Spectra of doubly heavy quark baryons

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(Received 22 September 1999; published 17 March 2000)

Baryons containing two heavy quarks are treated in the Born-Oppenheimer approximation. The Schrödinger equation for the two-center Coulomb plus harmonic oscillator potential is solved by the method of the ethalon equation at large intercenter separations. Asymptotical expansions for the energy term and wave function are obtained in analytical form. Using those formulas, the energy spectra of doubly heavy baryons with various quark compositions are calculated analytically.

PACS number(s): 24.85.+p, 03.65.Pm, 12.39.Pn, 12.40.Yx

INTRODUCTION

The investigation of the properties of hadrons containing one or more heavy quarks is very important for understanding the dynamics of quark and gluon interactions. Presently at the LHC, *B* factories, and the Tevatron with high luminosity, several experiments have been proposed, in which a detailed study of baryons containing two heavy quarks can be performed. In particular, in the forthcoming experiment at CERN, the COMPASS group is going to find the doubly charmed baryons and study their physical properties. In this connection, doubly heavy quark baryons are now becoming one of the most exciting subjects in particle physics. Therefore, theoretical predictions of the properties of doubly heavy quark baryons acquire a large significance for the forthcoming experimental study of these particles. So far there have been various approaches by which their mass spectra and other properties can be calculated. One of them is the nonrelativistic quark model which gives relatively accurate results for baryon spectra [1–3]. The possible quark compositions of doubly heavy quark baryons are *ccq*, *cbq*, and *bbq*, where q denotes a light u,d, or s quark. Note that the baryons containing the top quark(s) are not a practical subject here because the top quark is extremely heavy and hence we have no chance of finding them as stable hadrons. The doubly heavy quark baryons may be considered as an analog of the hydrogen molecular ion H_2^+, which has been treated successfully in the Born-Oppenheimer approximation. The same approximation is expected to be efficient even for doubly heavy quark baryons, though there exist some differences between these baryons and H_2^+ systems. One of them is, for this case, the appearance of the confining potential in addition to the QCD Coulomb potential. As is well known, the variables of the Schrödinger equation with two-center Coulomb plus confining potential cannot be separated for their kinematical variables, in general. To our knowledge, the two-center potential which allows the separation of variables is only the two-center Coulomb plus harmonic oscillator potential.

In this paper, we treat QQq baryons in the nonrelativistic approach by using the solution of the Schrödinger equation with two-center Coulomb plus harmonic oscillator potential, i.e., the well-known method of the ethalon equation which is widely used for solving the Schrödinger equation with two-center pure Coulomb potential in the physics of H_2^+ [4–6]. First, we give a general scheme of treatment QQq baryons in the Born-Oppenheimer approximation. Then, the two-center Schrödinger equation with two-center Coulomb plus harmonic oscillator potential is analytically solved with some approximation: the energy term of the light quark moving in the field of two heavy quarks is obtained in the form of an asymptotical expansion over the inverse power of the distance between heavy quarks. Finally, we give an analytical formula of the baryon energy spectrum for QQq.

DOUBLY HEAVY QUARK BARYON IN THE BORN-OPPENHEIMER APPROXIMATION

In the Born-Oppenheimer approximation the wave function is split into heavy- and light-quark degrees of freedom

$$\Psi(R,r)=\sum_n \phi_n(R)\psi_n(R,r),$$

where R is the distance between two heavy quarks and r is the distance between the light quark and center of mass of the heavy-quark pair. The light-quark wave function $\psi(r,R)$ and its energy term $E(R)$ can be found from the Schrödinger equation

$$\left[-\frac{1}{2m_q}\Delta+V(r_1)+V(r_2)\right]\psi=E(R)\psi,$$

where r_1 and r_2 are the distances between light and heavy quarks, Q_1 and Q_2, respectively. The binding energy of this system is approximated by the equation

$$\left[-\frac{1}{2\overline{M}_{QQ}}\Delta+V_{QQ}(R)+E(R)\right]\phi=\varepsilon\phi,$$

where $\overline{M}_{QQ}$ is the reduced mass of QQ.

A quark potential with Coulomb plus harmonic confinement for this baryon is given by [3]

$$V(r_{ij})=\sum_{i,j}\frac{1}{4}\lambda_i\lambda_j\left(V_0-Ar_{ij}^2+\frac{\alpha_s}{r_{ij}}\right)$$
$$=-\frac{2}{3}\sum_{i,j}\left(V_0-Ar_{ij}^2+\frac{\alpha_s}{r_{ij}}\right).$$

In the field of two heavy quarks with this potential, the motion of a light quark can be nonrelativistically described by the following Schrödinger equation:

$$\left[-\frac{1}{2}\Delta-\frac{Z}{r_1}-\frac{Z}{r_2}+\omega^2(r_1^2+r_2^2)-\frac{4}{3}V_0\right]\psi=E(R)\psi, \quad (1)$$

where $Z=2\alpha_s/3$ and $\omega^2=2A/3$.

In prolate spheroidal coordinates defined as

$$\xi=\frac{r_1+r_2}{R}(1<\xi<\infty), \quad \eta=\frac{r_1-r_2}{R}(-1<\eta<1),$$

the potential term in Eq. (1) can be written in the form

$$V(r_1,r_2)=-\frac{2}{R^2}\frac{a(\xi)+b(\eta)}{\xi^2-\eta^2}+\frac{\omega^2R^2}{2}-\frac{4}{3}V_0, \quad (2)$$

where

$$a(\xi)=2ZR-\frac{\omega^2R^4}{4}\xi^2(\xi^2-1),$$

$$b(\eta)=2ZR-\frac{\omega^2R^4}{4}\eta^2(\eta^2-1).$$

As is well known [4], the Schrödinger equation with potential in the form of Eq. (2) is separable in prolate spheroidal coordinates. Then it is convenient to use

$$\psi=\frac{U(\xi)}{\sqrt{\xi^2-1}}\frac{V(\eta)}{\sqrt{1-\eta^2}}\frac{e^{\pm im\phi}}{\sqrt{2\pi}},$$

where ϕ and m are azimuthal angle and azimuthal quantum number, respectively. After substituting this into Eq. (1), we obtain from the following ordinary differential equations connected with separation constants λ and m:

$$U''(\xi)+\left[\frac{h^2}{4}+\frac{h(\alpha\xi-\lambda)}{\xi^2-1}-h^4\gamma\xi^2+\frac{1-m^2}{(\xi^2-1)^2}\right]U(\xi)=0, \quad (3)$$

$$V''(\eta)+\left[\frac{h^2}{4}+\frac{h\lambda}{1-\eta^2}-h^4\gamma\eta^2+\frac{1-m^2}{(1-\eta^2)^2}\right]V(\eta)=0, \quad (4)$$

where $\alpha=2Z/\sqrt{2E'}$ and $\gamma=\omega^2/8E'^2$, and further

$$h=\sqrt{2E'}R, \quad (5)$$

with

$$E'=E-\frac{\omega^2R^2}{2}+\frac{4}{3}V_0.$$

The finiteness and continuity of the wave function ψ in the whole space lead to the following boundary conditions for the functions U and V:

$$U(\xi)|_{\xi=1}=0, \quad U(\xi)|_{\xi\to\infty}\to 0, \quad (6)$$

$$V(\eta)|_{\eta=\pm 1}=0. \quad (7)$$

ASYMPTOTICS OF THE QUASIANGULAR EQUATION

We will approximately solve Eqs. (3) and (4) for large R by the method of the ethalon equation. This method is successfully applied to the solution of the nonrelativistic two-center Coulomb problem [4–6] and in the theory of the diffraction of waves. Details of the method of the ethalon equation are given in [4–7], also briefly described in Appendix B.

Let us start from the angular equation (4). As an ethalon equation for Eq. (4), we choose the following Whittaker equation [8]:

$$W''+\left[-\frac{h^4}{4}+\frac{h^2k}{z}+\frac{1-m^2}{4z^2}\right]W=0 \quad (8)$$

and seek a solution in the form

$$V=[z'(\eta)]^{-1/2}M_{k,m/2}(h^2z), \quad (9)$$

where $M_{k,m/2}(h^2z)$ is the solution (regular at zero) of Eq. (8). Substituting Eq. (9) into Eq. (4) and taking into account Eq. (8), we get the following equation for z:

$$\frac{z'^2}{4}-\gamma(x-1)^2-\frac{1}{h^2}\left(\frac{1}{4}+\frac{kz'^2}{z}-\frac{\lambda}{2x(1-x/2)}\right)$$
$$+\frac{\tau}{h^2}\left(\frac{1}{x^2(1-x^2)}-\frac{z'^2}{z^2}\right)-\frac{1}{2h^2}\{z,x\}=0, \quad (10)$$

where

$$\{z,x\}=-\frac{3}{2}\left(\frac{z''}{z'}\right)^2+\frac{z'''}{z'},$$

and $\tau=(1-m^2)/4, x=1+\eta$.

The requirement of coincidence at the transition points [6,7],

$$z(x)|_{x=0}=0,$$

leads to the following ''quantum condition'':

$$\lambda=2kz'(0)+\frac{2\tau}{h^2}\left[\frac{z''(0)}{z'(0)}-1\right]. \qquad (11)$$

We will seek the solution of Eq. (10) and eigenvalues λ in the form of the following asymptotical expansion:

$$z=\sum_{k=0}^{\infty}\frac{z_k}{h^k}, \qquad \lambda=\sum_{k=0}^{\infty}\frac{\lambda_k}{h^k}.$$

Substitution of these expansions into Eq. (10) gives us the recurrence system of differential equations for z,

$$z_0'=2\gamma^{1/2}(x-1),$$

$$z_1'=0,$$

$$z_2'=\frac{1}{2z_0'}+\frac{2kz_0'}{z_0}-\frac{(z_1')^2}{2z_0'}-\frac{2\lambda_0}{z_0'x(1-x/2)}-\frac{z_2'^2}{2},\ \ldots,$$

and for λ,

$$\lambda_0=2kz_0'(0),$$

$$\lambda_1=2kz_1'(0),$$

$$\lambda_2=2kz_2'(0)+2\tau\left(\frac{z_0''(0)}{z_1'(0)}-1\right),\ \ldots\ .$$

Solving these recurrence equations, we obtain

$$\lambda^{(\eta)}=4k\gamma^{1/2}+\frac{2k\beta-4\tau}{h^2}+O\left(\frac{1}{h^4}\right), \qquad (12)$$

for λ, and

$$z=\gamma^{1/2}x(2-x)+\frac{1}{h^2}\beta\ln(1-x)+O\left(\frac{1}{h^4}\right), \qquad (13)$$

for z.

From boundary conditions one can obtain, for quantum number k [5,6],

$$k=q+\frac{m+1}{2},$$

where $q=0,1,2,\ldots$.

ASYMPTOTICS OF THE QUASIRADIAL EQUATION

As an ethalon equation for Eq. (3), we take the following equation:

$$W''+\left[h^2s-h^4y^2-\frac{4\tau+3}{4y^2}\right]W=0, \qquad (14)$$

a solution of which is expressed by the confluent hypergeometric functions [8,9]

$$W=y^c e^{-h^4y^2/2}F\left(\frac{s-2c-1}{4},c+\frac{1}{2},h^4y^2\right),$$

where $c=(1+\sqrt{m^2+3})/2$.

Boundary condition (6) and the properties of functions F [8] give rise to the following expression for s:

$$s=4n+\sqrt{m^2+3}+2.$$

Substituting

$$U=[y(\xi)]^{-1/2}W(y(\xi))$$

into Eq. (3), we obtain

$$\frac{y^2y'^2}{4}-\gamma\xi^2+\frac{1}{h^2}\left(\frac{1}{4}-sy'^2-\frac{\lambda}{\xi^2-1}\right)+\frac{1}{h^3}\frac{\alpha\xi}{\xi^2-1}$$
$$+\frac{4\tau}{h^4(\xi^2-1)^2}-\frac{3-4\tau}{4h^4}\frac{y'^2}{y^2}-\frac{1}{2h^4}\{y,\xi\}=0. \qquad (15)$$

After substitution of

$$\phi=\frac{y^2(t)}{4},$$

this equation can be reduced to the form

$$\phi'^2-\gamma(t+1)+\frac{1}{h^2}\left[\frac{1}{4}-\left(n+\frac{1}{2}\right)\frac{\phi'^2}{\phi}-\frac{\lambda}{t(t+2)}\right]$$
$$+\frac{1}{h^3}\frac{\alpha(t+1)}{t(t+2)}+\frac{\tau}{h^4}\left(\frac{\phi'^2}{\phi^2}-\frac{4}{t^2(t+2)^2}\right)-[\phi,t]=0, \qquad (16)$$

where $t=\xi-1$. The quantization condition which follows from $\phi(x)=0$ is written in the form

$$\lambda=-2s\phi'(0)+\frac{\alpha}{h}-\frac{1}{h^2}\left[\frac{\phi''}{\phi'}+1\right]\Bigg|_{t=0}. \qquad (17)$$

Inserting the asymptotical expansions

$$\phi=\sum_{k=0}^{\infty}\frac{\phi_k}{h^k}, \qquad \lambda=\sum_{k=0}^{\infty}\frac{\lambda_k}{h^k}$$

into Eq. (16) and solving the equations obtained herewith, we get the following result:

$$y=2\gamma^{1/4}(t^2+2t)^{1/2}+\frac{1}{h^2}\delta\gamma^{-1/4}(t^2+2t)^{-1/2}\ln(t+1)$$
$$+\frac{1}{h^3}\alpha\gamma^{-3/4}(t^2+2t)^{-1/2}\ln\frac{2(t+1)}{t+1}+O\left(\frac{1}{h^4}\right), \qquad (18)$$

for y, and

$$\lambda^{(\xi)}=-2s\gamma^{1/2}-\frac{\alpha}{h}+\frac{4\tau-s\delta}{h^2}-\frac{s\alpha\gamma^{-1/4}}{2h^3}+O\left(\frac{1}{h^4}\right), \tag{19}$$

for λ.

ASYMPTOTICAL EXPANSION FOR THE ENERGY

Asymptotical expansions (12) and (19) give us an expression for the energy term in the form of multipole expansion. In order to obtain this expansion one should insert

$$E'=E_0+\frac{E_1}{R}+\frac{E_2}{R^2}+\cdots$$

into Eqs. (12) and (19). Equating $\lambda^{(\eta)}$ to be $\lambda^{(\xi)}$ and taking into account Eq. (5), we get the following equations for coefficients $E_1, E_2, \ldots$:

$$E_1=\frac{1}{6Z}[(s\omega-2k\omega^{-1})(2E_0)^{5/2}+(4s^2-16k^2-16\tau)\times(2E_0)^{3/2}],$$

$$E_2=\frac{5}{2}E_1^2+2s\omega^{-1}E_0+E_1(2E_0)^{1/2}Z^{-1}(16\tau^2+16k^2-4s^2), \ldots .$$

Now we need to find E_0. In order to find this value, we note that for $R\to\infty, E'=E_0$, and hence we have

$$E=E_0+\frac{\omega^2R^2}{2}+\frac{4}{3}V_0. \tag{20}$$

On the other hand, for large R we have

$$\begin{aligned}V(r_1,r_2)&=\frac{2Z}{R}\sum_{l=0}^{\infty}\left(\frac{r}{R}\right)^l P_l(\cos\theta)+\omega^2\\&\quad\times\left[\left(r^2+2rR\cos\theta+\frac{R^2}{4}\right)\right.\\&\quad\left.+\left(r^2-2rR\cos\theta+\frac{R^2}{4}\right)\right]\\&\approx\omega^2\left(2r^2+\frac{R^2}{2}\right)-\frac{4}{3}V_0.\end{aligned} \tag{21}$$

Hence, for the energy term with this potential we obtain

$$E=2\omega\left(N+\frac{3}{2}\right)+\frac{\omega^2R^2}{2}+\frac{4}{3}V_0, \tag{22}$$

where $N=n+q+m+1$ is the principal quantum number. Comparing Eqs. (20) and (22), we obtain

$$E_0=2\omega\left(N+\frac{3}{2}\right).$$

Thus, the following asymptotical expansion is obtained for the energy term of a light quark in the field of two heavy quarks:

$$E=-\frac{4}{3}V_0+\frac{\omega^2R^2}{2}+E_0+\frac{E_1}{R}+\frac{E_2}{R^2}+\cdots .$$

QQq BARYON SPECTRA

As mentioned above, the QQq binding energy can be finally obtained by solving the Schrödinger equation

$$\left[-\frac{1}{2\bar{M}_{QQ}}\Delta+V_{QQ}(R)+E(R)\right]\phi=\varepsilon\phi. \tag{23}$$

If one takes $E(R)$ in the form

$$E=-\frac{4}{3}V_0+\frac{\omega^2R^2}{2}+E_0+\frac{E_1}{R},$$

for

$$V_{QQ}(R)=\omega^2R^2-\frac{Z}{R}-\frac{2}{3}V_0,$$

then Eq. (23) can be rewritten as

$$\left[-\frac{1}{2\bar{M}_{QQ}}\Delta+\omega'^2R^2-\frac{Z'}{R}-V_0'\right]\phi=\varepsilon\phi, \tag{24}$$

where $Z'=Z-E_1$, $\omega'^2=\frac{3}{2}\omega^2$, and $V_0'=2V_0-E_0$.

To solve this equation, we use the result of [10] where a method for an analytical solution of the Schrödinger equation with potential

$$V(R)=-\frac{Z}{R}+\lambda R^k$$

was offered. Details of this method and its application to our potential are given in Appendix A. Application of this method to Eq. (24) gives us

$$\varepsilon_{Nnl}=2\omega\left(N+\frac{3}{2}\right)+[Z'^2\omega'^6 r_{nl}]^{1/5}-2V_0, \tag{25}$$

where N is the principial quantum number of the light quark moving in the field of QQ, and r_{nl} is defined in Appendix A. Formula (25) describes the energy spectrum of the QQq baryon. In Tables I, II, and III, the mass spectra of ccq, bbq, and bcq baryons calculated using formula (25) are given, respectively. The following values of potential parameters are chosen in this calculation: $\alpha_s=0.39$, $\omega^2=0.174$ GeV^3, and $V_0=0.05$ GeV for the potential

$$V=\frac{2}{3}\left(-\frac{\alpha_s}{r}+\omega^2 r-V_0\right).$$

TABLE I. The mass spectrum of the ccq baryon (in GeV) calculated using formula (25); $m_q = 0.385$ GeV, $m_c = 1.486$ GeV, n_l and n_d are the principal quantum numbers of the light quark and cc diquark, respectively, and L is the orbital quantum number of the cc diquark.

n_l, n_d, L	Mass	n_l, n_d, L	Mass	n_l, n_d, L	Mass	n_l, n_d, L	Mass
1,1,0	3.661	1,1,1	3.613	1,2,2	3.649	1,3,3	3.694
1,2,0	3.730	1,2,1	3.708	1,3,2	3.764	1,4,3	3.825
1,3,0	3.816	1,3,1	3.799	1,4,2	3.872	1,5,3	3.949
1,4,0	3.914	1,4,1	3.901	1,5,2	3.988	1,6,3	4.077
1,5,0	4.024	1,5,1	4.012	1,6,2	4.110	1,7,3	4.210
2,1,0	3.839	2,1,1	3.791	2,2,2	3.828	2,3,3	3.873
2,2,0	3.908	2,2,1	3.887	2,3,2	3.942	2,4,3	4.003
2,3,0	3.994	2,3,1	3.978	2,4,2	4.051	2,5,3	4.127
2,4,0	4.093	2,4,1	4.079	2,5,2	4.166	2,6,3	4.255
2,5,0	4.202	2,5,1	4.190	2,6,2	4.289	2,7,3	4.388
3,1,0	4.018	3,1,1	3.970	3,2,2	4.006	3,3,3	4.051
3,2,0	4.086	3,2,1	4.065	3,3,2	4.120	3,4,3	4.182
3,3,0	4.172	3,3,1	4.156	3,4,2	4.229	3,5,3	4.305
3,4,0	4.271	3,4,1	4.257	3,5,2	4.344	3,6,3	4.433
3,5,0	4.380	3,5,1	4.369	3,6,2	4.467	3,7,3	4.567

CONCLUSION

In this work we have treated doubly heavy baryons in the Born-Oppenheimer approximation. The following two problems have been solved in the framework of this approximation: (1) the Schrödinger equation for the two-center Coulomb plus harmonic oscillator potential and (2) the Schrödinger equation for the central symmetric Coulomb plus harmonic oscillator potential. As the final result an analytical formula for the energy spectrum of baryons containing two heavy quarks is derived. The formula obtained is applied for the calculation of the mass spectra of doubly heavy quark baryons with various quark compositions. The above analytical results could be useful for further numerical calculations in the nonasymptotical region.

APPENDIX A: THE SCALING VARIATIONAL METHOD AND ITS APPLICATION TO THE COULOMB PLUS CONFINING POTENTIAL

Consider the following Hamiltonian

$$H = -\frac{1}{2}\Delta + V(r), \tag{A1}$$

which obeys the eigenvalue equation

$$H\psi_{nl} = E_{nl}\psi_{nl}, \quad \langle \psi_{nl} | \psi_{n'l'} \rangle = \delta_{nn'}\delta_{ll'}, \tag{A2}$$

TABLE II. The mass spectrum of the bbq baryon (in GeV) calculated using formula (25); $m_q = 0.385$ GeV, $m_b = 4.88$ GeV, n_l and n_d are the principal quantum numbers of the light quark and bb diquark, respectively, and L is the orbital quantum number of the bb diquark.

n_l, n_d, L	Mass	n_l, n_d, L	Mass	n_l, n_d, L	Mass	n_l, n_d, L	Mass
1,1,0	9.890	1,1,1	9.874	1,2,2	9.886	1,3,3	9.900
1,2,0	9.911	1,2,1	9.905	1,3,2	9.922	1,4,3	9.942
1,3,0	9.939	1,3,1	9.934	1,4,2	9.957	1,5,3	9.981
1,4,0	9.970	1,4,1	9.966	1,5,2	9.993	1,6,3	10.022
1,5,0	10.005	1,5,1	10.001	1,6,2	10.032	1,7,3	10.064
2,1,0	10.096	2,1,1	10.081	2,2,2	10.093	2,3,3	10.107
2,2,0	10.118	2,2,1	10.112	2,3,2	10.129	2,4,3	10.149
2,3,0	10.146	2,3,1	10.141	2,4,2	10.164	2,5,3	10.188
2,4,0	10.177	2,4,1	10.173	2,5,2	10.200	2,6,3	10.229
2,5,0	10.212	2,5,1	10.208	2,6,2	10.239	2,7,3	10.271
3,1,0	10.303	3,1,1	10.288	3,2,2	10.300	3,3,3	10.314
3,2,0	10.325	3,2,1	10.318	3,3,2	10.336	3,4,3	10.356
3,3,0	10.353	3,3,1	10.347	3,4,2	10.371	3,5,3	10.395
3,4,0	10.384	3,4,1	10.380	3,5,2	10.407	3,6,3	10.436
3,5,0	10.419	3,5,1	10.415	3,6,2	10.446	3,7,3	10.478

TABLE III. The mass spectrum of the *bcq* baryon (in GeV) calculated using formula (25); $m_q = 0.385$ GeV, $m_b = 4.88$ GeV, $m_c = 1.486$ GeV, n_l and n_d are the principal quantum numbers of the light quark and *bc* diquark, respectively, and L is the orbital quantum number of the *bc* diquark.

n_l, n_d, L	Mass	n_l, n_d, L	Mass	n_l, n_d, L	Mass	n_l, n_d, L	Mass
1,2,0	7.217	1,2,1	7.160	1,3,2	7.178	1,4,3	7.199
1,3,0	7.259	1,3,1	7.206	1,4,2	7.233	1,5,3	7.263
1,4,0	7.307	1,4,1	7.251	1,5,2	7.286	1,6,3	7.324
1,5,0	7.361	1,5,1	7.300	1,6,2	7.343	1,7,3	7.386
2,1,0	7.438	2,1,1	7.355	2,2,2	7.403	2,3,3	7.452
2,2,0	7.471	2,2,1	7.414	2,3,2	7.432	2,4,3	7.454
2,3,0	7.513	2,3,1	7.461	2,4,2	7.488	2,5,3	7.518
2,4,0	7.562	2,4,1	7.505	2,5,2	7.541	2,6,3	7.579
2,5,0	7.615	2,5,1	7.555	2,6,2	7.597	2,7,3	7.641
3,1,0	7.692	3,1,1	7.669	3,2,2	7.687	3,3,3	7.709
3,2,0	7.726	3,2,1	7.716	3,3,2	7.743	3,4,3	7.773
3,3,0	7.768	3,3,1	7.760	3,4,2	7.796	3,5,3	7.833
3,4,0	7.816	3,4,1	7.810	3,5,2	7.852	3,6,3	7.896
3,5,0	7.870	3,5,1	7.864	3,6,2	7.912	3,7,3	7.961

$$n, n' = 1,2, \ldots, \quad l, l' = 1,2, \ldots,$$

where n and l denote the principal and angular quantum numbers, respectively. To solve this equation, we start from a set of functions $\{\phi_{nl}\}$ which are the eigenfunctions of an arbitrary central field Hamiltonian H_0:

$$H_0 \phi_{nl} = \epsilon_{nl} \phi_{nl}, \quad \langle \phi_{nl} | \phi_{n'l'} \rangle = \delta_{nn'} \delta_{ll'}. \tag{A3}$$

Then, we construct the functionals

$$\varepsilon_{nl}(\alpha) = \langle \phi_{nl}^{\alpha} | H \phi_{nl}^{\alpha} \rangle, \tag{A4}$$

where

$$\phi_{nl}^{\alpha} = \alpha^{3/2} \phi_{nl}(\alpha r).$$

The value for α is determined by

$$\left(\frac{\partial \varepsilon}{\partial \alpha} \right)(\alpha = a) = 0. \tag{A5}$$

Let us now apply this method to our problem, i.e., to the Schrödinger equation with potential

$$V(r) = -\frac{Z'}{R} + \omega'^2 r^2.$$

As H_0, we choose a pure Coulomb Hamiltonian, i.e.,

$$H_0 = -\frac{1}{2} \Delta - \frac{Z'}{R}.$$

Then, $\epsilon_{nl} = -Z'^2/2n^2$ with $n = n_r + l + 1$, where n_r is the radial quantum number.

According to the above procedure, we have

$$\varepsilon_{nl}(\alpha) = \langle \phi_{nl}^{\alpha} | H_0 | \phi_{nl}^{\alpha} \rangle + \langle \phi_{nl}^{\alpha} | \omega^2 r^2 | \phi_{nl}^{\alpha} \rangle$$
$$= -\frac{Z'^2 \alpha^3}{2n^2} + \frac{\omega^2 \alpha^{-2} n^2}{2} \times [5n^2 + 1 - 3l(l+1)]. \tag{A6}$$

For calculation of the second matrix element, we have used the well-known expression for average value $\overline{r^2}$ in Coulomb field which is given in [11]. From $\partial \varepsilon(\alpha)/\partial \alpha = 0$, we obtain

$$\alpha_0 = \left\{ -\frac{2\omega^2 n^4}{3Z^2} [5n^2 + 1 - 3l(l+1)] \right\}^{1/5}.$$

Then, for the energy level one can get the following analytical formula:

$$\varepsilon_{nl} = [Z^2 \omega^6 r_{nl}]^{1/5}, \tag{A7}$$

where

$$r_{nl} = n^2 [5n^2 + 1 - 3l(l+1)]^3.$$

APPENDIX B: THE FORMAL PROCEDURE OF THE METHOD OF ETHALON EQUATION

Let us consider the following second order differential equation:

$$y''(x) + p^2 [\lambda - q(x)] y(x) = 0 \tag{B1}$$

in the interval $[a,b]$. Let Eq. (B1) in this interval have one transition point [poles and zeros of function $Q(x,\lambda) = q(x) - \lambda$ are called transition points of this equation].

The equation

$$w''(z) - p^2 R(z) w(z) = 0, \tag{B2}$$

which has the same or close transition points as Eq. (B2), is called the ethalon equation for Eq. (B1).

We will seek the solution of Eq. (B1) in the form

$$y(x)=[z'(x,p)]^{-1/2}w(z(x,p)), \tag{B3}$$

where w is the solution of Eq. (B2). Inserting Eq. (B3) into Eq. (B1) and taking into account Eq. (B2) we obtain the following (nonlinear) differential equation for $z(x,p)$:

$$R(z)z'^2-Q(x,\lambda)-\frac{1}{2p^2}\{z,x\}=0 \tag{B4}$$

where

$$\{z,x\}=-\frac{3}{2}\left(\frac{z''}{z'}\right)^2+\frac{z'''}{z'}.$$

In the case of Eq. (4) we have for Q

$$Q=-\left[\frac{h^2}{4}+\frac{h\lambda}{1-\eta^2}-h^4\gamma\eta^2+\frac{1-m^2}{(1-\eta^2)^2}\right]$$

and for R [from Eq. (8)]

$$R=-\left[-\frac{h^4}{4}+\frac{h^2k}{z}+\frac{1-m^2}{4z^2}\right].$$

So for z one obtains Eq. (10):

$$\frac{z'^2}{4}-\gamma(x-1)^2-\frac{1}{h^2}\left(\frac{1}{4}+\frac{kz'^2}{z}-\frac{\lambda}{2x(1-x/2)}\right)$$
$$+\frac{\tau}{h^2}\left(\frac{1}{x^2(1-x^2)}-\frac{z'^2}{z^2}\right)-\frac{1}{2h^2}\{z,x\}=0.$$

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