Relativistic modification of the Gamow factor

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In processes involving Coulomb-type initial- and final-state interactions, the Gamow factor has been traditionally used to take into account these additional interactions. The Gamow factor needs to be modified when the magnitude of the effective coupling constant increases or when the velocity increases. For the production of a pair of particles under their mutual Coulomb-type interaction, we obtain the modification of the Gamow factor in terms of the overlap of the Feynman amplitude with the relativistic wave function of the two particles. As a first example, we study the modification of the Gamow factor for the production of two bosons. The modification is substantial when the coupling constant is large.

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I. INTRODUCTION

Initial- and final- state interactions are important in many branches of theoretical physics involving the reaction or the production of particles. They have a great influence on the reaction rates or the production cross sections [1–11]. These initial- and final-state interactions lead to a large enhancement of the cross section if the particles are subject to a strong attractive interaction; they can lead to a large suppression under a strong repulsive interaction. We shall use the term ''the K factor'' to label the ratio of the cross section with the interaction to the corresponding quantity without the interaction.

As is well known, for interactions such as the electric-Coulomb and color-Coulomb interaction $\mathcal{A}(r)=-\alpha/r$ in nonrelativistic physics, the effect of the initial- or final-state interactions leads to a K factor given by the Gamow-Sommerfeld factor [2,3]. The Gamow-Sommerfeld factor (or simply called the Gamow factor) is given explicitly by

$$G(\eta)=\frac{2\pi\eta}{1-e^{-2\pi\eta}}, \tag{1}$$

where

$$\eta=\frac{\alpha}{v}. \tag{2}$$

The magnitude of the relative velocity v is the ratio of the asymptotic momentum p to the energy ϵ_w in the relative coordinate system [see Eqs. (28) and (31) below]. Following Todorov [12], Crater and Van Alstine [13], and Eqs. (21.13a)–(21.13c) of Crater *et al.* [14], the relative velocity for the particles a and b is related to their center-of-mass energy $\sqrt{s}$ by

$$v=\frac{(s^2-4sm^2)^{1/2}}{s-2m^2}. \tag{3}$$

This gives $v\sim 2\sqrt{1-4m^2/s}$ when $\sqrt{s}\sim 2m$ and $v\rightarrow 1$ when $s\rightarrow\infty$. This Gamow factor has been used to study initial- and final-state interactions in reaction processes.

There are physical processes in which the coupling constant of the interaction between the particles can be quite large and the use of the Gamow factor to correct the initial-state and final-state interactions may not be adequate. For example, in the annihilation or the production of $q\overline{q}$ pairs, the interaction arising from the exchange of a gluon leads to a color-Coulomb interaction with a coupling constant α about 0.2–0.4, depending on the renormalization scale of the reaction process. Another example is the modification of the Gamow factor in the presence of a final-state Coulomb interaction and its effects on the the Hanbury-Brown–Twiss effects of intensity interferometry as studied by Baym and Braun-Munzinger [15]. Another example with strong coupling occurs in the case of a negatively charged particle in a nucleus with a large Z number. Such a large coupling constant will also lead to a modification of the Gamow factor as there are higher-order effects of the potential which are important when the coupling constant becomes large. One can mention, for example, the well-known case of the ''Landau fall'' which is the relativistic nonperturbative collapse of the wave function for an attractive Coulomb-type potential when the coupling constant exceeds a certain limit [16]. Furthermore, although the effect of the interaction is very large for low relative velocities, it is useful to see how the effect varies as the velocity increases. For brevity of notation, we shall use the term ''Coulomb interaction'' with a variable coupling constant to refer to both the electric-Coulomb and color-Coulomb interactions.

While one sees the need to use the relativistic formalism to study the case with high relative velocities, one may wonder what special cases can be of interest to use a relativistic formalism for the case of low relative velocities. By the term ''the relative velocity,'' we usually refer to the relative velocity between the particles in the asymptotic region of $r\rightarrow\infty$ where there is no interaction. However, when there is a strongly attractive interaction, the actual relative velocity depends on the spatial location. One can envisage that if the coupling constant is large, the motion of the two particles at small distances can become relativistic, even though the relative velocity at $r\rightarrow\infty$ is small. Hence, it is necessary to use the relativistic formalism to study the effects of the mutual interaction with large coupling constants, even for the case of low asymptotic relative velocities at $r\rightarrow\infty$.

The K factor for the Coulomb potential can be studied by examining the two-body wave function in the Klein-Gordon or the Dirac equation involving a Coulomb potential. Compared with the nonrelativistic Schrödinger equation involving the Coulomb potential, there is an additional effective attractive potential, $-|\mathcal{A}(r)|^2/2m_w$, and a repulsive term from the spacelike part of the gauge interaction [see Eqs. (30) and (66) below], which lead to a nontrivial behavior when the coupling constant becomes large. In the case of fermions under the Coulomb interaction, there are further modifications associated with additional spin-dependent potential terms.

The question of initial- and final-state interactions is also related to the question of the decay and the production of bound states when the interactions lead to the formation of bound states [17,18]. Previously, the decay of the bound positronium $^1S_0$ state into two photons has been studied in the relativistic formalism by Crater [18]. In our present study, we are interested mainly in the case of two particles in the continuum. We wish to find out how the mutual interaction may affect their reaction or production rates.

The proper method to obtain the K factor is by taking the overlap of the relativistic wave function with the Feynman amplitude. When the Feynman amplitude is independent of the momentum, then one obtains the familiar result that the K factor is the absolute square of the wave function at the origin. However, this is not valid in the general case where the Feynman amplitude is a function of the relative momentum. It will then be necessary to evaluate the K factor by taking the overlap of the Feynman amplitude with the relativistic wave function.

As an illustration, we shall try out the method for the production of a pair of scalar particles interacting with a Coulomb-type final-state interaction. We shall first study the case with only the timelike part of the gauge interaction in Secs. III, IV, and V. We then study the addition of the transverse part of the interaction. The inclusion of the transverse component of the gauge interaction modifies the results only slightly and will be discussed subsequently in Secs. VI and VII.

II. *K* FACTOR

The effects of the final- and initial-state interaction depend on the physical process. Here we shall be interested in the class of processes involving the reaction or the production of a pair of particles a and b, subject to the mutual interaction between a and b. For definiteness, we shall study the production process, as the K factor is the same for production or reaction.

The simplest description of such a process is in terms of perturbation theory which gives the amplitude for the production of this pair of particles a and b. The state Φ_{ab} of the ab pair after the reaction $x+y\rightarrow a+b$ is represented by the state vector

$$|\Phi_{ab}\rangle=\mathcal{M}(xy\rightarrow a(P/2+q)b(P/2-q))\times|a(P/2+q)b(P/2-q)\rangle, \tag{4}$$

where $\mathcal{M}(xy\rightarrow ab)$ is the Feynman amplitude for the $x+y\rightarrow a+b$ process. For the two-particle system ab, we define the center-of-mass momentum $P=a+b$ and the relative momentum $q=(a-b)/2$.

On the other hand, under their mutual interaction which can be represented by a two-body potential $\mathcal{A}(r)$ between a and b, we can describe an ab pair with a center-of-mass momentum P as

$$|\Psi_V\rangle=\tilde{\psi}(q)|P\rangle. \tag{5}$$

The probability amplitude for the production of an ab pair under their mutual interaction is obtained by taking the overlap of the amplitude in Eq. (4) with the wave function in Eq. (5). The overlap is the simplest in the ab center-of-mass system where $P=(\sqrt{s},\mathbf{0})$ and $q=(0,\mathbf{q})$ and the (unnormalized) probability amplitude is [17–19]

$$\langle\Psi_V|\Phi_{ab}\rangle=\int\frac{d^3\mathbf{q}}{(2\pi)^3}\tilde{\psi}(\mathbf{q})\mathcal{M}(xy\rightarrow a(\boldsymbol{q})b(-\boldsymbol{q})). \tag{6}$$

[For a properly normalized probability amplitude, see Eqs. (29)–(31) of Ref. [19].] The K factor for the occurrence of ab in the state Ψ_V is then given by

$$K\equiv\frac{|\langle\Psi_V|\Phi_{ab}\rangle|^2}{|\langle\Psi_0|\Phi_{ab}\rangle|^2}=\frac{(\text{production cross section with final-state interaction})}{(\text{production cross section without final-state interaction})}, \tag{7}$$

where $|\Psi_0\rangle$ is the state of the ab pair without their mutual interaction. We have used the unnormalized amplitude in Eq. (6) as any normalization constant will cancel out in the definition of the K factor in Eq. (7). The result of the cross section calculated using the simple first-order diagram can be corrected to include the effects of the final-state interaction by multiplying by the K factor:

$$\begin{pmatrix}\text{production cross section}\\ \text{with final-state interaction}\end{pmatrix}=K\times\begin{pmatrix}\text{production cross section}\\ \text{without final-state interaction}\end{pmatrix}. \tag{8}$$

III. FEYNMAN AMPLITUDE AND THE OVERLAP WITH THE WAVE FUNCTION

To obtain the effect of the final-state interaction between a and b produced in a pair-production process, we consider the production of the pair of bosons from the fusion of two photons. Because the relevant factors associated with the mode of production will be canceled out at the end in Eq. (7), the results of the K factor depend only on the final-state interaction and is the same for a similar mode of production of the pair of bosons.

The diagrams we include are shown in Fig. 1 which give the amplitude

$$-i\mathcal{M}(xy\to a(P/2+q)b(P/2-q)) = ie^2\left[-\frac{(2\boldsymbol{q}+\boldsymbol{k})\cdot\boldsymbol{\epsilon}_1\,\boldsymbol{k}\cdot\boldsymbol{\epsilon}_2}{(\boldsymbol{q}+\boldsymbol{k})^2+m^2}+\frac{(2\boldsymbol{q}-\boldsymbol{k})\cdot\boldsymbol{\epsilon}_2\,\boldsymbol{k}\cdot\boldsymbol{\epsilon}_1}{(\boldsymbol{q}-\boldsymbol{k})^2+m^2}\right], \tag{9}$$

where $\boldsymbol{k}$ is the momentum of the photon, $\boldsymbol{q}$ is the momentum of one of the bosons, and $\boldsymbol{\epsilon}_i$ is the polarization vector of the ith photon. In addition to the two diagrams in Fig. 1, there is also the seagull diagram which is required to get gauge invariance and is of the same order. The seagull diagram is independent of the relative momentum and leads to a probability amplitude proportional to the wave function at the origin. When one properly takes into account the spacelike component of the Coulomb interaction (see Sec. VI), the spatial wave function is zero at the origin [20] and thus the contribution to the probability amplitude from the seagall term is zero.

The overlap of the wave function with the Feynman amplitude is then

$$\langle\Psi_V|\Phi_{ab}\rangle = ie^2\int\frac{d\boldsymbol{q}}{(2\pi)^3}\tilde{\psi}(\boldsymbol{q})\left[-\frac{(2\boldsymbol{q}+\boldsymbol{k})\cdot\boldsymbol{\epsilon}_1\,\boldsymbol{k}\cdot\boldsymbol{\epsilon}_2}{(\boldsymbol{q}+\boldsymbol{k})^2+m^2}+\frac{(2\boldsymbol{q}-\boldsymbol{k})\cdot\boldsymbol{\epsilon}_2\,\boldsymbol{k}\cdot\boldsymbol{\epsilon}_1}{(\boldsymbol{q}-\boldsymbol{k})^2+m^2}\right]. \tag{10}$$

It is useful to write the above integral in terms of the wave function in configuration space. The latter is given by

$$\psi(\boldsymbol{r}) = \int\frac{d\boldsymbol{q}}{(2\pi)^3}\tilde{\psi}(\boldsymbol{q})e^{-i\boldsymbol{q}\cdot\boldsymbol{r}}. \tag{11}$$

In conventional applications, one expands the Feynman amplitude (9) in powers of $\boldsymbol{q}$ and keeps only the lowest-order q-independent term $\mathcal{M}_0$:

$$\mathcal{M}\approx\mathcal{M}_0+O(|\boldsymbol{q}|). \tag{12}$$

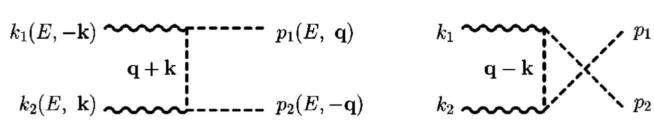

FIG. 1. Feynman diagrams included in the calculation.

In this approximation of keeping only the leading term, Eqs. (7) and (11) then give the usual K factor as the absolute square of the wave function $\psi(r)$ at the origin:

$$K=|\psi(r=0)|^2. \tag{13}$$

However, we are interested in the improvement over this approximation and use the full Feynman amplitude to evaluate the overlap integral and the K factor in Eqs. (9) and (7).

In terms of the spatial wave function, the overlap integral (10) is

$$\langle\Psi_V|\Phi_{ab}\rangle = e^2\int d\boldsymbol{r}\,\psi(\boldsymbol{r})\left[e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\left\{2\hat{\boldsymbol{\epsilon}}_1\cdot\frac{\nabla}{i}-\hat{\boldsymbol{\epsilon}}_1\cdot\boldsymbol{k}\right\}\frac{e^{-mr}}{4\pi r}\hat{\boldsymbol{\epsilon}}_2\cdot\boldsymbol{k} - e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\left\{2\hat{\boldsymbol{\epsilon}}_2\cdot\frac{\nabla}{i}+\hat{\boldsymbol{\epsilon}}_2\cdot\boldsymbol{k}\right\}\frac{e^{-mr}}{4\pi r}\hat{\boldsymbol{\epsilon}}_1\cdot\boldsymbol{k}\right]. \tag{14}$$

We shall specialize to the S-wave case with $l=0$. The overlap integral (14) becomes

$$\langle\Psi_V|\Phi_{ab}\rangle = e^2\,\hat{\boldsymbol{\epsilon}}_1\cdot\boldsymbol{k}\,\hat{\boldsymbol{\epsilon}}_2\cdot\boldsymbol{k}[A-B], \tag{15}$$

where

$$A=\frac{4}{k}\int_0^\infty dr\, j_1(kr)\psi(r)e^{-mr}(1+mr) \tag{16}$$

and

$$B=2\int_0^\infty r\,dr\, j_0(kr)\psi(r)e^{-mr}. \tag{17}$$

For the probability amplitude for the case without the final-state interaction, we use the wave function $\psi_0(r)=\sin pr/pr$ for the S state and we obtain the probability amplitude as given by

$$\langle\Psi_0|\Phi_{ab}\rangle = 2e^2(\hat{\boldsymbol{\epsilon}}_1\cdot\boldsymbol{k})(\hat{\boldsymbol{\epsilon}}_2\cdot\boldsymbol{k})\mathcal{B}, \tag{18}$$

where the factor $\mathcal{B}$ is

$$\mathcal{B}=\mathrm{Im}\{(\cot\theta^*-2m/k)(\theta^*\cot\theta^*-1)\}/pk, \tag{19}$$

and θ^* is a complex conjugate of θ. Here, we have introduced the complex angle variable

$$\theta=\tan^{-1}\frac{k}{m+ip}=\frac{\pi}{4}-i\frac{1}{4}\ln\frac{k+p}{k-p}, \tag{20}$$

which is a relativistic measure of the relative motion between particles a and b. The real part of θ is always $\pi/4$, and the imaginary part is negative, with a magnitude that is half of the rapidity of the produced particle in the center-of-mass system.

Thus, the K factor is given by

$$K=\left|\frac{A-B}{2\mathcal{B}}\right|^2. \tag{21}$$

IV. KLEIN-GORDON EQUATION FOR THE TIMELIKE PART OF THE GAUGE INTERACTION

In this first study, in order to illustrate the main features of the effect and to avoid complications brought on by the spinor algebra, we shall carry out the procedures outlined above for the production of two scalar particles.

We need to separate out the center-of-mass motion and the relative motion for these two particles. Consider first the case without a mutual interaction as in the region where the two particles are far apart. The two particles have four-momenta p_1 and p_2 and rest masses m_1 and m_2. We introduce the total momentum P [13,14],

$$P=p_1+p_2, \tag{22}$$

and the relative momentum q,

$$q=\frac{\epsilon_2 p_1-\epsilon_1 p_2}{\sqrt{s}}, \tag{23}$$

where

$$\epsilon_1=\frac{s-m_2^2+m_1^2}{2\sqrt{s}} \tag{24}$$

and

$$\epsilon_2=\frac{s-m_1^2+m_2^2}{2\sqrt{s}}, \tag{25}$$

with $s=P^2$. We have the following identity:

$$p_1^2-m_1^2+p_2^2-m_2^2=\frac{(\epsilon_1^2+\epsilon_2^2)P^2}{s}+2q^2-m_1^2-m_2^2=0. \tag{26}$$

We can choose to work in the center-of-mass system in which $P=(\sqrt{s},\mathbf{0})$ and $q=(0,\boldsymbol{q})$. The above equation can be written in terms of an effective energy ϵ_w, and a generalized reduced mass m_w as

$$\epsilon_w^2-\boldsymbol{q}^2-m_w^2=0, \tag{27}$$

where

$$\epsilon_w=\frac{s-m_1^2-m_2^2}{2\sqrt{s}} \tag{28}$$

and

$$m_w=\frac{m_1 m_2}{\sqrt{s}}. \tag{29}$$

Next, we study first the system of masses $m_1=m_2=m$ interacting only with a timelike part of the gauge interaction $\mathcal{A}(r)=-\alpha/r$ because the results can be written out in an analytical form. We shall postpone the discussion of the full interaction to Sec. VI. The equation of motion can be obtained from Eq. (27) by the canonical method of replacing the timelike component ϵ_w with $\epsilon_w-\mathcal{A}(r)$. The Klein-Gordon equation for the two-particle system under a mutual Coulomb-type interaction $\mathcal{A}(r)$ is

$$\{[\epsilon_w-\mathcal{A}(r)]^2-\boldsymbol{q}^2-m_w^2\}\psi(\boldsymbol{r})=0. \tag{30}$$

We introduce the dimensionless variable $\mathbf{z}=p\mathbf{r}$ where p is the asymptotic momentum at $r\to\infty$ given by

$$p=\sqrt{\epsilon_w^2-m_w^2}. \tag{31}$$

Writing $\psi(\boldsymbol{r})=R_{nl}(z)Y_{lm}(\theta,\phi)$ in Eq. (30), the equation for $R_{nl}(z)$ is

$$\left[\frac{d}{dz^2}+\frac{2}{z}\frac{d}{dz}-\frac{l(l+1)}{z^2}+\frac{2\eta}{z}+\frac{\alpha^2}{z^2}+1\right]R_{nl}(z)=0. \tag{32}$$

The wave function $R_{nl}(z)$ is characterized by two dimensionless parameters: $\eta=\alpha/v$ and α^2, where $v=p/\epsilon_w$ is given by Eq. (3).

The solution of Eq. (32) is

$$R_{nl}(z)=\frac{|\Gamma(a)|}{\Gamma(b)}e^{\pi\eta/2}(2iz)^{\mu-1/2}e^{-iz}{}_1F_1(a,b,2iz), \tag{33}$$

where

$$a=\mu+\frac{1}{2}+i\eta, \tag{34}$$

$$b=2\mu+1, \tag{35}$$

$$\mu=\sqrt{\left(l+\frac{1}{2}\right)^2-\alpha^2}, \tag{36}$$

${}_1F_1$ is the confluent hypergeometric function, and the normalization constant has been determined by using the boundary condition that at $r\to\infty$, $R_{nl}(z)\to(i)^{\mu-1/2}\sin(z+\delta_l)/z$ with the Coulomb phase shift δ_l. For the S state, the critical value of α is 1/2.

Using the wave function of Eq. (33), we carry out the integrations of Eqs. (16) and (17) and obtain

$$\langle\Psi_V|\Phi_{ab}\rangle=2e^2(\hat{\boldsymbol{\epsilon}}_1\cdot\boldsymbol{k})(\hat{\boldsymbol{\epsilon}}_2\cdot\boldsymbol{k})\frac{|\Gamma(a)|}{\Gamma(b)}e^{\pi\eta/2}\sum_{n=0}^{\infty}\frac{(a)_n\Gamma\left(\frac{3}{2}+\mu+n\right)}{(b)_n\,n!}\left(\frac{2ip}{\sqrt{\delta^2+k^2}}\right)^{n+\mu-1/2}\frac{1}{\delta^2+k^2}$$
$$\times\left[\frac{2}{3}F\left(\frac{3}{4}+\frac{\mu+n}{2},\frac{5}{4}-\frac{\mu+n}{2};\frac{5}{2};\xi^2\right)+\frac{2m}{3}\frac{\frac{3}{2}+\mu+n}{\sqrt{\delta^2+k^2}}F\left(\frac{5}{4}+\frac{\mu+n}{2},\frac{3}{4}-\frac{\mu+n}{2};\frac{5}{2};\xi^2\right)\right.$$
$$\left.-F\left(\frac{3}{4}+\frac{\mu+n}{2},\frac{1}{4}-\frac{\mu+n}{2};\frac{3}{2};\xi^2\right)\right],\tag{37}$$

where

$$\delta=m+ip,\tag{38}$$

$$\xi^2=\frac{k^2}{(m+ip)^2+k^2}.\tag{39}$$

In Eq. (37), $(a)_n=a(a+1)(a+2)\cdots(a+n-1)$, with $(a)_0=1$. The quantity $(b)_n$ is similarly defined.

V. RESULTS FOR THE K FACTOR FOR THE TIMELIKE PART OF THE GAUGE INTERACTION

In terms of the angle variable of Eq. (20), Eq. (37) can be transformed as follows:

$$\langle\Psi_V|\Phi_{ab}\rangle=2e^2(\hat{\boldsymbol{\epsilon}}_1\cdot\boldsymbol{k})(\hat{\boldsymbol{\epsilon}}_2\cdot\boldsymbol{k})\frac{|\Gamma(a)|}{\Gamma(b)}e^{\pi\eta/2}\mathcal{D},\tag{40}$$

where the factor $\mathcal{D}$ is

$$\mathcal{D}=\sum_{n=0}^{\infty}\frac{(a)_n\Gamma(2\nu)}{(b)_n\,n!}\left(\frac{2ip}{\sqrt{\delta^2+k^2}}\right)^{n+\mu-1/2}\frac{1}{k^2\sin\theta}\left[\frac{2}{2\nu-3}\left\{\frac{\sin(2\nu-2)\theta}{2\nu-2}\cos\theta-\frac{\sin(2\nu-1)\theta}{2\nu-1}\right\}\right.$$
$$\left.+\frac{2m}{k}\frac{2\nu}{2\nu-2}\sin\theta\left\{\frac{\sin(2\nu-1)\theta}{2\nu-1}\cos\theta-\frac{\sin2\nu\theta}{2\nu}\right\}-\sin^2\theta\frac{\sin(2\nu-1)\theta}{2\nu-1}\right],\tag{41}$$

and $\nu=3/4+(\mu+n)/2$. On the other hand, the overlap between the Feynman amplitude and the wave function without the final-state interaction is given by Eq. (18). Thus, the ratio between the absolute squares of Eqs. (40) and (18) is the relativistic expression of the K factor,

$$K=\left|\frac{\Gamma(a)}{\Gamma(b)}e^{\pi\eta/2}\frac{\mathcal{D}}{\mathcal{B}}\right|^2.\tag{42}$$

We can identify the factor $|\Gamma(a)e^{\pi\eta/2}/\Gamma(b)|^2$ as closely related to the Gamow factor $G(\eta)$. One can show that

$$\left|\frac{\Gamma(a)}{\Gamma(b)}e^{\pi\eta/2}\right|^2=G(\eta)\left|\frac{\Gamma(\mu+1/2)}{\Gamma(2\mu+1)}\right|^2\prod_{j=0}^{\infty}\left(1+\frac{\mu-1/2}{1+j}\right)^2\left(1+\frac{\left(\frac{3}{2}+\mu+2j\right)\left(\frac{1}{2}-\mu\right)}{\left(\mu+\frac{1}{2}+j\right)^2+\eta^2}\right).\tag{43}$$

Therefore, the proper treatment of the dynamics of the interacting particles leads to the modification of the Gamow factor $G(\eta)$ of Eq. (1) by a factor κ given by

$$K=G(\eta)\kappa,\tag{44}$$

where

$$\kappa=\left|\frac{\Gamma(\mu+1/2)}{\Gamma(2\mu+1)}\right|^2\prod_{j=0}^{\infty}\left(1+\frac{\mu-1/2}{1+j}\right)^2\left(1+\frac{\left(\frac{3}{2}+\mu+2j\right)\left(\frac{1}{2}-\mu\right)}{\left(\mu+\frac{1}{2}+j\right)^2+\eta^2}\right)\left|\frac{\mathcal{D}}{\mathcal{B}}\right|^2 \tag{45}$$

and

$$\left|\frac{\mathcal{D}}{\mathcal{B}}\right|^2=\left|\sum_{n=0}^{\infty}\frac{(a)_n\Gamma(2\nu)}{(b)_n n!\sin\theta}\left(\frac{2ip}{\sqrt{\delta^2+k^2}}\right)^{n+\mu-1/2}\left[\frac{2}{2\nu-3}\left\{\frac{\sin(2\nu-2)\theta}{2\nu-2}\cos\theta-\frac{\sin(2\nu-1)\theta}{2\nu-1}\right\}+\frac{2m}{k}\frac{2\nu}{2\nu-2}\right.\right.$$
$$\left.\left.\times\sin\theta\left\{\frac{\sin(2\nu-1)\theta}{2\nu-1}\cos\theta-\frac{\sin 2\nu\theta}{2\nu}\right\}-\sin^2\theta\frac{\sin(2\nu-1)\theta}{2\nu-1}\right]\right|^2\Bigg/\left|(k/p)\mathrm{Im}\left\{\left(\cot\theta^*-\frac{2m}{k}\right)(\theta^*\cot\theta^*-1)\right\}\right|^2. \tag{46}$$

In the limit of $\alpha\to 0$ or $v\to 0$, the factor κ goes to 1 and is consistent with the Gamow factor.

We note that the center-of-mass energy $\sqrt{s}$ in units of the rest mass of the produced particle is a function of η/α:

$$\frac{\sqrt{s}}{m}=\sqrt{2\left(1+\frac{\eta/\alpha}{\sqrt{\eta^2/\alpha^2-1}}\right)}. \tag{47}$$

Various other kinematic variables, such as $k/m=\sqrt{s}/2m$ and $p/m=\sqrt{s/4m^2-1}$, can be similarly expressed as a function of η/α. From these relations and the relation between the K factor and η and α, we can study the K factor for the production of a pair of particles in any specific kinematic configuration.

We show the behavior of the K factor as a function of η in Fig. 2 for various values of α. The solid curve gives the K factor for $\alpha=0.02$ and the dotted curve gives the K factor for $\alpha=0.32$. For a fixed value of α, the K factor decreases as η decreases. This is consistent with the expectation that the effects of the final-state interaction diminish as the velocity becomes relativistic. The limiting value is $K=1$ as $\eta=\alpha/v\to\alpha$. Figure 2 also shows that for a given value of $\eta=\alpha/v$, the K factor decreases as α increases. It should be noted that the same value of η corresponds to different velocities v for different values of α. To see the effect of the final-state interaction as a function of α for a fixed value of v, we plot in Fig. 3 the K factor as a function of v. As one observes, when the velocity is fixed, the K factor increases as the coupling constant increases, indicating a greater effect of the final-state interaction as α increases. For all values of α, the K factor decreases as v increases and goes to unity as v approaches 1. The decrease is very rapid for small values of α.

It is of interest to see how the K factor obtained here is different from the Gamow factor. In Fig. 4, we showed the ratio between the K factor and the Gamow factor for various values of α. As we expect, the ratio is almost 1 for weak coupling and the use of the Gamow factor is relatively safe there. However, if we increase α to 0.32, the ratio decreases significantly. The Gamow factor overestimates the magnitude of the final-state interaction. It cannot be used for the

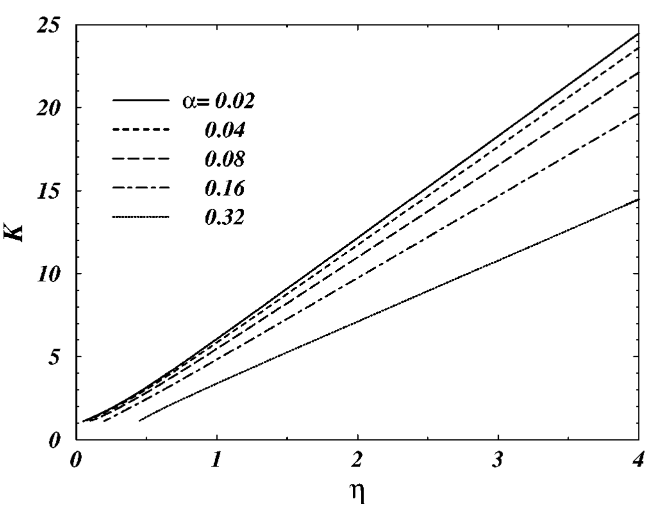

FIG. 2. The K factor versus η for various values of α, with only the timelike component of the gauge interaction.

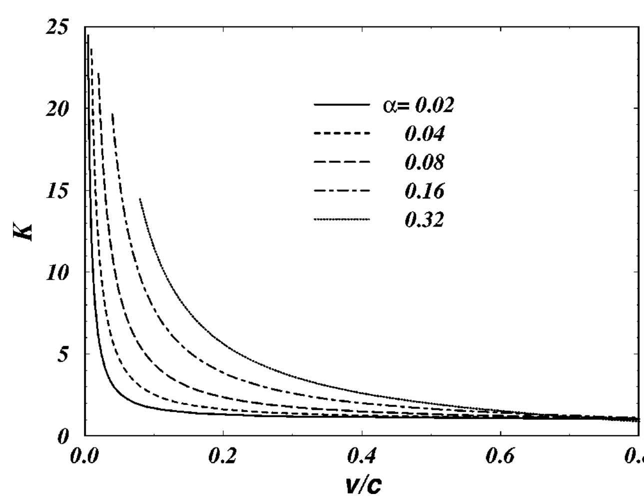

FIG. 3. The K factor versus the velocity v for various values of α, with only the timelike component of the gauge interaction.

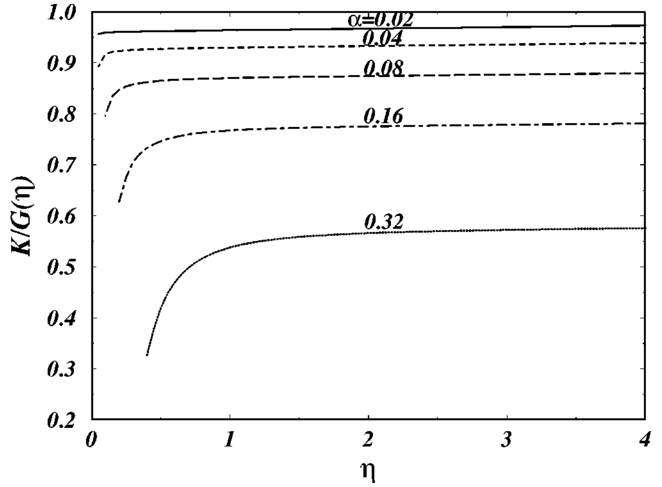

FIG. 4. The ratio between the K factor and the Gamow factor $G(\eta)$ for various values of α, with only the timelike component of the gauge interaction.

case with strong coupling. There is an effective screening of the long-range Coulomb interaction. As a consequence, the enhancement due to the long-range Coulomb-type interaction is reduced. It can also be observed in Fig. 4 that the ratio of $K/G(\eta)$ is a relatively slowly varying function of η for $\eta>1$ but drops down rapidly as η decreases in the region of small η.

It is worth pointing out that the expansion of $\mathcal{D}$ in Eqs. (41) and (46) is given as a series in powers of $p/\sqrt{\delta^2+k^2}$ which increases as the velocity v increases. We still obtain convergent results for v up to about 0.8, but there is a limit on using such an expansion for greater velocities where $p/\sqrt{\delta^2+k^2}$ is too large to allow for a convergent term-by-term summation. A different expansion method is needed for $v>0.8$.

VI. INCLUSION OF THE SPACELIKE PART OF THE GAUGE INTERACTION

In the above discussions, we consider only the timelike part of the gauge interaction, $A^0\equiv\mathcal{A}$. Two charged particles interact not only through the static timelike part of the gauge interaction, but also through the currents and the spacelike part of their mutual interaction $\mathbf{A}$. It is of interest to see how the additional spacelike gauge interaction modifies the K factor.

The spacelike part (the transverse part) of the gauge interaction can be included as formulated by Crater and Van Alstine [20]. We briefly summarize their results here. In classical field theory, the interacting two particles satisfy two equations of motion:

$$\mathcal{H}_1=\pi_1^2-m_1^2=0, \tag{48}$$

$$\mathcal{H}_2=\pi_2^2-m_2^2=0. \tag{49}$$

In the above two equations, the spacelike component $\boldsymbol{\pi}_i$ is

$$\boldsymbol{\pi}_1=\mathbf{q}-\mathbf{A}=-\boldsymbol{\pi}_2. \tag{50}$$

The transverse field $\mathbf{A}$ is proportional to $\mathbf{q}$, and the two terms in the above equation can be grouped together into a function G:

$$\boldsymbol{\pi}_1=G\mathbf{q}=-\boldsymbol{\pi}_2, \tag{51}$$

where the factor G should be chosen to give the correct potential in the weak-field limit and to make Eqs. (48) and (49) compatible (i.e., $[\mathcal{H}_1,\mathcal{H}_2]=0$) [20,21]. The timelike momentum π_i^0 is proportional to $(\epsilon_i-\mathcal{A})$ and the proportional factor turns out to be the same function G as for the spacelike components:

$$\pi_1^0=G(\epsilon_1-\mathcal{A}), \tag{52}$$

$$\pi_2^0=G(\epsilon_2-\mathcal{A}), \tag{53}$$

where ϵ_1 and ϵ_1 are given by Eqs. (24) and (25). The function G is a function of the coordinate $x_\perp$ where

$$(x_\perp)^\mu=x^\mu-(x\cdot\hat{P})\hat{P}^\mu \tag{54}$$

and $\hat{P}$ is the unit vector of the total momentum P. In the center-of-mass system, $\mathbf{x}_\perp=\mathbf{r}$.

Crater and Van Alstine showed that a consistent choice of the function $G(r)$ which gives all the correct properties of the weak field limit and makes the system of Eqs. (48) and (49) compatible is

$$G(r)=\left[1-\frac{2\mathcal{A}}{\sqrt{s}}\right]^{-1/2}. \tag{55}$$

In particular, in addition to giving the correct Darwin term of the two-body electromagnetic interaction, such a function of G leads to an effective potential which, in the weak-field limit, is canonically equivalent to the Liénert-Wiechert potential up to order $(v/c)^2$ [20,21].

In covariant form, the four-momentum of the two particles can be related to the total momentum P and the relative momentum q by

$$\pi_1^\mu=G[\hat{P}^\mu(\epsilon_1-\mathcal{A})+q^\mu] \tag{56}$$

and

$$\pi_2^\mu=G[\hat{P}^\mu(\epsilon_2-\mathcal{A})-q^\mu]. \tag{57}$$

When one quantizes the two-body systems, the momentum operator needs to be Hermitized and one has

$$\pi_1^\mu=G\left[\hat{P}^\mu(\epsilon_1-\mathcal{A})+q^\mu+\frac{1}{2}i\{\partial^\mu\ln G\}\right] \tag{58}$$

and

$$\pi_2^\mu=G\left[\hat{P}^\mu(\epsilon_2-\mathcal{A})-q^\mu-\frac{1}{2}i\{\partial^\mu\ln G\}\right]. \tag{59}$$

The total Hamiltonian is

$$\mathcal{H}=\frac{\epsilon_1}{\sqrt{s}}\mathcal{H}_1+\frac{\epsilon_2}{\sqrt{s}}\mathcal{H}_2. \tag{60}$$

The two-body equation of motion for the system is

$$\mathcal{H}\Psi=0=\mathcal{H}_1\Psi=\mathcal{H}_2\Psi. \tag{61}$$

Upon working out the operators in the above equation, one obtains

$$\mathcal{H}\Psi=G^2\left[\left(\mathbf{q}+\frac{1}{i}\{\nabla\ln G\}\right)^2+\frac{1}{4}\{(\nabla\ln G)^2\}\right.$$
$$\left.+\frac{1}{2}\{\nabla^2\ln G\}-(\epsilon_w-\mathcal{A})^2+m_w^2\right]\Psi=0. \tag{62}$$

Upon making the scale transformation

$$\Psi=\frac{\psi}{G}, \tag{63}$$

we have

$$\mathcal{H}\Psi=G\left[-\nabla^2-(\epsilon_w-\mathcal{A})^2+\frac{1}{4}\{(\nabla\ln G)^2\}\right.$$
$$\left.+\frac{1}{2}\{\nabla^2\ln G\}+m_w^2\right]\psi=0. \tag{64}$$

Thus the transverse field leads to an additional effective interaction of the form

$$\frac{1}{4}(\nabla\ln G)^2+\frac{1}{2}\nabla^2\ln G, \tag{65}$$

in addition to the scale transformation, Eq. (63). It gives the Darwin term and the proper perturbation expansion limits, as explained in detail in Ref. [20].

Using the form of $G(r)$ as given by Eq. (55), we obtain

$$\mathcal{H}\Psi=G\left[\mathbf{q}^2-(\epsilon_w-\mathcal{A})^2+m_w^2+\frac{5\alpha^2}{4r^2(r\sqrt{s}+2\alpha)^2}\right]\psi(r)=0, \tag{66}$$

which differs from Eq. (30) for the timelike component only by the additional last term, aside from the overall factor G. This additional term goes as $5/16r^2$ at small values of r to make the effective interaction repulsive at the origin. As a result, the wave function near $r\to 0$ goes as

$$\psi(r)\sim r^{l'}, \tag{67}$$

with

$$l'=\frac{-1+\sqrt{1+(5-16\alpha^2)/4}}{2}. \tag{68}$$

The wave function $\psi(r)$ therefore approaches zero at the origin.

What is the orthonomality condition for the amplitude in terms of Ψ or ψ? For the region of energy ϵ_{wi} close to ϵ_{wj} we have

$$\int d\mathbf{r}(\Psi_i\mathcal{H}\Psi_j-\Psi_j\mathcal{H}\Psi_i)$$
$$=(\epsilon_{wi}^2-\epsilon_{wj}^2)\int d\mathbf{r}\,\psi_i(\mathbf{r})\psi_j(\mathbf{r})\approx 0. \tag{69}$$

The approximate equality arises because of the dependence of the interaction on energy. Equality is attained when ϵ_{wi} approaches ϵ_{wj}. The wave function can therefore be normalized in a local energy region according to

$$\int d\mathbf{r}\,\psi_i(\mathbf{r})\psi_j(\mathbf{r})=\delta_{ij}. \tag{70}$$

Thus, the amplitude $\psi(\mathbf{r})$ [and not $\Psi(\mathbf{r})=\psi(\mathbf{r})/G(\mathbf{r})$] is the probability amplitude to be used in calculating the overlap integral in Eq. (14).

VII. NUMERICAL SOLUTION OF THE WAVE FUNCTION

If we use the dimensionless variable $\mathbf{z}=p\mathbf{r}$ as before, then Eq. (66) becomes

$$[-\nabla_z^2+U(z)-1]\psi(\mathbf{z})=0, \tag{71}$$

where

$$U(z)=-\frac{2\eta}{z}-\frac{\alpha^2}{z^2}+\frac{5}{4}\frac{\alpha^2}{z^2(z\sqrt{s}/p+2\alpha)^2}. \tag{72}$$

To obtain the wave function $\psi(z)$, we follow the phase-angle method discussed in detail by Calogero [22]. We write the wave function as

$$\psi_{lm}(\boldsymbol{r})=\frac{u_l(z)}{z}\mathcal{Y}_{lm}(\hat{\boldsymbol{r}}). \tag{73}$$

We represent the wave function $u_l(z)$ in terms of the amplitude $\alpha_l(z)$ and the phase shift $\delta_l(z)$:

$$u_l(r)=\frac{\alpha_l(z)}{\alpha_l(\infty)}\hat{D}_l(z)\sin[\hat{\delta}_l(z)+\delta_l(z)]\sqrt{\frac{4\pi}{2l+1}}, \tag{74}$$

with the boundary condition that $\delta_l(z\to 0)=0$. The functions $\hat{D}(z)$ and $\hat{\delta}(z)$ are known functions [22]:

$$\hat{D}_0(z)=1,\quad \hat{D}_1(x)=(1+1/z^2)^{1/2} \tag{75}$$

and

$$\hat{\delta}_0(z)=z,\quad \hat{\delta}_1(z)=z-\tan^{-1}z. \tag{76}$$

The equation for $\delta_l(z)$ is [22]

$$\frac{d}{dz}\delta_l(z)=-U(z)\hat{D}_l^2(z)\{\sin[\hat{\delta}_l(z)+\delta_l(z)]\}^2. \tag{77}$$

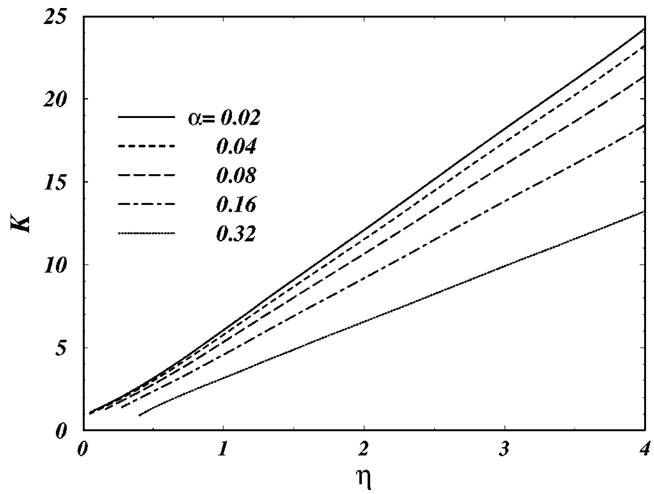

FIG. 5. The K factor versus η for various values of α, with the inclusion of both the timelike and the spacelike components of the gauge interaction.

After the function $\delta_l(z)$ is evaluated, the amplitude can be obtained from $\delta_l(z)$ by

$$\alpha_l(z)=\exp\left\{\frac{1}{2}\int_0^z ds\, U(s)\hat{D}_l^2(s)\sin 2[\hat{\delta}_l(s)+\delta_l(s)]\right\}. \tag{78}$$

We shall again specialize to the S wave. After we obtain the wave function $\psi(r)$ by solving the above equations numerically, we use it in Eqs. (16) and (17) to get the overlap integrals A and B and the K factor in Eq. (21). The results are shown in Figs. 5–7. In Fig. 5, we show the results of the K factor as a function of η for various values of α when the spacelike part component of the gauge interaction is also taken into account. The K factor increases with η and is much greater than 1 for large values of η. Its behavior is very similar to those with only the timelike component of the interaction.

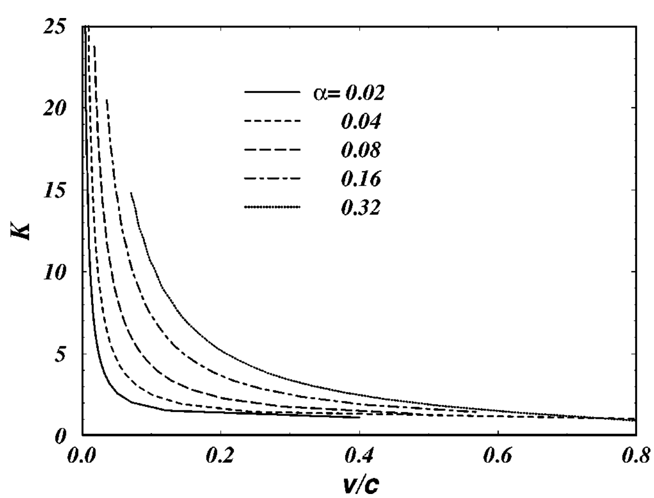

FIG. 6. The K factor versus the velocity v for various values of α, with the inclusion of both the timelike and the spacelike components of the gauge interaction.

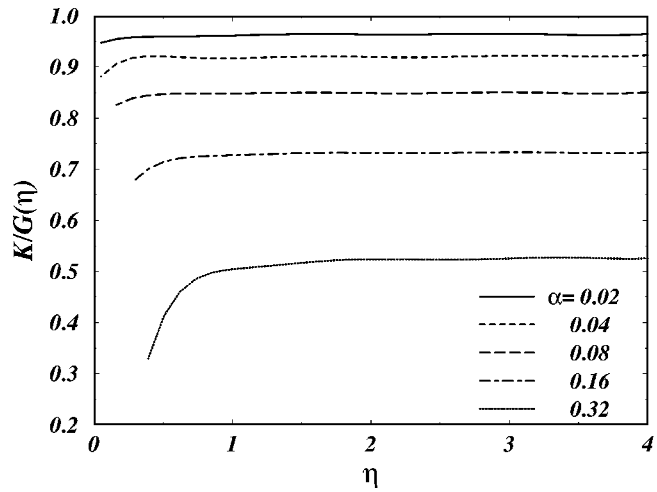

FIG. 7. The ratio between the K factor and the Gamow factor $G(\eta)$ for various values of α, with the inclusion of both the timelike and the spacelike components of the gauge interaction.

In Fig. 6, we show the behavior of K as a function of velocity for different values of α, for the case with both the timelike and the spacelike components of the gauge interaction. The K factor is much larger than unity for small velocities, and it decreases to unity as the velocity increases. In Fig. 7, we plot K/G as a function of η for various values of α. The results in Fig. 7 show significant differences from the Gamow factor and the difference is greater at smaller η (i.e., at higher velocities). The gross features of Figs. 5–7 are very similar to those with only the timelike part of the gauge interaction. For example, the K factor is getting closer to the Gamow factor and much greater than unity as the velocity becomes small, and the K factor is getting to 1 as the velocity becomes greater. However, there exist some small differences between the results with and without the spacelike component of the gauge interaction. Comparing Fig. 2 and Fig. 5, for all cases of α and for the whole range of η, the K factor with the spacelike component of the gauge interaction is smaller than that for the case without the spacelike component. The same conclusion can be obtained by comparing Fig. 4 with Fig. 7. This indicates that the spacelike component of the gauge interaction reduces the attractive final- or initial-state interactions. The spacelike part of the gauge interaction produces an effective repulsive interaction and it reduces the magnitude of the K factor. This effect is stronger for higher values of η or for slower velocities. This effect is also stronger in higher values of α, corresponding to a stronger coupling. For example, at $\eta=4$ the spacelike component leads to a reduction of the K factor by only 0.8% for $\alpha=0.02$ but 9% for $\alpha=0.32$.

VIII. CONCLUSIONS AND DISCUSSIONS

The mutual final-state interaction between the produced particles has an effect on their rate of production. There will be similar effects if the particles interact via the initial-state interaction. The effects are simplest to be taken into account by using the method of the K factor. One calculates the rate for the process as though there were no initial- or final-state interactions, using, for example, the perturbation theory. The additional initial- or final-state interactions can be included by multiplying a K factor as given by Eq. (8).

For Coulomb-type interactions, the K factor has been traditionally taken to be the Gamow factor obtained as the absolute square of the wave function at the origin of the relative coordinate. With relativistic Coulomb wave functions, the K factor can be obtained as the overlap of the wave function with the Feynman amplitude.

Our investigation of the K factor for the case of the production of a pair of scalar particles indicates that there are substantial deviations from the Gamow factor when the strength of the coupling is large. In particular, the proper treatment reduces the magnitude of the Gamow factor significantly. The reason for this reduction is that in the pair production, there is an effective screening of the Coulomb-type interaction arising from the effective ''exchange'' of one of the produced particles.

We have presented an explicit formula for the relativistic modification of the Gamow factor for the production of a pair of bosons for the case of an attractive timelike part of the gauge interaction. Numerical results are also obtained to show the magnitude of the K factor. The results of the K factor can be applied to a class of processes in which the boson particles are produced and interacting with a Coulomb-type interaction.

We have also studied the effects of the additional spacelike part (the transverse part) of the gauge interaction. It leads to an effective repulsive potential and it serves to decrease the magnitude of the K factor. This reduction is larger for smaller values of η (or for lower velocities) and for stronger coupling constants. However, the magnitude of this reduction of the K factor is small (less than about 10% and smaller in most cases). To the extent that a 10% error can be tolerated, the analytical results in Sec. V obtained without the transverse component are an adequate representation of the K factor. More accurate results will require the inclusion of the transverse component of the gauge interaction as discussed in Secs. VI and VII.

The large modification of the Gamow factor for the production of two bosons studied here indicates the need to extend the present formalism to study the case of two fermions or two gluons. The application to fermions or gluons will be useful in the problem of production or reaction of quarks and gluons. As a pair of quarks or gluons interact with a strong color-Coulomb interaction with a coupling constant α about 0.2–0.4, the simple results from the present study indicate that the effects of the initial- or final-state interaction for quarks and gluons will be large, and the K factor will be substantially different from what one obtains using the Gamow factor. We hope to study the modification of the Gamow factor for two produced fermions in our next investigation.

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