Constraints on spin observables in $\bar{p}p \to \bar{\Lambda}\Lambda$

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It is recalled that spin observables in the strangeness-exchange reaction $\bar{p}p \to \bar{\Lambda}\Lambda$ are not independent, but are related to each other by simple algebraic relations. This provides constraints on the existing data on polarization and spin-correlation coefficients and also on forthcoming data obtained using a polarized proton target.

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Antihyperon-hyperon production ($\bar{Y}Y$) in antiproton-proton ($\bar{p}p$) collisions has been studied by the PS185 Collaboration [1,2] at CERN. Experimental data on the integrated cross section, differential distribution I_0, polarization P and spin-correlation coefficients C_{ij} at various energies have already been published. The experiment has been resumed using a polarized proton target, and the results of the analysis are expected to be published soon [3].

The algebra of observables involved in the scattering of two spin-1/2 particles is rather straightforward and has been extensively studied in dedicated articles [4–6]. This knowledge has, however, been somewhat lost, and it seems desirable to adapt the general formalism to the special case of the set of observables available for $\bar{p}p \to \bar{\Lambda}\Lambda$.

Furthermore, there is considerable interest in the study of this reaction. Models based on K, K^* exchange [7], quark-pair creation, or polarized strange sea-quarks in the nucleon [8] give different predictions on the transfer of spin polarization. This was the main motivation for extending the study of strangeness production. The problems encountered there are intimately related to those of deep inelastic scattering or violation of the Okubo-Zweig-Iizuka (OZI) rule in $\bar{p}p$ annihilation [8].

In Ref. [9], it was recalled that the existing data on correlation coefficients give constraints on the transfer of polarization from p to Λ or from p to $\bar{\Lambda}$. In the present paper, we wish to provide further inequalities, which hopefully will be useful for analyzing the data.

We start from the decomposition of the transition matrix $\mathcal{M}$ into six (complex) amplitudes a, b, c, d, e, and g. In the c.m. system (c.m.s), $\mathcal{M}$ can be written as [4,5]

$$\begin{aligned}\mathcal{M} = {} & (a+b)I + (a-b)\boldsymbol{\sigma}_1\cdot\hat{\mathbf{n}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{n}} + (c+d)\boldsymbol{\sigma}_1\cdot\hat{\mathbf{k}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{k}} \\ & + (c-d)\boldsymbol{\sigma}_1\cdot\hat{\mathbf{p}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{p}} + e(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\hat{\mathbf{n}} \\ & + g(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{k}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{p}} + \boldsymbol{\sigma}_1\cdot\hat{\mathbf{p}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{k}}), \end{aligned} \tag{1}$$

where the kinematical unit vectors are defined from the momentum $\mathbf{p}_i$ of $\bar{p}$ and $\mathbf{p}_f$ of $\bar{\Lambda}$:

$$\hat{\mathbf{p}} = \frac{\mathbf{p}_f}{|\mathbf{p}_f|}, \quad \hat{\mathbf{n}} = \frac{\mathbf{p}_i \times \mathbf{p}_f}{|\mathbf{p}_i \times \mathbf{p}_f|}, \quad \hat{\mathbf{k}} = \hat{\mathbf{n}} \times \hat{\mathbf{p}}. \tag{2}$$

Neglecting an overall flux and phase-space factor, the differential cross section I_0 and the spin observables are given by

$$\begin{aligned} I_0 &= \mathrm{Tr}[\mathcal{M}\mathcal{M}^\dagger], \\ P_n I_0 &= \mathrm{Tr}[\boldsymbol{\sigma}_1\cdot\hat{\mathbf{n}}\mathcal{M}\mathcal{M}^\dagger], \\ A_n I_0 &= \mathrm{Tr}[\mathcal{M}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{n}}\mathcal{M}^\dagger], \\ C_{ij} I_0 &= \mathrm{Tr}[\boldsymbol{\sigma}_1\cdot\hat{\mathbf{i}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{j}}\mathcal{M}\mathcal{M}^\dagger], \\ D_{ij} I_0 &= \mathrm{Tr}[\boldsymbol{\sigma}_2\cdot\hat{\mathbf{i}}\mathcal{M}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{j}}\mathcal{M}^\dagger], \\ K_{ij} I_0 &= \mathrm{Tr}[\boldsymbol{\sigma}_1\cdot\hat{\mathbf{i}}\mathcal{M}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{j}}\mathcal{M}^\dagger]. \end{aligned} \tag{3}$$

More explicitly (again, up to an overall factor),

$$\begin{aligned} I_0 &= |a|^2+|b|^2+|c|^2+|d|^2+|e|^2+|g|^2, \\ P_n I_0 &= 2\,\mathrm{Re}(ae^*) + 2\,\mathrm{Im}(dg^*), \\ A_n I_0 &= 2\,\mathrm{Re}(ae^*) - 2\,\mathrm{Im}(dg^*), \\ C_{nn} I_0 &= |a|^2-|b|^2-|c|^2+|d|^2+|e|^2+|g|^2, \\ C_{xx} I_0 &= -2\,\mathrm{Re}(ad^*+bc^*) - 2\,\mathrm{Im}(ge^*), \\ C_{zz} I_0 &= 2\,\mathrm{Re}(ad^*-bc^*) + 2\,\mathrm{Im}(ge^*), \\ C_{xz} I_0 &= -2\,\mathrm{Re}(ag^*) - 2\,\mathrm{Im}(ed^*), \\ D_{nn} I_0 &= |a|^2+|b|^2-|c|^2-|d|^2+|e|^2-|g|^2, \\ D_{xx} I_0 &= 2\,\mathrm{Re}(ab^*+cd^*), \\ D_{zz} I_0 &= 2\,\mathrm{Re}(ab^*-cd^*), \\ D_{xz} I_0 &= 2\,\mathrm{Re}(cg^*) + 2\,\mathrm{Im}(be^*), \\ K_{nn} I_0 &= |a|^2-|b|^2+|c|^2-|d|^2+|e|^2-|g|^2, \\ K_{xx} I_0 &= -2\,\mathrm{Re}(ac^*+bd^*), \\ K_{zz} I_0 &= -2\,\mathrm{Re}(ac^*-bd^*), \\ K_{xz} I_0 &= -2\,\mathrm{Re}(bg^*) + 2\,\mathrm{Im}(ec^*). \end{aligned} \tag{4}$$

To project out the spins of the particles, we follow here the usual convention that for $\bar{\Lambda}$, $\{\hat{\mathbf{x}},\hat{\mathbf{n}},\hat{\mathbf{z}}\}$ coincides with $\{\hat{\mathbf{k}},\hat{\mathbf{n}},\hat{\mathbf{p}}\}$, while for p or Λ, the axes $\{\hat{\mathbf{x}},\hat{\mathbf{n}},\hat{\mathbf{z}}\}$ coincide with $\{-\hat{\mathbf{k}},\hat{\mathbf{n}},-\hat{\mathbf{p}}\}$.

In principle, a polarized target gives access to rank-3 observables of the type

$$C_{0\alpha ij}I_0=\mathrm{Tr}[\boldsymbol{\sigma}_1\cdot\hat{\mathbf{i}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{j}}\mathcal{M}\boldsymbol{\sigma}_2\cdot\hat{\alpha}\mathcal{M}^\dagger]. \tag{5}$$

For instance,

$$C_{0nzz}I_0=2\ \mathrm{Re}(de^*)-2\ \mathrm{Im}(ag^*),$$
$$C_{0nxx}I_0=-2\ \mathrm{Re}(de^*)+2\ \mathrm{Im}(ag^*),$$
$$C_{0nzx}I_0=-2\ \mathrm{Re}(ge^*)-2\ \mathrm{Im}(ac^*+bd^*), \tag{6}$$

C_{0nnn} being equal to A_n. It is not yet sure, however, whether a statistically significant measurement of these rank-3 observables will be possible from the data accumulated during the last run of the CERN experiment (PS 185/3) [3].

There are linear relations among observables. For instance, it is shown in the Appendix that

$$2|A_n|-C_{nn}\leqslant 1,$$
$$2|P_n|-C_{nn}\leqslant 1,$$
$$2|C_{xz}|-C_{nn}\leqslant 1. \tag{7}$$

From the proof, it becomes clear that similar inequalities exist among combinations of observables, one example being

$$2|A_n+P_n|-(K_{nn}+D_{nn})\leqslant 2. \tag{8}$$

If one now concentrates on the set $\{I_0,C_{nn},D_{nn},K_{nn}\}$

$$I_0=|a|^2+|b|^2+|c|^2+|d|^2+|e|^2+|g|^2,$$
$$C_{nn}I_0=|a|^2-|b|^2-|c|^2+|d|^2+|e|^2+|g|^2,$$
$$D_{nn}I_0=|a|^2+|b|^2-|c|^2-|d|^2+|e|^2-|g|^2,$$
$$K_{nn}I_0=|a|^2-|b|^2+|c|^2-|d|^2+|e|^2-|g|^2, \tag{9}$$

one deduces

$$4(|a|^2+|e|^2)=I_0(1+C_{nn}+D_{nn}+K_{nn})\geqslant 0,$$
$$4(|d|^2+|g|^2)=I_0(1+C_{nn}-D_{nn}-K_{nn})\geqslant 0,$$
$$4|c|^2=I_0(1-C_{nn}-D_{nn}+K_{nn})\geqslant 0,$$
$$4|b|^2=I_0(1-C_{nn}+D_{nn}-K_{nn})\geqslant 0. \tag{10}$$

With the help of A_n and P_n, the positivity of $|a|^2+|e|^2$ is refined into the separate positivity of $|a\mp e|^2$, and similarly for $|d|^2+|g|^2$ into $|d\mp ig|^2$. One easily derives

$$1+C_{nn}+D_{nn}+K_{nn}+2P_n+2A_n\geqslant 0,$$
$$1+C_{nn}+D_{nn}+K_{nn}-2P_n-2A_n\geqslant 0,$$
$$1+C_{nn}-D_{nn}-K_{nn}+2P_n-2A_n\geqslant 0,$$
$$1+C_{nn}-D_{nn}-K_{nn}-2P_n+2A_n\geqslant 0. \tag{11}$$

The variables

$$a'=(a+d)/\sqrt{2},\quad b'=(b+c)/\sqrt{2},\quad e'=(e+ig)/\sqrt{2},$$
$$d'=(a-d)/\sqrt{2},\quad c'=(b-c)/\sqrt{2},\quad g'=(e-ig)/\sqrt{2} \tag{12}$$

allow one to rewrite the set $\{I_0,C_{nn},C_{xx},C_{zz}\}$ as

$$I_0=|a'|^2+|b'|^2+|c'|^2+|d'|^2+|e'|^2+|g'|^2,$$
$$-C_{xx}I_0=|a'|^2+|b'|^2-|c'|^2-|d'|^2-|e'|^2+|g'|^2,$$
$$C_{nn}I_0=|a'|^2-|b'|^2-|c'|^2+|d'|^2+|e'|^2+|g'|^2,$$
$$C_{zz}I_0=|a'|^2-|b'|^2+|c'|^2-|d'|^2-|e'|^2+|g'|^2, \tag{13}$$

and so deduce

$$4(|a'|^2+|e'|^2)=I_0(1+C_{nn}-C_{xx}+C_{zz})\geqslant 0,$$
$$4(|d'|^2+|g'|^2)=I_0(1+C_{nn}+C_{xx}-C_{zz})\geqslant 0,$$
$$4|c'|^2=I_0(1-C_{nn}+C_{xx}+C_{zz})\geqslant 0,$$
$$4|b'|^2=I_0(1-C_{nn}-C_{xx}-C_{zz})\geqslant 0. \tag{14}$$

Note that the third of the above relations is nothing but the spin-singlet fraction

$$F_0=\frac{1}{4}(1+C_{xx}-C_{yy}+C_{zz})=\frac{1}{2I_0}|b-c|^2, \tag{15}$$

being positive. The normalization is such that $F_0=1/4$ in the absence of a spin-dependent interaction.

Let us now provide some examples of quadratic inequalities. In Ref. [9], it was recalled that Eqs. (4) imply

$$C_{zz}^2+D_{nn}^2\leqslant 1, \tag{16}$$

and a number of similar inequalities. The proof is given in the Appendix. Table I summarizes the pairs of rank-1 or rank-2 observables which satisfy a quadratic relation similar to Eq. (16).

Other relations can be written down, involving combinations of more than two or three observables. For instance, it will be shown below that $D_{nn}=K_{nn}$ when $F_0=0$. This suggests that D_{nn} cannot differ too much from K_{nn} when F_0 is small. It can be shown that

$$\left(\frac{D_{nn}-K_{nn}}{2}\right)^2+(2F_0-1)^2\leqslant 1, \tag{17}$$

which relates D_{nn}, K_{nn}, C_{nn}, C_{xx}, and C_{zz}. As a consequence, $D_{nn}=K_{nn}$ also in the (unphysical) limit of a pure spin-singlet reaction.

TABLE I. Pairs of observables fulfilling an inequality such as $C_{zz}^2+D_{nn}^2\leqslant 1$.

A_n	C_{nn}	C_{xx}	C_{zz}	C_{xz}	D_{nn}	D_{xx}	D_{zz}	D_{xz}	K_{nn}	K_{xx}	K_{zz}	K_{xz}	
		×	×	×		×	×	×		×	×	×	P_n
						×	×	×		×	×	×	A_n
						×	×	×		×	×	×	C_{nn}
				×	×		×		×		×		C_{xx}
				×	×	×		×	×	×		×	C_{zz}
					×	×	×	×	×	×	×	×	C_{xz}
										×	×	×	D_{nn}
								×	×		×	×	D_{xx}
								×	×	×		×	D_{zz}
									×	×	×		D_{xz}
													K_{nn}
												×	K_{xx}
												×	K_{zz}

The most general method for writing a number of quadratic equalities has been given in Refs. [5,6]. One can solve Eqs. (4) and similar for higher-rank observables to extract $aa^*, ab^*,\ldots$ in terms of experimental quantities. Then any identity of the type

$$(ab^*)(cd^*)=(ad^*)(cb^*) \tag{18}$$

translates into a relation between observables. This usually involves quantities such as $D_{0\alpha\beta\gamma}$, which are hardly measurable. We refer to Ref. [6] for more details in the case of *pp* elastic scattering.

Special cases. There are great simplifications in situations where one or more amplitudes (or combinations) vanish, in particular the following.

(1) *Pure spin-triplet production*. $b=c$, and as a consequence

$$D_{xx}=-K_{xx}, \quad D_{zz}=-K_{zz}, \quad D_{xz}=-K_{xz}, \quad D_{nn}=K_{nn}. \tag{19}$$

This is not a surprise. If the final state contains only components which are symmetric under exchange of Λ and $\bar{\Lambda}$ spins, then the same correlation is expected between *p* and Λ spins as between *p* and $\bar{\Lambda}$ spins. In the case of pure spin-triplet production, we also have

$$C_{nn}-C_{xx}\geqslant 0, \quad C_{nn}-C_{zz}\geqslant 0, \quad C_{xx}+C_{zz}\geqslant 0, \tag{20}$$

as well as several inequalities similar to those of Eq. (7),

$$2|D_{zz}|+C_{zz}\leqslant 1, \quad 2|K_{xx}|+C_{xx}\leqslant 1, \tag{21}$$

etc., which are listed in Table II.

(2) *Forward production.* This case was discussed recently in Ref. [10]. In the forward limit $\theta_{\mathrm{c.m.}}=0$, the transition matrix $\mathcal{M}$ becomes invariant under any rotation around the beam axis. In our notation, this means $e=g=0$ and $a-b=c+d$. As a consequence, the spin parameters are related. In particular,

$$C_{xx}=-C_{nn}, \quad D_{xx}=D_{nn}, \quad K_{xx}=-K_{nn}, \tag{22}$$

and

$$C_{zz}=1-2|b+c|^2/I_0, \quad D_{zz}=1-2|a-b|^2/I_0,$$

$$K_{zz}=-1+2|a-c|^2/I_0, \tag{23}$$

implying

$$K_{zz}-D_{zz}\leqslant 0, \quad C_{zz}+D_{zz}\geqslant 0, \quad \text{etc.} \tag{24}$$

The relation between C_{xx} and C_{nn} was already noticed [7].

(3) Furthermore, in the combined case of pure spin-triplet and forward production, the expression for the longitudinal spin observables simplifies to

TABLE II. Pairs of observables fulfilling an inequality $2|\alpha|\pm\beta\leqslant 1$. The relations involving $(\alpha,\beta)=(P_n,C_{nn})$, (A_n,C_{nn}) and (C_{xz},C_{nn}) are general, as per Eq. (7): the others are specific of a pure spin-triplet reaction.

	α													
β	P_n	A_n	C_{nn}	C_{xx}	C_{zz}	C_{xz}	D_{nn}	D_{xx}	D_{zz}	D_{xz}	K_{nn}	K_{xx}	K_{zz}	K_{xz}
C_{nn}	−	−				−	−				−			
C_{xx}								+		+		+		+
C_{zz}									+				+	
C_{xz}														

$$\frac{1-C_{zz}}{2}=D_{zz}=-K_{zz}=4|b|^2/I_0\geqslant 0. \tag{25}$$

Within the error limits, the linear constraints (14) and the quadratic inequalities of type (16), in particular those between P_n and C_{ij}, seem to be satisfied by the data [2] except at a few points. Also, $C_{xx}\simeq -C_{nn}$ is observed in the forward and backward regions. So from this point of view, the most recent data [2] are more consistent than the former ones [1].

Let us summarize. Several spin observables can be measured for the $\overline{p}p\to\overline{\Lambda}\Lambda$ reaction since the weak decay of Λ (or $\overline{\Lambda}$), which gives an indication on its spin. This offers the possibility to test in great detail the mechanisms by which strangeness is created. The experiment is however, delicate, and its analysis might use the consistency checks provided by the linear or quadratic inequalities listed in this paper. It is hoped that reliable spin observables will help probe the mechanisms proposed for this strangeness-exchange reaction. In particular, the hypothesis of a pure spin-triplet $\overline{\Lambda}\Lambda$ production, suggested by early LEAR data on unpolarized targets, can be tested accurately.

Note added. After the completion of this work, we received a paper by Paschke and Quinn [11], where it is shown explicitly that data taken with a tranversally polarized target provide in principle the possibility of a full reconstruction of the amplitudes (of course, to an overall phase). In the case of elastic nucleon-nucleon interactions, such a full reconstruction has already been achieved, as seen, e.g., in Ref. [12] and references therein.

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APPENDIX

The quadratic inequality (16) can be derived using the vectors

$$\mathbf{V}_1=(|a|^2-|d|^2,2|ad|),\quad \mathbf{V}_2=(|b|^2-|c|^2,2|bc|),$$
$$\mathbf{V}_3=(|e|^2-|g|^2,2|eg|), \tag{A1}$$

with normalization $|\mathbf{V}_1|=|a|^2+|d|^2$, $|\mathbf{V}_2|=|b|^2+|c|^2$ and $|\mathbf{V}_3|=|e|^2+|g|^2$. Given that $C_{zz}I_0\leqslant 2(|ad|+|bc|+|eg|)$, one can deduce

$$I_0^2(C_{zz}^2+D_{nn}^2)\leqslant(\mathbf{V}_1+\mathbf{V}_2+\mathbf{V}_3)^2\leqslant(|\mathbf{V}_1|+|\mathbf{V}_2|+|\mathbf{V}_3|)^2 \tag{A2}$$

and thus the desired $C_{zz}^2+D_{nn}^2\leqslant 1$.

Similarly, one can introduce the vectors

$$\mathbf{V}_1=(|a|,|d|),\quad \mathbf{V}_2=(|b|,|c|),\quad \mathbf{V}_3=(|e|,|g|), \tag{A3}$$

with normalization $|\mathbf{V}_1|^2=|a|^2+|d|^2$, $|\mathbf{V}_2|^2=|b|^2+|c|^2$, and $|\mathbf{V}_3|^2=|e|^2+|g|^2$. Given that $I_0A_n\leqslant 2\mathbf{V}_1\cdot\mathbf{V}_3$ and $I_0D_{xx}\leqslant 2\mathbf{V}_1\cdot\mathbf{V}_2$, one can deduce

$$\begin{aligned}I_0^2(A_n^2+D_{xx}^2)&\leqslant 4|\mathbf{V}_1|^2(|\mathbf{V}_2|^2+|\mathbf{V}_3|^2)\\&\leqslant 4I_0^2(|\mathbf{V}_1|^2/I_0)[1-(|\mathbf{V}_1|^2/I_0)]\\&\leqslant I_0^2\end{aligned} \tag{A4}$$

and thus $A_n^2+D_{xx}^2\leqslant 1$.

To prove the inequality between C_{zz} and D_{zz} in the case of pure spin-triplet production, one can start from a simplified problem where the amplitudes are real and

$$I_0=a^2+2b^2+d^2,\quad C_{zz}I_0=2(ad-b^2),\quad D_{zz}I_0=b(a-d). \tag{A5}$$

Then

$$\begin{aligned}I_0(C_{zz}+2D_{zz})&=a^2+2b^2+d^2-(a-d)^2-4b^2+4b(a-d)\\&=I_0-(a-d-2b)^2\\&\leqslant I_0.\end{aligned} \tag{A6}$$

It is easily shown that restoring the complex character of the amplitudes and possible nonvanishing of e and g cannot do anything but strengthen the inequality. A similar reasoning holds for $C_{zz}-2D_{zz}$. The proof is analogous for the other inequalities in Table II and Eq. (7).

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