentum transfer squared in the range considered in these experiments [0.5 to 3.5 $(\mathrm{GeV}/c)^2$], the question arises as to whether the same radiative correction used in the case of unpolarized beams and targets can be applied in the case of polarized electron beams when the polarization of the recoil proton is measured. We show here that if the approximations inherent in the calculations developed in [1] and given in [2] are maintained, then the same radiative correction applies both in the case of initially polarized and unpolarized electrons. In Sec. II we present the cross section for the scattering of polarized electrons from unpolarized protons in the absence of radiative corrections. In Sec. III we give each of the matrix elements associated with the radiative correction and discuss the significant approximations that are made in [1] to evaluate their contribution to the cross section. We then show that with these approximations the radiative corrections do not depend on the polarization of either the electron or the proton in the initial or final state.

II. DIFFERENTIAL CROSS SECTION FOR SCATTERING OF POLARIZED ELECTRONS

The differential cross section for the scattering of polarized electrons from unpolarized protons can be derived using standard techniques of quantum electrodynamics. We follow the conventions of Bjorken and Drell [4]; the metric is defined by $p_i\cdot p_j=\epsilon_i\epsilon_j-\mathbf{p}_i\cdot\mathbf{p}_j$. Further, $\alpha=e^2/4\pi=1/137.036$; m is the electron rest mass; M is the target nucleus rest mass; κ the anomalous magnetic moment of the proton; p_1 and p_3 the initial and final electron four-momenta respectively; p_2 and p_4 the initial and final target nucleus four-momenta respectively; $q=p_1-p_3=p_4-p_2$ is the four-momentum transfer to the target nucleus for elastic scattering.

For one-photon exchange the matrix element is

$$M_0=e^2\overline{u}(p_3)\gamma^\mu u(p_1)\frac{(-i)}{q^2+i\epsilon}\overline{u}(p_4)\Gamma_\mu(q^2)u(p_2), \qquad (2.1)$$

whose magnitude squared summed over final electron spin and averaged over initial proton spin is

$$|\overline{M}_0|^2=\frac{1}{2}\mathrm{Tr}\{\gamma^\nu\Lambda_3\gamma^\mu\Lambda_1\Sigma_1\}\mathrm{Tr}\{\Sigma_4\Lambda_4\Gamma_\mu\Lambda_2\tilde{\Gamma}_\nu\}, \qquad (2.2)$$

where $\Lambda_i=(\not{p}_i+m_i)/(2m_i)$ and $\Sigma_i=(1+\gamma_5\not{s}_i)/2$ are energy and spin projection operators and

$$\Gamma_\mu=F_1(q^2)\gamma_\mu+\kappa F_2(q^2)\frac{i\sigma_{\mu\alpha}q^\alpha}{2M}, \quad (\tilde{\Gamma}_\nu\equiv\gamma^0\Gamma_\nu^\dagger\gamma^0) \qquad (2.3)$$

is the proton-current operator. We assume high energies for the initial and final electrons ($\epsilon_1,\epsilon_3\gg m$) and large momentum transfers ($-q^2\gg m^2$). Further, we express the cross section in terms of the Sachs form factors $G_E(q^2)$ and $G_M(q^2)$, which are defined in terms of F_1 and F_2 by

$$G_E=F_1-\tau\kappa F_2, \quad G_M=F_1+\kappa F_2, \qquad (2.4)$$

where $\tau=-q^2/4M^2$. Finally we express the spin polarization four-vectors of the initial electron and final proton, s_1 and s_4, respectively, in terms of the three-dimensional unit vectors specifying the spin direction of the particles in their respective rest frames $\boldsymbol{\zeta}_1$ and $\boldsymbol{\zeta}_4$. In general for a particle of mass m and four-momentum $p=(\epsilon,\mathbf{p})$, the four-vector s is given in terms of $\boldsymbol{\zeta}$ by [5,6]

$$s_0=\frac{\boldsymbol{\zeta}\cdot\mathbf{p}}{m}, \qquad (2.5)$$

$$\mathbf{s}=\boldsymbol{\zeta}+\mathbf{p}\left[\frac{\boldsymbol{\zeta}\cdot\mathbf{p}}{m(m+\epsilon)}\right].$$

For the initial electron we have, neglecting terms of relative order m/ϵ_1,

$$s_1 \doteq h p_1/m, \tag{2.6}$$

where $h=\boldsymbol{\zeta}_1\cdot\hat{\mathbf{p}}$. The cross section for the scattering of high energy polarized electrons into the direction θ by unpolarized protons initially at rest is then

$$\frac{d\sigma}{d\Omega}=\frac{\alpha^2\epsilon_3\cos^2\frac{\theta}{2}}{4\epsilon_1^3\sin^4\frac{\theta}{2}}\frac{1}{(1+\tau)}\left\{\begin{array}{c}G_E^2+\tau G_M^2+2\tau(1+\tau)G_M^2\tan^2\frac{\theta}{2}\\ +h\left[\frac{\epsilon_1+\epsilon_3}{M}\sqrt{\tau(1+\tau)}G_M^2\tan^2\frac{\theta}{2}\,\boldsymbol{\zeta}_4\cdot\hat{\mathbf{z}}-2\sqrt{\tau(1+\tau)}G_MG_E\tan\frac{\theta}{2}\,\boldsymbol{\zeta}_4\cdot\hat{\mathbf{x}}\right]\end{array}\right\}, \tag{2.7}$$

where we take the unit vector $\hat{\mathbf{z}}$ in the direction of $\mathbf{p}_4$, the unit vector $\hat{\mathbf{y}}$ in the direction of $\mathbf{p}_1\times\mathbf{p}_3$ (i.e., perpendicular to the scattering plane), and the unit vector $\hat{\mathbf{x}}$ in the scattering plane and defined by $\hat{\mathbf{x}}=\hat{\mathbf{y}}\times\hat{\mathbf{z}}$.

In Eq. (2.7) the spin-independent terms give the well-known Rosenbluth cross section. The remaining terms determine the longitudinal and perpendicular polarization of the recoil proton [8].

III. RADIATIVE CORRECTIONS TO ELASTIC ELECTRON-PROTON SCATTERING

In this section we consider each of the terms contributing to the radiative correction to elastic electron-proton scattering as treated in the generally used analysis given in [1] and [2]. We show that if one makes the approximations which are inherent to the derivation given in these references then the radiative correction to elastic electron-proton scattering is the same for polarized and unpolarized electrons and protons.

The radiative correction is comprised of the purely elastic amplitudes (electron and proton vertex corrections, electron and proton self energies, box and crossed box diagrams and vacuum polarization terms) and inelastic amplitudes (emission of soft bremsstrahlung photons by any of the charged particles). Let us consider each of these in turn. The cross section for emission of soft photons $d\sigma_{\rm brem}$, is simply equal to a factor which multiplies the one-photon exchange cross section $d\sigma$, and that factor is independent of the spins of the electrons and protons:

$$d\sigma_{\rm brem}=-\frac{\alpha}{4\pi^2}d\sigma\int{}'\frac{d^3k}{\omega}\left(\frac{p_3}{p_3\cdot k}-\frac{p_1}{p_1\cdot k}-\frac{p_4}{p_4\cdot k}+\frac{p_2}{p_2\cdot k}\right)^2. \tag{3.1}$$

Consider next the radiative corrections to the purely elastic cross section. To lowest order in α these are found from the cross product of the matrix element for one-photon exchange, M_0, and the matrix elements for each of the higher order processes:

$$|\mathcal{M}|^2=|M_0|^2+2\,\mathrm{Re}\{M_0^\dagger(M_1+M_2+\cdots)\}. \tag{3.2}$$

Thus, provided the matrix elements $M_1, M_2, \ldots$ can be expressed as M_0 times a factor which is independent of the spin of the particles, the radiative correction for elastic scattering will factor as a spin-independent term.

The matrix element for vacuum polarization, M_1, is, after charge renormalization, related simply to the matrix element M_0 by

$$M_1=M_0\sum_i \Pi(q^2/m_i^2) \tag{3.3}$$

in which $\Pi(q^2/m_i^2)$ is independent of the spins of the particles [7,4] and the sum is carried over the electron and higher mass particle-antiparticle loops.

The matrix element for the electron vertex correction, M_2, is given by

$$M_2=Ze^2\overline{u}(p_3)\Lambda^\mu(p_3,p_1)u(p_1)\frac{(-i)}{q^2+i\epsilon}\overline{u}(p_4)\Gamma_\mu(q^2)u(p_2), \tag{3.4}$$

where

$$\Lambda^\mu(p_3,p_1)=-ie^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2-\lambda^2+i\epsilon}\times\gamma^\nu\frac{1}{(\not p_3-\not k-m+i\epsilon)}\times\gamma^\mu\frac{1}{(\not p_1-\not k-m+i\epsilon)}\gamma_\nu. \tag{3.5}$$

Comparing Eq. (3.4) with Eq. (2.1) we see that if the spin-operator dependence in $\Lambda^\mu(p_3,p_1)$ reduces to γ^μ, then M_2 will be a multiple of M_0, the factor being independent of the spins of the particles. As it stands, the integral for $\Lambda^\mu(p_3,p_1)$ is divergent. However, if we introduce a convergence factor, $-\Lambda^2/(k^2-\Lambda^2+i\epsilon)$, in the integrand then the integration can be carried out, and taking the limit $\Lambda\to\infty$ we find that $\Lambda^\mu(p_3,p_1)$ has the form $G_1(q^2)\gamma^\mu+G_2(q^2)(i\sigma^{\mu\nu}q_\nu/2m)$ where

$$G_1^{(e)}(q^2)=\frac{\alpha}{4\pi}\left\{-2(2m^2-q^2)\phi_1(\lambda^2)+\left(\frac{3\rho^2-4m^2}{\rho\rho_1}\right)\ln x+\frac{1}{2}+\ln\left(\frac{\Lambda^2}{m^2}\right)\right\} \quad (3.6)$$

and

$$G_2^{(e)}(q^2)=\frac{\alpha}{4\pi}\left\{\frac{4m^2}{\rho\rho_1}\ln x\right\} \quad (3.7)$$

in which

$$\phi_1(\lambda^2)\xrightarrow[\lambda\to 0]{}\frac{1}{\rho\rho_1}\left\{-2L\left(-\frac{1}{x}\right)-\frac{\pi^2}{6}-\frac{1}{2}\ln^2 x+\ln x\,\ln\left(\frac{\rho^2}{\lambda^2}\right)\right\},$$
$$L(z)=-\int_0^z\frac{\ln(1-t)}{t}dt, \quad (3.8)$$

with $\rho^2=-q^2+4m^2$, $\rho_1^2=-q^2$, and $x=(\rho+\rho_1)/(\rho-\rho_1)=(\rho+\rho_1)^2/4m^2$. Thus for $-q^2\gg m^2$ the term $G_2(q^2)$ is of order $m^2/(-q^2)$ relative to $G_1(q^2)$ and hence may be neglected so that we have $M_2=G_1(q^2)M_0$. The inclusion of the self-energy contribution for the electron is obtained by subtracting $\Lambda^\mu(p_1,p_1)$ from the expression given in Eq. (3.5), giving

$$\tilde{M}_2=[G_1(q^2)-G_1(0)]M_0, \quad (3.9)$$

where for $-q^2\gg m^2$

$$G_1(q^2)-G_1(0)=\frac{\alpha}{2\pi}\left\{-\frac{1}{2}\ln^2\left(\frac{-q^2}{m^2}\right)+\frac{\pi^2}{6}-\left[\ln\left(\frac{-q^2}{m^2}\right)-1\right]\ln\left(\frac{m^2}{\lambda^2}\right)+\frac{3}{2}\ln\left(\frac{-q^2}{m^2}\right)-2\right\}. \quad (3.10)$$

Finally, we consider the proton vertex correction and the box and crossed box contributions M_3, M_4, and M_5, respectively. The matrix elements for these corrections are given by

$$M_3=Z^3e^2\overline{u}(p_3)\gamma^\mu u(p_1)\frac{(-i)}{q^2+i\epsilon}\overline{u}(p_4)\Lambda_\mu(p_4,p_2)u(p_2), \quad (3.11)$$

with

$$\Lambda_\mu(p_4,p_2)=-ie^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2-\lambda^2+i\epsilon}\times\Gamma^\nu(k^2)\frac{1}{(\not{p}_4-\not{k}-M+i\epsilon)}\Gamma_\mu(q^2)\times\frac{1}{(\not{p}_2-\not{k}-M+i\epsilon)}\Gamma_\nu(k^2), \quad (3.12)$$

$$M_4=(Ze^2)^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2-\lambda^2+i\epsilon}\frac{1}{(k-q)^2-\lambda^2+i\epsilon}\times\left[\overline{u}(p_3)\gamma^\nu\frac{1}{\not{p}_1-\not{k}-m+i\epsilon}\gamma^\mu u(p_1)\right]\times\left[\overline{u}(p_4)\Gamma_\nu((k-q)^2)\frac{1}{\not{p}_2+\not{k}-M+i\epsilon}\times\Gamma_\mu(k^2)u(p_2)\right], \quad (3.13)$$

and

$$M_5=(Ze^2)^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2-\lambda^2+i\epsilon}\frac{1}{(k-q)^2-\lambda^2+i\epsilon}\times\left[\overline{u}(p_3)\gamma^\nu\frac{1}{\not{p}_1-\not{k}-m+i\epsilon}\gamma^\mu u(p_1)\right]\times\left[\overline{u}(p_4)\Gamma_\mu(k^2)\frac{1}{\not{p}_4-\not{k}-M+i\epsilon}\times\Gamma_\nu((k-q)^2)u(p_2)\right]. \quad (3.14)$$

In general these matrix elements depend on the initial and final spin states and are not proportional to M_0 times a spin independent factor.

Now consider the approximation used in [1] to evaluate these matrix elements which we call here the soft-photon approximation. The integrands in M_4 and M_5 have two infrared divergent factors $[(k^2-\lambda^2+i\epsilon)((k-q)^2-\lambda^2+i\epsilon)]^{-1}$ and are thus peaked when either of the two exchanged photons is soft, becoming divergent when $k\to 0$ or when $k\to q$. We therefore first rationalize the propagators so that all spin matrices are in the numerator and then evaluate the *numerators* in M_4 and M_5 at these two points [first setting $k=0$ and then setting $k=q$; note that $\Gamma_\mu(0)=\gamma_\mu$] but make no changes to the denominators. A simple calculation using the fact that we have on-shell particles shows that in fact each of the numerators has the same value for $k=0$ as for $k=q$, viz., $4ip_1\cdot p_2q^2M_0$ in the case of M_4 and $4ip_3\cdot p_2q^2M_0$ in the case of M_5. Taking this factor outside of the integral we are left with a scalar four-point function, independent of the particle spins:

$$M_4 = 8iZe^2q^2M_0p_1\cdot p_2\int \frac{d^4k}{(2\pi)^4} \times\frac{1}{k^2-\lambda^2+i\epsilon}\,\frac{1}{(k-q)^2-\lambda^2+i\epsilon} \times\frac{1}{(k^2-2k\cdot p_1+i\epsilon)}\,\frac{1}{(k^2+2k\cdot p_2+i\epsilon)} \tag{3.15}$$

and

$$M_5 = 8iZe^2q^2M_0p_3\cdot p_2\int \frac{d^4k}{(2\pi)^4} \times\frac{1}{k^2-\lambda^2+i\epsilon}\,\frac{1}{(k-q)^2-\lambda^2+i\epsilon} \times\frac{1}{(k^2-2k\cdot p_1+i\epsilon)}\,\frac{1}{(k^2-2k\cdot p_4+i\epsilon)}. \tag{3.16}$$

We note that in [1] an approximation is also made in the denominators of these integrals, reducing these four-point functions to three-point functions, but this is not needed for the conclusions of the present paper.

In the case of M_3 the integrand is peaked when $k=0$; we therefore set $k=0$ in all terms of the *numerator* of M_3 again using the fact that we have on-shell particles and find

$$M_3 = -4iZ^2e^2p_4\cdot p_2M_0\int \frac{d^4k}{(2\pi)^4}\,\frac{1}{(k^2-\lambda^2+i\epsilon)} \times\frac{1}{(k^2-2k\cdot p_4+i\epsilon)}\,\frac{1}{(k^2-2k\cdot p_2+i\epsilon)}. \tag{3.17}$$

With the soft-photon approximation the proton vertex correction is a multiple of M_0 and, as with M_2, the factor is independent of the spins of the particles. Again because of the soft-photon approximation, the self-energy contribution is essentially the same as that obtained for the electron: since the virtual photon in the self-energy diagrams is assumed to be soft, its interaction with the proton is given by γ_μ, as in the case of the electron, so that the self-energy contribution is obtained by subtracting $\Lambda_\mu(p_2,p_2)$ from the expression given in Eq. (3.12).

Thus, substituting the expressions for M_1, M_2, M_3, M_4, and M_5 given in Eqs. (3.3) (3.9), (3.17), (3.15), (3.16) in Eq. (3.2) and adding the contribution from real soft photons (3.1) we see that the cross section can be written in the form

$$d\sigma_{\rm corr} = d\sigma(1+\delta), \tag{3.18}$$

in which the radiative correction term δ is independent of the spins of the particles.

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