Calculations for populations of selected isotopes in intermediate energy heavy ion collisions

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We compute the populations of isotopes of boron, carbon, and nitrogen measured experimentally in intermediate energy heavy ion collisions. A two-component soluble statistical model is used to find the initial populations of different nuclei at a finite temperature. These initial populations are both in particle stable and particle unstable states. The particle unstable states then decay. The final populations after these decays are computed and compared with experimental data.

I. INTRODUCTION

In this work we attempt to calculate the populations of various isotopes of boron, carbon, and nitrogen that were measured in a number of experiments [1,2]. The calculation proceeds in two stages. In the first, primary populations are calculated in a two-component statistical model. The calculations in the first part are exact although numerical. These populations are both in particle stable and unstable states. In the second stage the particle unstable states are allowed to decay. This is done in a Weisskopf formalism. Exact calculations are very long and some approximations had to be introduced. These approximations will be discussed. After the decays, the populations are compared with experiments.

One motivation for this calculation was that it serves as an application of the two-component statistical model where an exact calculation can be done. This, therefore, could serve as a benchmark of how far one can trust the predictions of the model. Unfortunately, the particular predictions we are looking for are also affected by the subsequent decays. This effect is not small. Hence, the predictions are the result of the combination of two models which had to be applied in tandem before experimental data could be compared. A recent application of the two-component model was the computation of the caloric curve [3] in nuclei.

The sections are organized as follows. Section II gives a brief description of the two-component statistical model. After presenting, in Sec. III, in words and simple formulas, the overview of the secondary decay calculation, we present in Sec. IV the formalism that we use to model secondary decay. In Sec. V we present some calculational details and Sec. VI presents the results of the calculation. Summary and conclusion are presented in Sec. VII. A short appendix of the more complicated formula is also presented.

II. THE TWO-COMPONENT SOLUBLE STATISTICAL MODEL

For completeness, we present here the essential details of the two-component statistical model. The one-component model was described elsewhere [4,5]. The formalism of the two-component model can also be found in [3].

Assume that the system which breaks up after two heavy ions hit each other can be desribed as a hot, equilibrated nuclear system characterized by a temperature T and a freeze-out volume V within which there are A nucleons (A

$=Z+N$). The partition function of the system is given by

$$Q_{Z,N} = \sum \Pi_{i,j} \frac{\omega_{i,j}^{n_{i,j}}}{n_{i,j}!}. \tag{2.1}$$

Here $n_{i,j}$ is the number of composites with proton number i and neutron number j, and $\omega_{i,j}$ is the partition function of a single composite with proton, neutron numbers i,j, respectively. There are two constraints: $\Sigma_{i,j} i n_{i,j} = Z$ and $\Sigma_{i,j} j n_{i,j} = N$. These constraints would appear to make the computation of $Q_{Z,N}$ prohibitively difficult, but a recursion relation exists [3,4] which allows numerical computation of $Q_{Z,N}$ quite easy even for large Z and N. Three equivalent recursion relations exist, any one of which could be used. For example, one such relation is

$$Q_{z,n} = \frac{1}{z} \sum_{i,j} i \omega_{i,j} Q_{z-i,n-j}. \tag{2.2}$$

All nuclear properties are contained in $\omega_{i,j}$. It is given by

$$\omega_{i,j} = \frac{V_f}{\hbar^3} \left(\frac{mT}{2\pi} \right)^{3/2} (i+j)^{3/2} \times q_{i,j,\text{int}}. \tag{2.3}$$

Here V_f is the free volume within which the particles move; V_f is related to V through $V_f = V - V_{\text{ex}}$ where V_{ex} is the excluded volume due to finite sizes of composites. We take V_f to be a variable of the calculation, it is set to be equal to $f V_0$ where V_0 is the normal volume for $(Z+N)$ nucleons, f is then varied to obtain the best fit with experimental data. The quantity $q_{i,j,\text{int}}$ is the internal partition function of the composite:

$$q_{i,j,\text{int}} = \sum_{k}^{E_{\text{max}}} (2J_k+1) e^{(-E_k/T)} + q_{i,j,\text{cont}}, \tag{2.4}$$

where the summation on the right-hand side is the contribution from the discrete spectrum (the cutoff E_{max} is simply the highest energy level that has been resolved for the given nucleus and is available from data tables [6]); and $q_{i,j,\text{cont}}$ is the contribution from the continuum. Without loss of generality we can write

$$q_{A,\text{int}} = \int \rho_A(E) e^{-\beta E} dE, \tag{2.5}$$

where we have used the abbreviation $A = i + j$, to stand for both i and j; $\rho_A(E)$ is usually partly discrete and partly continuous.

We will need both $q_{A,\text{int}}$ and $\rho_A(E)$. Volumes of work are available on $\rho_A(E)$. This is dealt with in detail in Appendix 2B of [7]. The saddle-point approximation for the density of states assuming a Fermi-gas model is [see Eq. (2B-14) in [7]]

$$\rho_A(E) = \rho_A^0(E) \times \exp(\ln z_{gr} - \alpha_0 A + \beta_0 E). \quad (2.6)$$

For explanations of how α_0 and β_0 are to be chosen see [7]. In the Fermi-gas model the quantity which is exponentiated is simply the total entropy $S = As$. Thus the density of states is given by a familiar expression $\rho_A(E) = \rho_A^0(E)\exp(S)$ where $\rho_A^0(E)$ is the prefactor. Approximate values of $\rho_A^0(E)$ are known provided one does not have to be concerned with a very low value of E (which we do need). At the temperatures with which we will be concerned, $\exp(S)$ in the Fermi-gas model is given quite accurately by $\exp[\pi(AE/\epsilon_F)^{1/2}]$.

In the bulk of this paper we adopt this prescription. For up to ^{20}F we write the density of states as $\rho_A(E) = \rho_A^0 \times \exp(S)$, where the low-temperature Fermi-gas expression for S as written above is used. The energy-independent value of the prefactor is fixed from experimentally known levels:

$$\sum_{k=0}^{E_{\text{max}}} (2J_K + 1)e^{-E_k/T} = \rho_A^0 \int_0^{E_{\text{max}}} e^{(S(E) - \beta E)} dE. \quad (2.7)$$

While objections can be raised against this procedure, it achieves three objectives which we wanted to have: (a) we did not want to lose all information of the experimentally measured discrete excited states; (b) we did want to take into account the contribution from the continuum; and (c) with this procedure calculations are fairly simple. Although we will not report on all other formulas for density of states that we also used, our final results for the isotope populations are quite stable within the reasonable variations that were tried.

We estimate the continuum contribution as a similar integral from E_{max} to infinity:

$$q_{i,j,\text{cont}} = \int_{E_{\text{max}}}^{\infty} \rho_A^0 \rho_A(E) e^{-\beta E} dE. \quad (2.8)$$

This process is continued up to ^{20}F wherein we can read off energy levels from data tables. For elements above ^{20}F, a parametrized version was used, which is given as

$$q_{i,j,\text{int}} = \exp\left[\left(W_0(i+j) - \sigma(i+j)^{2/3} - \kappa\frac{i^2}{(i+j)^{1/3}}\right.\right.$$
$$\left.\left. - s\frac{(j-i)^2}{j+i} + T^2(i+j)/e\right) \Big/ T\right], \quad (2.9)$$

where $W_0 = 15.8$ MeV, $\sigma = 18.0$ MeV, $\kappa = 0.72$ MeV, $s = 23.5$ MeV, and $e = 16.0$ MeV. The first four terms in the right-hand side of Eq. (2.9) arise from a parametrized version of the binding energy of the ground state. The last term

arises from an approximation to the Fermi-gas formula for level density. This was also used in [8]. For protons and neutrons q is 1.

The average number of particles of a composite is given by

$$\langle n_{i,j} \rangle = \omega_{i,j} \frac{Q_{Z-i,N-j}}{Q_{Z,N}}. \quad (2.10)$$

However, this population is partly over particle stable states and partly over particle unstable states which will decay into other nuclei before reaching the detectors.

III. SECONDARY DECAY

In keeping with the way experimental data are presented, we will compute ratios of yields of different isotopes of boron, carbon, and nitrogen. To lowest order one can consider the $\langle n_{i,j} \rangle$ obtained from Eq. (2.10) above, remove the particle unstable fractions, and compare them directly with experiment. This is shown in the figures as the dotted line with a filled triangle plotting symbol. These populations contain only particle stable states.

Next we consider decay of the particle unstable states. We restrict the secondary decay to be due to emission of six species: neutron, proton, deuteron, ^{3}He, triton, and α particles. Any given nucleus (i,j) from a particle unstable state can, in principle, go to at most six other nuclei. As the populations are canonically distributed among the various energy levels, we can calculate the fraction that are in particle stable or unstable states. If the fraction of nuclei (i,j) at the first stage in unstable states is $f_{i,j}^0$, then the number of nuclei (i,j) left in particle stable states at the stage we call ''up to single decay'' is given by

$$\langle n_{i,j} \rangle^1 = (1 - f_{i,j}^0)\langle n_{i,j} \rangle$$
$$+ \sum_{a,b} (1 - f_{i,j}^1)\frac{\Gamma_{a,b}}{\Gamma_T}f_{i+a,j+b}^0\langle n_{i+a,j+b} \rangle, \quad (3.1)$$

where $f_{i,j}^1$ is the fraction of the once decayed nuclei in unstable states. We will indicate how to calculate $f_{i,j}^1$ in the next section. $\Gamma_{a,b}$ is the width for emission by (a,b) from $(i+a,j+b)$ and Γ_T is the total width.

We can then take these revised populations $\langle n_{i,j} \rangle^1$ and again compute the ratios. We label these ''up to single decay.'' These are reported in the plots as the small dashed line with the diamond plotting symbol. Note: this is just the stable fraction of the population after one stage of decay, the actual population is possibly greater.

After the first decay there may be some fraction in particle unstable states. These can decay, thereby changing the population of (i,j) to $\langle n_{i,j} \rangle^2$. If we take the ratios now we get what we call ''up to double decay,'' this is denoted by the dot-dashed line and the square plotting symbol. Again at this stage the $\langle n_{i,j} \rangle^2$ represent only the sum of the stable fractions of the populations obtained from the initial distribution, single, and double decays.

It is clear the procedure can be continued. The fraction remaining in particle unstable states will continue to decrease. We found no significant difference between the up to triple decay and the up to quadruple decay calculation. Thus we do not continue beyond. Once again it should be noted that all the plotted populations, $\langle n_{i,j} \rangle$, $\langle n_{i,j} \rangle^1$, $\langle n_{i,j} \rangle^2$, $\langle n_{i,j} \rangle^3$ etc., quote only the stable fractions at freeze-out, after single, double, and triple decay, respectively.

The formalism for the decay calculation is given in the next section, there the quantities $f_{i,j}$, $\Gamma_{a,b}$ will be calculated in somewhat greater detail. The reader who is only interested in the final results could jump to Secs. VI and VII.

IV. THE DECAY FORMALISM

As the heated clusters stream out from the hot source, many of them will be in particle unstable states, these will decay by particle emission, for example, by emitting a neutron, proton, α particle, etc. They will then leave a residue nucleus which may be particle stable or unstable; if it is unstable then it will decay further into another isotope and this process will continue till the residue is produced in a particle stable state.

The primary calculation assumes that thermal equilibrium is achieved at freeze-out; if this is true then the number of composites with i protons and j neutrons with an energy in the interval E and $E+dE$ is given by the canonical factor

$$dn_A(E) = C_{i,j}\rho_A(E)e^{-\beta E}dE, \tag{4.1}$$

where we have abbreviated A to mean (i,j), and $\rho_A(E)$, from Sec. II is given as $\rho_A(E) = \rho_A^0\exp(S)$. The multiplicative constant ρ_A^0 will, henceforth, be absorbed into the overall normalization constant $C_{i,j}$. Thus from now on the density function is given simply as

$$\rho_A(E) = \exp\left[\pi\left(\frac{(i+j)E}{\epsilon_F}\right)^{1/2}\right]. \tag{4.2}$$

$C_{i,j}$ is a normalization constant such that

$$\int_0^\infty C_{i,j}\rho_A(E)e^{-\beta E}dE = \langle n_{i,j} \rangle. \tag{4.3}$$

Now of the various levels in a particular nucleus, some will be at a very low energy and as a result will be stable to any form of particle decay. Those that lie above an energy

$$E_{x,y} = (M_{x,y} + M_{i-x,j-y} - M_{i,j}) + V_{x,y} \tag{4.4}$$

will, in general, be unstable to decay via emission of a particle (x,y) (i.e., a particle with neutron number y and proton number x) , where $M_{x,y}$ is the mass of the particle, $M_{i,j}$ is the mass of the decaying nucleus (i,j), $M_{i-x,j-y}$ is the mass of the residue left over after decay and $V_{x,y}$ is the Coulomb barrier for that particle. Note that (x,y) could represent a variety of particles; in this note we will consider ''six'' such particles, as mentioned in the Introduction.

As is evident from Eq. (4.4), different particle decays have different energy thresholds. Consider an isotope (i,j),

as an example let us take ^{12}C ($i=6, j=6$). As we start from the ground-state level and move upwards, we will encounter different thresholds. The lowest will be the ^{4}He decay threshold at an energy $L_1 = E_{2,2}$ (in ^{12}C it is at 9.6 MeV approximately), the next higher threshold is for proton decay at $L_2 = E_{1,0}$ (in ^{12}C it is at 18.14 MeV approximately), and so on; we will get different thresholds one after the other (note: the order of different thresholds is different for different isotopes).

All nuclei of type (i,j) which are formed between the ground state and the lowest threshold L_1, will remain as isotopes (i,j), this number is given by

$$n_{i,j}(0\leftrightarrow L_1) = \int_0^{L_1} C_{i,j}\rho(E)e^{-\beta E}dE. \tag{4.5}$$

Those that are formed between L_1 and the next threshold L_2, will all completely decay by ^{4}He emission, and these nuclei will then appear as nuclei of type $(i-2, j-2)$ and must be added on to the population of isotopes $(i-2, j-2)$.

Then, those nuclei of type (i,j) which are formed between L_2 and the next threshold L_3, will decay both by ^{4}He emission and by proton emission. In the next zone there will be three kinds of decay, and so on. We now ask, how many of the initial nuclei formed in a particular zone will decay by each of the channels that are available, and how many of the residues formed will be stable or unstable?

To answer the above questions: we start by writing down the number of particles of type (x,y) with energy between $(\varepsilon, \varepsilon+d\varepsilon)$ that are emitted, in a time interval between t and $t+dt$, by nuclei of type (i,j), lying between an energy $(E, E+dE)$, leaving behind a residue nucleus $(i-x, j-y)$ [we may alternatively refer to (x,y), (i,j), and $(i-x, j-y)$ by simply their mass numbers a, A, and B where $a = x+y$, $A = i+j$, and $B = i-x+j-y$]

$$d^3N_a = W(E, \varepsilon)d\varepsilon\, dt\, dN(E,t), \tag{4.6}$$

where $dN(E,t)$ is the number of nuclei of type (i,j) initially formed at an energy $(E, E+dE)$ which are still left undecayed after a time t, given by

$$dN(E,t) = C_{i,j}e^{-\Gamma_T(E)t}\rho_A(E)e^{-\beta E}dE \tag{4.7}$$

and $W(E, \varepsilon)d\varepsilon$ is the Weisskopf decay probability per unit time [9] given by the expression

$$W(E, \varepsilon)d\varepsilon = g_a\gamma_a\varepsilon\sigma_{[a+B\to A]}\frac{\rho_B(E-B_a-\varepsilon)\rho_B^0}{\rho_A(E)\rho_A^0}. \tag{4.8}$$

In Eq. (4.7), $\Gamma_T(E)$ is the the total decay probability per unit time from an energy level E of the isotope A. In Eq. (4.8), g_a is the spin degeneracy factor of the emitted particle, γ_a is a constant of a particular decay [9,10], given by

$$\gamma_a = \frac{m_p}{\pi^2\hbar^3}\times\frac{a(A-a)}{A}, \tag{4.9}$$

where m_p is the mass of a nucleon. In Eq. (4.8), $B_a = M_{i-x,j-y} + M_{x,y} - M_{i,j}$, is the separation energy of the decay; $\sigma_{[a+B \to A]}$ is the cross section for the reverse reaction to occur, (i.e., $a+B \to A$). It is given semiclassically for uncharged particles as

$$\sigma_{[a+B \to A]} = \pi R_a^2 \qquad (4.10)$$

and for charged particles as

$$\sigma_{[a+B \to A]} = \pi R_a^2 \frac{\varepsilon - V}{\varepsilon} \theta(\varepsilon - V), \qquad (4.11)$$

where R_a is the radius associated with the geometrical cross section of the formation of A from B and a. Following the prescription of Friedmann and Lynch [10], R_a is given by

$$R_a = \begin{cases} [(A-a)^{1/3} + (a)^{1/3}]r_0, & \text{for } a \geq 2, \\ r_0(A-1)^{1/3}, & \text{for } a=1, \end{cases}$$

where $r_0 = 1.2$ fm. In Eq. (4.11), V is the Coulomb barrier for the formation of A from B and a. Again following [10], this is written in the touching sphere approximation as

$$V_a = \begin{cases} \dfrac{x(i-x)e^2}{((A-a)^{1/3} + (a)^{1/3})r_c}, & \text{for } a \geq 2, \\ \dfrac{(i-1)e^2}{r_c(A)^{1/3}}, & \text{for protons,} \end{cases}$$

where $r_c = 1.44$ fm. Also in Eq. (4.8), $\rho_A(E)$, $\rho_B(E - B_a - \varepsilon)$ are the respective density of states of the two nuclei. They have the same form as in Eq. (4.2). Also ρ_A^0 and ρ_B^0 are the respective multiplicative constants for the density of states, as mentioned in Sec. II.

We note that $d^3 N_a$ in Eq. (4.6) is also equal to the number of nuclei that were initially formed as nuclei of type A at an energy between E and $E + dE$, and then decayed into nuclei B with an excitation energy of $E - B_a - \varepsilon$. To get the total number of states that decayed from a level E by emission of a particle of any allowed energy, we integrate over ε from its minimum value V_a to its maximum value $E - B_a$, and get

$$d^2 N = \Gamma_a(E) C_{i,j} e^{-\Gamma_T(E)t} \rho_A(E) e^{-\beta E} dE dt, \qquad (4.12)$$

where

$$\Gamma_a(E) = \int_{V_a}^{E - B_a} W(E, \varepsilon) d\varepsilon, \qquad (4.13)$$

which on integration gives

$$\Gamma_a(E) = \frac{2\gamma_a' \rho_B^0}{\rho_A(E) \rho_A^0} \left[(E - B_a - V_a) \left(\frac{e^{C\mathcal{B}}}{C} (\mathcal{B} - 1/C) \right. \right.$$
$$\left. - \frac{e^{C\mathcal{A}}}{C} (\mathcal{A} - 1/C) \right) - \frac{e^{C\mathcal{B}}}{C} \left(\mathcal{B}^3 - \frac{3\mathcal{B}^2}{C} + \frac{6\mathcal{B}}{C^2} - \frac{6}{C^3} \right)$$
$$\left. + \frac{e^{C\mathcal{A}}}{C} \left(\mathcal{A}^3 - \frac{3\mathcal{A}^2}{C} + \frac{6\mathcal{A}}{C^2} - \frac{6}{C^3} \right) \right] \qquad (4.14)$$

(the derivation of the above equation is given in the Appendix), where $C = \pi[(i+j-x-y)/\epsilon_F]^{1/2}$, $\mathcal{B} = \sqrt{E - B_a - V_a}$, $\mathcal{A} = 0$. In the above equation g_a [Eq. (4.8)], and some of the factors of σ [Eq. (4.11)] have been absorbed into γ_a' thus

$$\gamma_a' = \gamma_a g_a \pi R_a^2. \qquad (4.15)$$

We may now integrate out the time to get

$$dN_a = \frac{\Gamma_a(E)}{\Gamma_T(E)} C_{i,j} \rho_A(E) e^{-\beta E} dE. \qquad (4.16)$$

To get the total number of states that have decayed from nuclei of type A by channel a we must integrate over E from L_1 to ∞,

$$N_a = \int_{L_1}^{\infty} \frac{\Gamma_a(E)}{\Gamma_T(E)} C_{i,j} \rho_A(E) e^{-\beta E} dE. \qquad (4.17)$$

This integration is quite involved for as we crossover from one zone of decay (L_1, L_2) to another zone (L_2, L_3), $\Gamma_T(E)$ changes discontinously as a new channel of decay becomes accessible to the nuclei. Thus we break up the integration into six zones, corresponding to the six real decay zones, and integrate within each zone independently. Note that the last zone extends from L_6 to $L_7 = \infty$, and is thus considerably larger than the other zones. However, at the low temperatures that will be encountered, this zone will be sparsely populated. Thus the following approximation is valid. Within each zone, with an energy from L_k to L_{k+1}, the integral can be replaced by a mean value expression

$$N_a(L_k, L_{k+1}) = \frac{L_k \langle \Gamma_a \rangle_{L_{k+1}}}{L_k \langle \Gamma_T \rangle_{L_{k+1}}} \Delta n_{i,j}(L_k, L_{k+1}), \qquad (4.18)$$

where $N_a(L_k, L_{k+1})$ is the mean number of nuclei of type (i,j) (or A) that were initially formed at an energy between L_k and L_{k+1}, and decayed by the (x,y) (or a) channel. In the above equation

$$L_k \langle \Gamma_a \rangle_{L_{k+1}} = \int_{L_k}^{L_{k+1}} \Gamma_a(E) C_{i,j} \rho_A(E) e^{-\beta E} dE. \qquad (4.19)$$

Of course, the left-hand side is zero if channel a is not open in the region L_k to L_{k+1}. The mean decay rate over all channels is

$$L_k \langle \Gamma_T \rangle_{L_{k+1}} = \int_{L_k}^{L_{k+1}} \Gamma_T(E) C_{i,j} \rho_A(E) e^{-\beta E} dE \qquad (4.20)$$

and

$$\Delta n_{i,j}(L_k, L_{k+1}) = \int_{L_k}^{L_{k+1}} C_{i,j}\rho_A(E)e^{-\beta E}dE. \quad (4.21)$$

Thus by summing up all the contributions from the six different zones, we get the total number of nuclei that have decayed from isotope A by the a channel as

$$N_a = \sum_{k=1}^{6} N_a(L_k, L_{k+1}). \quad (4.22)$$

To find out how many of these have decayed to stable isotopes, we must first calculate from Eq. (4.6) the stable decay rate $\Gamma_a^s(E)$. Two cases emerge in this calculation. If $E - B_a - V_a \geqslant E_{A-a}^s$, $\Gamma_a^s(E)$ is obtained by integrating over ε, from $(E - B_a - E_{A-a}^s)$ to its maximum value $(E - B_a)$, where E_{A-a}^s is the stable level or the lowest threshold L_1 of the residue nucleus B above which B is unstable. The expression for $\Gamma_a^s(E)$ is obtained from that of $\Gamma_a(E)$ in Eq. (4.14) by replacing $\mathcal{B} = \sqrt{E_{A-a}^s}$. If $E - B_a - V_a < E_{A-a}^s$, then $\Gamma_a^s(E) = \Gamma_a(E)$. Then, following a similar procedure as above for Γ_a, we get the total number of nuclei A [or (i,j)] lying in an energy range between (L_k, L_{k+1}), that decay by the a channel to a stable state as

$$N_a^s(L_k, L_{k+1}) = \frac{L_k \langle \Gamma_a^s \rangle_{L_{k+1}}}{L_k \langle \Gamma_T \rangle_{L_{k+1}}} \Delta n_{i,j}(L_k, L_{k+1}). \quad (4.23)$$

The unstable decay rate from a particular level or zone is the probability of a decay per unit time from A to an unstable level or levels of B from which further decay can take place. It is easy to see that they are given simply as the difference of the total decay rate and the stable decay rate, i.e.,

$$\Gamma_a^u = \Gamma_a - \Gamma_a^s. \quad (4.24)$$

The derivations and expressions for the full decay rates are given in the Appendix.

After a decay has taken place $(A \rightarrow B + a)$, we ask what is the population distribution of the residue as a function of its energy $(x = E - B_a - \varepsilon)$. This can, in principle, be calculated from Eq. (4.6) by integrating over E and ε, such that $(x = E - B_a - \varepsilon)$, the energy of the residue, is a constant. First we make a change of variables from (E, ε) to (E, x) and then integrate over E only. We get

$$dN_a(x)$$

$$= \left(\int_{B_a + V_a + x}^{\infty} dE\, \gamma_a' \frac{E - B_a - V_a - x}{\Gamma_T(E)} \rho_B(x) \frac{\rho_B^0}{\rho_A^0} C_{i,j}e^{-\beta E} \right)$$

$$\times dx. \quad (4.25)$$

This integration is quite involved. We assume that the residue population is canonically distributed, but with a new temperature $1/\beta'$, i.e.,

$$dN_a(x) = \mathcal{D}_{i,j \rightarrow k,l}\rho_B(x)e^{-\beta' x}. \quad (4.26)$$

There are two unknowns in this formula, the new temperature $1/\beta'$ and the overall normalization constant $\mathcal{D}_{i,j \rightarrow k,l}$. To find these two constants we will impose that the total population of this interim stage (i.e., $N_{i,j \rightarrow k,l}$), and the mean energy of the distribution $\langle x \rangle$, be reproduced by this new temperature.

We can obtain formal expressions for the total population of the residue B as contributed by the decay of A, as well as its mean energy $\langle x \rangle$, from Eq. (4.26) as

$$N_a(\beta', \mathcal{D}, C) = \int_0^{\infty} dN_a(x)$$

$$= \frac{\mathcal{D}_{i,j \rightarrow k,l}}{\beta'} \left[1 + C\sqrt{\frac{\pi}{4\beta'}} e^{C^2/4\beta'} \right.$$

$$\left. \times \left(1 - \mathrm{erf}\left(\frac{C}{2\sqrt{\beta'}}\right) \right) \right], \quad (4.27)$$

$$\langle x(\beta', C) \rangle = \frac{1}{N_a(\beta', \mathcal{D}, C)} \int_0^{\infty} x\, dN_a(x)$$

$$= \mathcal{D}_{i,j \rightarrow k,l} \left[\frac{1}{\beta'^2} + \frac{3\sqrt{\pi}C}{\beta'^{5/2}} \left\{ 1 + \mathrm{erf}\left(\frac{C}{2\sqrt{\beta'}}\right) \right\} e^{C^2/4\beta'} \right.$$

$$\left. + \frac{C^2}{4\beta'^3} + \frac{C^3\sqrt{\pi}}{8\beta'^{7/2}} \left\{ 1 + \mathrm{erf}\left(\frac{C}{2\sqrt{\beta'}}\right) \right\} e^{C^2/4\beta'} \right], \quad (4.28)$$

where the formal expression for $N_a(\beta', \mathcal{D}, C)$ is used in Eq. (4.28). The numerical value of N_a is taken from Eq. (4.22). The numerical value of $\langle x \rangle$ is found by explicit use of Eq. (4.25). From these two equations we obtain the two constants $\mathcal{D}_{i,j \rightarrow k,l}$ and β'.

The numerical value of $\langle x \rangle$ is derived from Eq. (4.25) as follows:

$$\langle x \rangle = \frac{1}{N_a} \int_0^{\infty} dx\, x \left(\int_{B_a + V_a + x}^{E_{\max}} dE\, \gamma_a' \frac{E - B_a - V_a - x}{\Gamma_T(E)} \right.$$

$$\left. \times \rho_B(x) \frac{\rho_B^0}{\rho_A^0} C_{i,j}e^{-\beta E} \right). \quad (4.29)$$

In the above equation, the numerical value of N_a is taken from Eq. (4.22). We may now change the order of integration to get

$$\langle x \rangle = \frac{1}{N_a} \int_{B_a + V_a}^{\infty} dE \int_0^{E - B_a - V_a} dx\, x\, \gamma_a' \frac{E - B_a - V_a - x}{\Gamma_T(E)}$$

$$\times \rho_B(x) \frac{\rho_B^0}{\rho_A^0} C_{i,j}e^{-\beta E}. \quad (4.30)$$

The x integration is now done simply to obtain

$$\langle x \rangle = \frac{1}{N_a} \int_{B_a+V_a}^{\infty} dE\, \gamma_a' \frac{I(E)}{\Gamma_T(E)} C_{i,j} \rho_A(E) e^{-\beta E}, \quad (4.31)$$

where $I(E)$ is given by

$$I(E) = \frac{1}{\rho_A(E)} \frac{\rho_B^0}{\rho_A^0} \left[\frac{4(E-B_a-V_a)^2 e^{C\sqrt{E-B_a-V_a}}}{C^2} \right.$$

$$- \frac{28(E-B_a-V_a)^{3/2} e^{C\sqrt{E-B_a-V_a}}}{C^3}$$

$$+ \frac{108(E-B_a-V_a) e^{C\sqrt{E-B_a-V_a}}}{C^4}$$

$$- \frac{240(E-B_a-V_a)^{1/2} e^{C\sqrt{E-B_a-V_a}}}{C^5}$$

$$\left. + \frac{240 e^{C\sqrt{E-B_a-V_a}}}{C^6} + \frac{12(E-B_a-V_a)}{C^4} - \frac{240}{C^6} \right].$$

$$(4.32)$$

In the ensuing integration over E, we, once again, replace the integral with its mean value expression:

$$\langle x \rangle = \frac{1}{N_a} \int_{B_a+V_a}^{\infty} dE\, \gamma_a' \frac{{}_{B_a+V_a}\langle I(E) \rangle_{\infty}}{{}_{B_a+V_a}\langle \Gamma_T(E) \rangle_{\infty}} C_{i,j} \rho_A(E) e^{-\beta E},$$

$$(4.33)$$

where

$${}_{B_a+V_a}\langle \Gamma_T(E) \rangle_{\infty} = \sum_{E_k > B_a+V_a} {}_{L_k}\langle \Gamma_T(E) \rangle_{L_{k+1}} \quad (4.34)$$

and

$${}_{B_a+V_a}\langle I(E) \rangle_{\infty} = \int_{B_a+V_a}^{\infty} dE\, I(E) C_{i,j} \rho_A(E) e^{-\beta E}$$

$$= C_{i,j} e^{-\beta G_a} \frac{\rho_B^0}{\rho_A^0} \left[\frac{1}{\beta^4} + \frac{3C\sqrt{\pi}}{4\beta^{9/2}} \right.$$

$$\times \left\{ 1 + \text{erf}\left(\frac{C}{2\sqrt{\beta}}\right) \right\} e^{C^2/4\beta}$$

$$+ \frac{C^2}{4\beta^5} + \frac{C^3\sqrt{\pi}}{8\beta^{11/2}}$$

$$\left. \times \left\{ 1 + \text{erf}\left(\frac{C}{2\sqrt{\beta}}\right) \right\} e^{C^2/4\beta} \right]. \quad (4.35)$$

Thus the formal expressions for $N_a(\beta', \mathcal{D}, C)$ [Eq. (4.27)], and $\langle x(\beta', C) \rangle$ [Eq. (4.28)], are compared to the actual values obtained for N_a [Eq. (4.22)], and $\langle x \rangle$ [Eq. (4.33)], and the two unknowns of Eq. (4.26) are evaluated. We can now proceed with further decays following the same

procedure as before with decay occurring from a canonically distributed population at a temperature $1/\beta'$.

We can thus model an n-step decay process by assuming that at each intermediate stage the population is canonically distributed with a new temperature and overall normalization constant. The decay rates to the next stage are calculated with the new temperature. Following this, the fraction of the population that decays through a particular channel, and the mean energy of the resultant residue nucleus, are calculated. These are then used to secure the temperature and normalization constant of the next stage of decay. This process will continue till the fraction of decay to particle unstable states becomes negligible.

V. THE CALCULATION

From the primary calculation we obtain that $\langle n_{i,j} \rangle$ nuclei of type (i,j) (or A) are formed from the initial multifragmentation. The population $\langle n_{i,j} \rangle$ is distributed canonically among the various energy levels as demonstrated by Eq. (4.1). If a particular nucleus is at a sufficiently excited state then it will emit a particle (x,y) (or a) and leave a residue $(i-x=k, j-y=l)$ (or B), which may again decay by emitting a particle (u,v) (or b) leaving a nucleus $(k-u=m, l-v=n)$ (or D), and so on until it finally reaches a nucleus (p,q) (or Z) in a stable state. We ask the question that if $\langle n_{i,j} \rangle$ nuclei of type A were intially formed, then how many of these will finally end up as stable nuclei of type A, B, $D \ldots Z$. The contribution of $\langle n_{i,j} \rangle$ to the final stable population of A is given simply by Eq. (4.5) as

$$n_A^f = \int_0^{L_1} C_{i,j} \rho_A(E) e^{-\beta E} dE = \Delta n_{i,j}(0, L_1). \quad (5.1)$$

The number of nuclei initially formed as (i,j) which decay to $(i-x, j-y) = (k,l)$ is given as

$$n_{A \to B} = \sum_{k=1}^{6} \frac{{}_{L_k}\langle \Gamma_a \rangle_{L_{k+1}}}{{}_{L_k}\langle \Gamma_T \rangle_{L_{k+1}}} \Delta n_{i,j}(L_k, L_{k+1}). \quad (5.2)$$

The mean energy of the newly formed residue nucleus is given by

$$\langle x \rangle = \frac{1}{n_{A \to B}} \gamma_a' \frac{{}_{B_a+V_a}\langle I(E) \rangle_{\infty}}{{}_{B_a+V_a}\langle \Gamma_T(E) \rangle_{\infty}} \Delta n_{i,j}(B_a+V_a, \infty).$$

$$(5.3)$$

We assume that this population is canonically distributed from an excitation energy of $E_0 = 0$ to ∞ with a new temperature $1/\beta'$ [Eq. 4.26]. Extraction of the new temperature $1/\beta'$ and the overall normalization constant $\mathcal{D}_{i,j \to k,l}$ is done as detailed in Sec. IV. In most cases, where this procedure was implemented, we obtained a new temperature $1/\beta'$ which was lower than the initial temperature $1/\beta$; however, in about 3% of the cases $1/\beta'$ turned out to be higher than $1/\beta$; this occurs when the residue of the decay process is far from the valley of stability. We can then calculate the number of nuclei that initially started out as A's and finally ended up as "stable" B's as

$$n_{A \to B}^f = \int_0^{L_1} \mathcal{D}_{i,j \to k,l} \rho_B(x) e^{-\beta' x} dx. \tag{5.4}$$

Note that in the above equation $\rho_B(x)$ and L_1 are the density of states and lowest decay threshold for the nucleus of type $(i - x = k, j - y = l)$. This number can also be calculated directly by using the stable decay rates [Eq. (4.23)]:

$$n_{A \to B}^f = \sum_{k=1}^{6} \frac{L_k \langle \Gamma_a^s \rangle_{L_{k+1}}}{L_k \langle \Gamma_T \rangle_{L_{k+1}}} \Delta n_{i,j}(L_k, L_{k+1}). \tag{5.5}$$

The second equation is more correct as it does not depend on the assumption that the residue is canonically distributed. A comparison of the $n_{A \to B}^f$ obtained from the above two equations gives an estimate of the error involved in the assumption of a canonically distributed residue population. Now we ask, what is the number of nuclei of the B's just formed which will decay by emitting a particle b to a nucleus of type D; this is calculated simply as

$$n_{A \to B \to D} = \sum_{k=1}^{L_6} \frac{L_k \langle \Gamma_b \rangle_{L_{k+1}}}{L_k \langle \Gamma_T \rangle_{L_{k+1}}} \Delta n_{A \to B}(L_k, L_{k+1}). \tag{5.6}$$

The decay rates in the above equation are calculated with the temperature $1/\beta'$. We then calculate the mean energy $\langle y \rangle$ of the new distribution as

$$\langle y \rangle = \frac{1}{n_{A \to B \to D}} \gamma_b' \frac{B_b + V_b \langle I(x) \rangle_\infty}{B_b + V_b \langle \Gamma_T(x) \rangle_\infty} \Delta n_{A \to B}(B_b + V_b, \infty). \tag{5.7}$$

Using these, we continue the process, by again calculating the temperature and norm of a canonical distribution, which when summed from excitation energy 0 to ∞ is equal to $n_{A \to B \to D}$, and whose mean energy is equal to $\langle y \rangle$. We can then proceed to find how many of these will be in stable states, how many will decay on further, etc. We continue this process till the contribution from this decay chain, $A \to B \to D \to \ldots$, will give numbers of nuclei negligible compared to the already present number in stable states.

VI. RESULTS OF THE CALCULATION

Our objective is to calculate the yields of the boron, carbon, and nitrogen isotopes measured in the S + Ag heavy-ion collision at an energy of 22.3A MeV [1]. In Figs. 1 to 5 the data are shown as empty squares. The method of calculation is simple, first we calculate the primary populations of the isotopes using Eq. (2.10). We then remove the unstable fraction of the population, and quote only the stable part. This is denoted by the dotted line and triangle plotting symbol. We then incorporate secondary decay by adding on all the populations of nuclei that can reach a stable level of the isotopes by emitting only one of the six particles considered. We call these the up to single decay populations and denote them by the small dashed line and diamond plotting symbol. We then add on all those unstable nuclei which can reach a stable level of the given isotopes by sequentially emitting
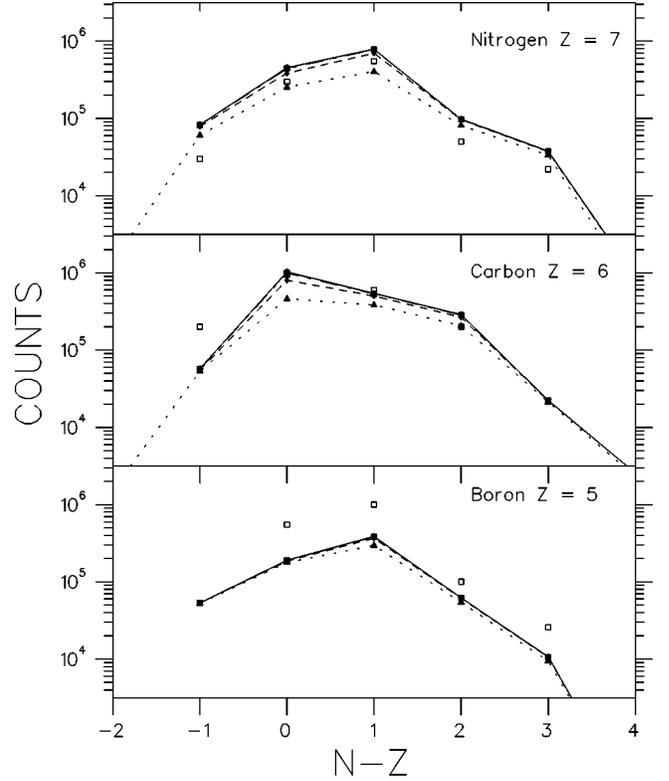
FIG. 1. Neutron number (N), proton number (Z) vs *counts* for the three cases of boron, carbon, and nitrogen. The experimental data are from [1] S+Ag at 22.3A MeV. The fits show varying stages of decay for a total $Z_T = 50$, $A_T = 110$, $T = 3.0$ MeV, $V_{fr}/V_0 = 3.0$, and $\log_{10} \mathcal{H} = 6.82$. The empty squares are the experimental data. The dotted line with the triangle plotting symbol is the primary calculation. The small dashed line with diamond plotting symbol is the up to single decay calculation. The dot-dashed line with square plotting symbol is the up to double decay calculation. The dashed line with the star plotting symbol is the up to triple decay calculation. The solid line with circle plotting symbol is the up to quadruple decay calculation.

any two particles of the six considered. We call these the up to double decay populations and denote them by the dot-dashed line and square plotting symbol. We then add on all those that can reach the isotopes by three particle emissions, called the up to triple decay population and denoted by the large dashed line and star plotting symbol. And finally we add on the up to quadruple decay population denoted by the solid line and circle plotting symbol. As there is negligible difference between up to triple decay and up to quadruple decay we stop after quadruple decay.

To fit with experimental data, we have four parameters to tune, the obvious ones being the initial temperature β or T, the free volume V_f of the primary calculation, the ratio A/Z (as one does not know how much loss due to pre-equilibrium emission has taken place), and an overall multiplicative constant $\mathcal{H}$ (as we do not know how many nuclei collided in the experiment). The plots are noted to be most sensitive to β and A/Z. Thus in fitting the data we first set particular values of β and A/Z, and calculate the multiplicities at all stages of decay (V_f is varied to get the best possible fit at this tem-
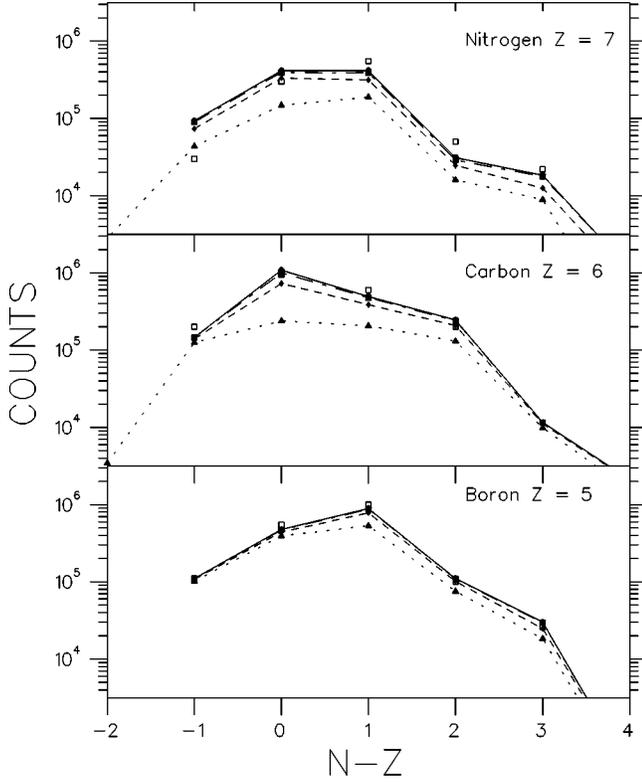
FIG. 2. Same as Fig. 1 but with $Z_T=50$, $A_T=110$, $T=5.0$ MeV, $V_{fr}/V_0=5.5$, and $\log_{10}\mathcal{H}=6.34$. The best fit with the data has been obtained with these parameters.
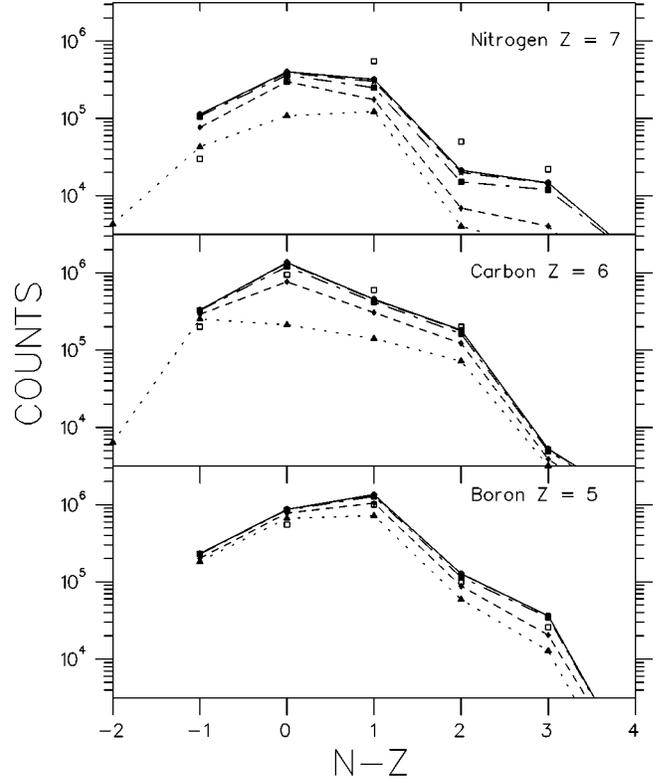
FIG. 3. Same as Fig. 1 but with $Z_T=50$, $A_T=110$, $T=7.0$ MeV, $V_{fr}/V_0=3.0$, and $\log_{10}\mathcal{H}=6.82$.

perature and A/Z). We then multiply all the multiplicities by an appropriate $\mathcal{H}$ and take the logarithm. These are then plotted and compared with the *counts* obtained from the experiment. We then vary β, A/Z, and repeat the above procedure till a good fit is obtained. We present fits for three different temperatures, and different A/Z for each temperature. V_f and $\mathcal{H}$ are set to obtain the best fit possible for a given β and A/Z.

We note that the S+Ag system is one with $A=139$ and $Z=63$ thus $A/Z=2.2$. The authors of [1] state that some pre-equilibrium emission may have taken place. As we do not know what proportion of neutrons and protons are lost in such a process, we start the calculation with the same A/Z as the S+Ag system. We start with a $Z=50$ and $A/Z=2.2$, i.e., $A=110$. We start the calculation with a low temperature of 3 MeV in Fig. 1 (V_f and $\mathcal{H}$ are varied to get the best fit). We note that overall there is a slight excess of the heavier nitrogen isotopes as compared to data and a deficit of the lighter boron isotopes, this implies that the temperature is too low and enough of the light isotopes are not being formed. We proceed by raising the temperature to 5 MeV, maintaining the same A/Z. By now varying V_f and $\mathcal{H}$ we find an excellent fit with the data (Fig. 2).

One may ask, at this point, if there are more than one set of parameters which fit the data well. To answer this question we increase, first, the temperature to 7 MeV, maintain the same A/Z and redo the calculation. We get a bad fit (Fig. 3). There is an overall deficit in the nitrogen population and an excess in the boron population. Also we note that within a
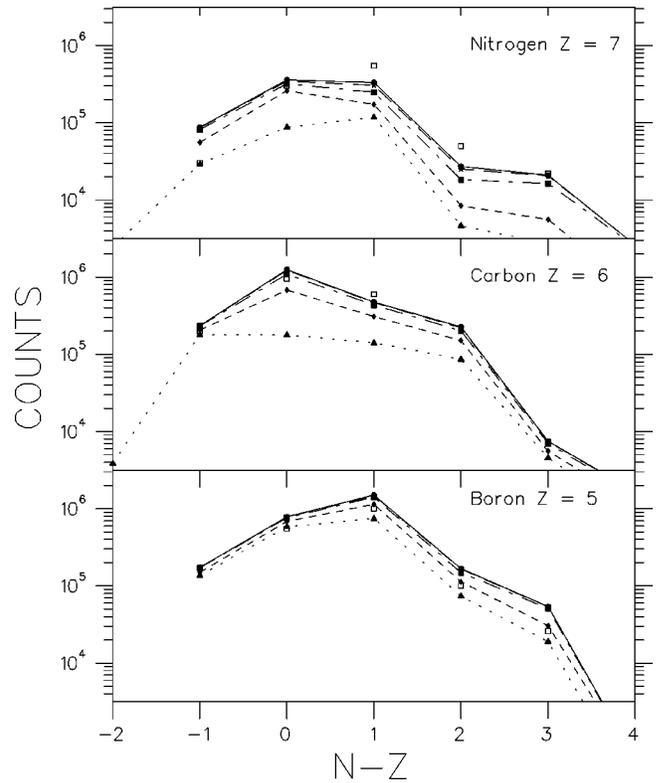
FIG. 4. Same as Fig. 1 but with $Z_T=50$, $A_T=115$, $T=7.0$ MeV, $V_{fr}/V_0=3.0$, and $\log_{10}\mathcal{H}=6.82$.
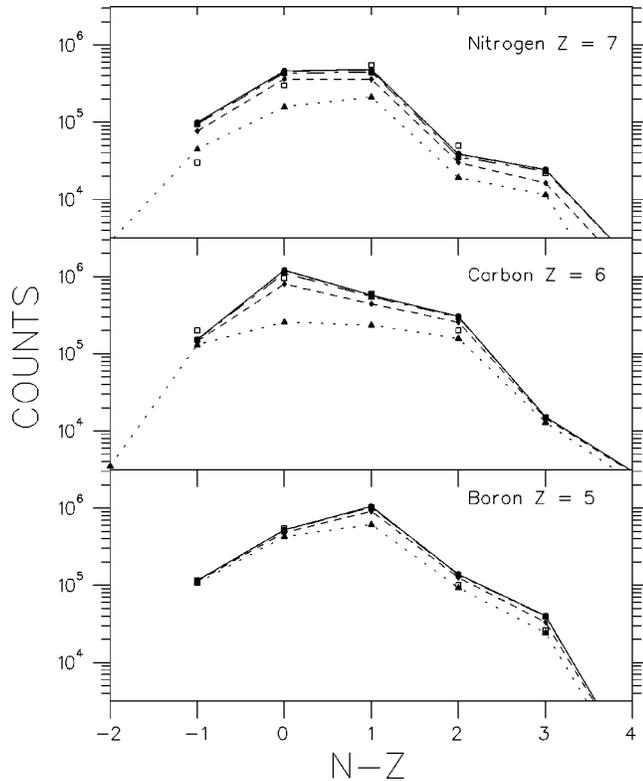
FIG. 5. Same as Fig. 2 but with $Z_T = 63$, $A_T = 140$, $T = 5.0$ MeV, $V_{fr}/V_0 = 5.5$, and $\log_{10} \mathcal{H} = 6.30$.

particular Z there is a deficit in the neutron-rich isotopes. We try to remedy this situation by increasing the A/Z ratio. The best fit at this temperature is obtained at an $A/Z = 2.3$ (Fig. 4), but we still obtain an overal deficit in the nitrogen population; the carbon fit is good, but there still remains an excess in the boron isotopes especially in the neutron-rich isotopes.

On inspection of the fits (Figs. 1–4), we note that the best fit is obtained at Fig. 2. In this fit $T = 5.0$ MeV, $A/Z = 2.2$, $V_{fr}/V_0 = 5.5$, and $\log_{10}(\mathcal{H}) = 6.34$. In this figure we note that, for the boron populations we get an excellent agreement with the data. In this case there seems to be little change after single decay. For the carbon isotopes the agreement is good. For nitrogen, we have a good fit except for the case of ^{13}N.

Another property of the fits noticed is that they do not seem to depend on A and Z independently but rather on the ratio A/Z. As a demonstration of this, we plot in Fig. 5 a fit for $A = 140$ and $Z = 63$ (i.e., $A/Z = 2.22$). We note that we are able to obtain a fit very similar to Fig. 2, with the same temperature and V_{fr}/V_0 as in Fig. 2, but with a slightly lower $\mathcal{H}$. This is very much expected, as in this case each source has a larger number of nucleons than before.

VII. DISCUSSONS AND CONCLUSION

In this paper we have presented a secondary decay formalism and performed calculations to fit the populations of various isotopes measured in [1]. We obtain very good fits (Fig. 2) with experiment for the boron and carbon isotopes. In the nitrogen isotopes, we obtain a good fit except for the

case of ^{13}N. No particular reason could be found for this, but let us go over several approximations (introduced to keep the calculation at a resonably simple level) which may have contributed.

Actual energy levels from data tables were used only up to $A = 20$ [Eq. (2.4)] for the primary populations. For higher masses, the emperical mass formula [Eq. (2.9)] was used. The secondary decay is very approximate, instead of calculating decay level to level, we have blurred out such details by using a smoothed level density.

For the capture cross section [Eq. (4.11)], we have used a simple semiclassical formula, assuming that all nuclei are spherically symmetric which is definitely not true. A more precise calculation involving level to level decay would use a more accurate expression for the cross sections, e.g., the Hauser-Feshbach formalism [11,12].

Still another problem lies in the assumption made in calculating the effects of higher-order decay, that the interim populations can be taken to be canonically distributed. This is true only in first-order decay, thus making the higher-order contributions subject to some error.

There is also an experimental problem according to the authors of [1], the angular distributions were forward peaked, indicating significant emission prior to attainment of thermodynamic equilibrium. Such an emission could affect the populations of the various isotopes.

No doubt, incorporating changes to correct the above-mentioned problems will improove the accuracy of the calculation. However, such changes may make the expressions analytically intractible and one would have to resort to numerical means. This may slow down the calculation considerably. The calculations presented in this note take minimal computer time. Inspite of the shortcomings of the calculation presented above, this still remains a good test of the two-component statistical model, and shows that such a model can definintely be used to explain certain experimental data quite accurately.

APPENDIX

1. Derivation of Eq. (4.14)

The full decay rate from a particular energy level E of a nucleus A [or (i,j)], which is decaying by emitting a particle a [or (x,y)], is given as [Eq. (4.13)]

$$\Gamma_a = \int_{V_a}^{E-B_a} W(E, \varepsilon) d\varepsilon, \tag{A1}$$

where $W(E, \varepsilon)$ is the Weisskopf decay probability per unit time given by Eq. (4.8). On writing down the full expression for W we get

$$\Gamma_a = \int_{V_a}^{E-B_a} \frac{\gamma_a' \rho_B^0}{\rho_A(E)\rho_A^0}(\varepsilon - V_a)$$

$$\times \exp\left[\frac{\pi}{\sqrt{\epsilon_F}}\{\sqrt{(i+j-x-y)(E-B_a-\varepsilon)}\}\right]d\varepsilon. \tag{A2}$$

Now we substitute $z = (\sqrt{E-B_a-\varepsilon})$ and integrate over z and let $C = (\pi/\sqrt{\epsilon_F}\sqrt{i+j-x-y})$, then

$$\Gamma_a(E) = \int_0^{\sqrt{E-B_a-V_a}} \frac{\gamma_a' \rho_B^0}{\rho_A(E)\rho_A^0} 2z(E-B_a-V_a-z^2)$$

$$\times \exp(Cz)dz. \tag{A3}$$

On carrying out this simple integration we get

$$\Gamma_a(E) = \frac{2\gamma_a' \rho_B^0}{\rho_A(E)\rho_A^0}\left[(E-B_a-V_a)\left(\frac{e^{C\mathcal{B}}}{C}(\mathcal{B}-1/C)\right.\right.$$

$$\left. - \frac{e^{C\mathcal{A}}}{C}(\mathcal{A}-1/C)\right) - \frac{e^{C\mathcal{B}}}{C}\left(\mathcal{B}^3 - \frac{3\mathcal{B}^2}{C} + \frac{6\mathcal{B}}{C^2} - \frac{6}{C^3}\right)$$

$$\left. + \frac{e^{C\mathcal{A}}}{C}\left(\mathcal{A}^3 - \frac{3\mathcal{A}^2}{C} + \frac{6\mathcal{A}}{C^2} - \frac{6}{C^3}\right)\right] \tag{A4}$$

with $\mathcal{B} = \sqrt{E-B_a-V_a}$, $\mathcal{A} = 0$. The stable decay rate, i.e., the decay rate from an energy level E of the nucleus A to any of the allowed stable levels of B is given simply from the above expression by replacing the upper limit to $\mathcal{B} = \sqrt{E_B^s}$ where E_B^s is the stable threshold of the residue nucleus B. However, in the event that $E-B_a-V_a \lesssim E_B^s$ then the above mentioned replacement should not be made. In this case the total decay rate is the same as the stable decay rate.

2. Derivation of the generic expression for $_{L_k}\langle\Gamma_a\rangle_{L_{k+1}}$

From Eq. (4.19) we obtain the definition of $_{L_k}\langle\Gamma_a\rangle_{L_{k+1}}$ as

$$_{L_k}\langle\Gamma_a\rangle_{L_{k+1}} = \int_{L_k}^{L_{k+1}} \Gamma_a(E)C_{i,j}\rho(E)e^{-\beta E}dE. \tag{A5}$$

Now taking the expression of $\Gamma_a(E)$ from Eq. (A4) and substituting $z = \mathcal{B} = (\sqrt{E-B_a-V_a})$ we get

$$_{L_k}\langle\Gamma_a\rangle_{L_{k+1}} = \int_{\sqrt{L_k-B_a-V_a}}^{\sqrt{L_{k+1}-B_a-V_a}} 4\gamma_a' C_{i,j}$$

$$\times \frac{\rho_B^0}{\rho_A^0}\left[\left\{\frac{2z^3}{C^2} - \frac{6z^2}{C^3} + \frac{6z}{C^4}\right\}e^{Cz}\right.$$

$$\left. + \frac{z^3}{C^2} - \frac{6z}{C^4}\right]e^{-\beta(z^2+B_a+V_a)}dz. \tag{A6}$$

Now we may separate the integration into three parts

$$_{L_k}\langle\Gamma_a\rangle_{L_{k+1}} = (I^1_{L_k,L_{k+1}} + I^2_{L_k,L_{k+1}} + I^3_{L_k,L_{k+1}})\frac{\rho_B^0}{\rho_A^0}, \tag{A7}$$

where

$$I^1_{L_k,L_{k+1}} = \int_{\sqrt{L_k-B_a-V_a}}^{\sqrt{L_{k+1}-B_a-V_a}} -C_{i,j}24\gamma_a'\frac{z}{C^4}$$

$$\times e^{-\beta(z^2+B_a+V_a)}dz, \tag{A8}$$

$$I^2_{L_k,L_{k+1}} = \int_{\sqrt{L_k-B_a-V_a}}^{\sqrt{L_{k+1}-B_a-V_a}} C_{i,j}4\gamma_a'\frac{z^3}{C^4}$$

$$\times e^{-\beta(z^2+B_a+V_a)}dz, \tag{A9}$$

$$I^3_{L_k,L_{k+1}} = \int_{\sqrt{L_k-B_a-V_a}}^{\sqrt{L_{k+1}-B_a-V_a}} 2\gamma_a'C_{i,j}$$

$$\times \left\{\frac{4z^3}{C^2} - \frac{12z^2}{C^3} + \frac{12z}{C^4}\right\}e^{Cz-\beta(z^2+B_a+V_a)}dz. \tag{A10}$$

The three integrals can be done simply to give

$$I^1_{L_k,L_{k+1}} = \frac{-12\gamma_a'C_{i,j}}{C^4\beta}[e^{-\beta L_k} - e^{-\beta L_{k+1}}], \tag{A11}$$

$$I^2_{L_k,L_{k+1}} = \frac{2\gamma_a'C_{i,j}}{C^2\beta}[L_k e^{-\beta L_k} - L_{k+1}e^{-\beta L_{k+1}}$$

$$+ \{1/\beta - B_a - V_a\}(e^{-\beta L_k} - e^{-\beta L_{k+1}})], \tag{A12}$$

$$I^3_{L_k,L_{k+1}} = \frac{4\gamma_a'C_{i,j}}{C^2\beta}e^{-\beta(B_a+V_a)}e^{C^2/4\beta^2}\left[M_k^2 e^{-\beta M_k^2}\right.$$

$$- M_{k+1}^2 e^{-\beta M_{k+1}^2} + e^{-\beta M_k^2} - e^{-\beta M_{k+1}^2}$$

$$+ \left(\frac{3C^2}{2\beta} - \frac{3}{C}\right)\{M_k e^{-\beta M_k^2} - M_{k+1}e^{-\beta M_{k+1}^2}\}$$

$$+ \left(\frac{3C^2}{4\beta^2} - \frac{3}{\beta} + \frac{3}{C^2}\right)\{e^{-\beta M_k^2} - e^{-\beta M_{k+1}^2}\}$$

$$\left. + \frac{C^3\sqrt{\pi}}{8\beta^2}\{\text{erf}(\sqrt{\beta}M_{k+1}) - \text{erf}(\sqrt{\beta}M_k)\}\right], \tag{A13}$$

where $M_k = \sqrt{L_k-B_a-V_a} - C/(2\beta)$ and $M_{k+1} = \sqrt{L_{k+1}-B_a-V_a} - C/(2\beta)$ and C is the same as in Eq. (A4).

The calculation of the stable decay rate is a bit more involved in the limits of integration and three cases emerge. If $L_k - B_a - V_a < E_B^s$, and $L_{k+1} - B_a - V_a \leq E_B^s$, then

$$_{L_k}\langle\Gamma_a^s\rangle_{L_{k+1}} = {}_{L_k}\langle\Gamma_a\rangle_{L_{k+1}}. \tag{A14}$$

If $L_k - B_a - V_a < E_B^s$, but $L_{k+1} - B_a - V_a > E_B^s$, then the calculation of $_{L_k}\langle\Gamma_a^s\rangle_{L_{k+1}}$ has to be done in two parts

$$_{L_k}\langle\Gamma_a^s\rangle_{L_{k+1}} = (I_1^s + I_2^s)\frac{\rho_B^0}{\rho_A^0}, \tag{A15}$$

where

$$I_1^s = \int_{L_k}^{E_B^s + B_a + V_a} \Gamma_a(E) C_{i,j} \rho(E) e^{-\beta E} dE. \tag{A16}$$

The expression for this is the same as Eq. (A6) with the appropriate change of limits:

$$I_2^s = \int_{E_B^s + B_a + V_a}^{L_{k+1}} \Gamma_a^s(E) C_{i,j} \rho(E) e^{-\beta E} dE. \tag{A17}$$

If, however, $L_k - B_a - V_a \geq E_B^s$, and $L_{k+1} - B_a - V_a > E_B^s$, then

$$_{L_k}\langle\Gamma_a^s\rangle_{L_{k+1}} = \int_{L_k}^{L_{k+1}} \Gamma_a^s(E) C_{i,j} \rho(E) e^{-\beta E} dE. \tag{A18}$$

The above two integrals are rather trivial and thus detailed expressions are not presented.

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