Phase-equivalent energy-dependent potentials

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(Received 13 August 1999; published 4 January 2000)

Supersymmetric transformations, i.e., factorizations of the radial Schrödinger equation, are generalized to potentials with a linear dependence on energy. These transformations are equivalent to generalized Darboux transformations and do not affect the energy dependence of the potential. A pair of transformations allows the removal of a bound state without modification of the phase shift. The method is applied to the removal of the forbidden bound states of a deep energy-dependent $\alpha+{}^{16}\mathrm{O}$ potential generating a phase-equivalent shallow potential with an r^{-2} repulsive core at the origin and the same energy dependence as the initial potential.

PACS number(s): 24.10.Ht, 03.65.Nk, 25.70.Bc

I. INTRODUCTION

Local central potentials provide a simple description of the interaction between two particles and of their collisions [1]. This description can also be applied to composite-particle systems, e.g., in nuclear physics. The corresponding interactions are complex and depend on energy and on angular momentum [2–4] thus taking into account approximately the characteristic features of the many-body system. Specifically, the energy and angular-momentum dependences of these so-called optical potentials reflect the reduction of the microscopic many-body Hamiltonian to a one-body operator and are inherent features of this simplified model. Because of its simplicity the optical model has become an important tool in nuclear physics and there has been considerable effort for its determination.

The extraction of optical potentials from elastic scattering cross sections suffers from ambiguities. The problem is twofold. First, the *phase problem* which denotes the fact that the scattering cross section does not contain information on the phase of the scattering amplitude necessary for a unique determination of the potential [5]. Second, even if the phase of the scattering amplitude is known there is still freedom in the discrete spectrum, in particular in the number of bound states. This phenomenon is known as the *discrete ambiguity* [6] which is characterized by a set of (almost) *phase-equivalent* potentials sustaining different numbers of bound states.

Microscopic and semimicroscopic calculations of nucleus-nucleus interactions [7] suggest *deep* potentials [8,9], which is confirmed by the evaluation of simple deep phenomenological potentials, e.g., for the $\alpha+\alpha$ [10], ${}^{16}\mathrm{O}+{}^{16}\mathrm{O}$ [11], and $\alpha+$nucleus [12] systems. An important prediction of the microscopic theory is the number of bound states sustained by these potentials. Among these bound states the lowest ones correspond to relative motions which are suppressed in the microscopic model because of antisymmetrization. They are called Pauli forbidden states [13] and do not correspond to physical bound states of the many-particle system. These forbidden bound states have no influence on the on-shell properties of the two-body interaction but may influence its off-shell properties. For instance, the $\alpha+\alpha$ bremsstrahlung cross sections are very sensitive to the presence of forbidden bound states in the $\alpha+\alpha$ potential used in the calculation [14]. Removing the forbidden states from a deep potential thus allows interesting comparisons between off-shell properties of phase-equivalent potentials. In three- and many-body problems, these states have even stronger effects: they lead to spurious bound states which make the interpretation of the many-body spectra difficult [15]. Removing the forbidden bound states from the two-body interactions before performing the many-body calculations is thus very advantageous since it greatly simplifies the interpretation of the obtained spectra. For instance, this technique is applied in Ref. [16] in three-body calculations of halo nuclei. Hence, although deep potentials are physically well founded, it is useful to eliminate their bound states corresponding to Pauli forbidden states in different contexts.

The formalism of supersymmetric quantum mechanics [17,18] provides an elegant algebraic technique for the elimination of bound states. This technique is equivalent to the factorization method [19] and to Darboux transformations [20,21] of the Schrödinger equation. It was shown in Ref. [22] that *two successive* supersymmetric transformations allow the elimination of the ground state of a given Hamiltonian without modifying its scattering behavior. Actually, this phase-equivalent bound-state elimination is not limited to the ground state [23]; moreover, additions and modifications of bound states are also possible within the supersymmetric formalism [24,25]. The application of the phase-equivalent bound-state removal to the forbidden states of a deep potential leads to a shallow potential which has the same scattering properties but sustains only physical bound states. The transformed potential exhibits an r^{-2} singularity at the origin and is in general l dependent.

A few years ago, the supersymmetric formalism was only available for real single-channel energy-independent potentials. Since more complicated potentials are used very frequently, particularly in nuclear physics, we are trying to generalize this formalism to wider categories of potentials. Indeed, generalizations to complex (optical) potentials [26,27] and to coupled-channel systems [28] have been presented. As outlined above an important issue in nuclear physics is the removal of unphysical bound states of Hamil-

tonians with energy-dependent potentials. This problem has been considered first by Fiedeldey *et al.* [29] who constructed via inverse scattering techniques for each partial wave an energy-independent potential from the scattering phase shifts generated by the original Hamiltonian; this provides a deep energy-independent potential for which the standard supersymmetric transformation for the elimination of forbidden states can be applied. The final shallow potential is energy-independent but strongly l dependent. An alternative procedure for the removal of bound states in systems with energy-dependent potentials was given in Ref. [27]. There one considers the energy-dependent potential at a given energy as an energy-independent one and applies the standard procedure for the removal of bound states. This procedure has the drawback that the bound-state removal has to be performed at each energy. The final shallow potential depends on both the energy and the angular momentum.

In this article we present a new algebraic method to perform a phase-equivalent removal of bound states from a system with linearly energy-dependent potentials. Such linear energy dependences are very frequent in nuclear physics, particularly for complex potentials, the imaginary part of which increases with energy to simulate the opening of new channels. Unlike in preceding methods, we neither use inversion techniques nor intermediate energy-independent potential: the method is as direct and as elegant as for the simple case of energy-independent potentials. In particular, the final potential is expressed in closed analytical form directly from the initial potential and has the same energy dependence. The method has been suggested by Refs. [30,31], where Darboux transformations for specific Sturm-Liouville equations are presented. Considering this problem in terms of the formalism of supersymmetric quantum mechanics allowed us to develop a direct method for the removal of bound states and the construction of phase-equivalent potentials.

In Sec. II, we generalize the supersymmetric formalism to the case of linearly energy-dependent potentials. In Sec. III, we recall the principle of phase-equivalent bound-state removal with a pair of supersymmetric transformations and establish the corresponding formulas in the case of linearly energy-dependent potentials. In Sec. IV, we apply this formalism to the removal of forbidden bound states from the real part of the deep $\alpha+{}^{16}\mathrm{O}$ potential of Ref. [32] for $l=0$ and compare the result with that obtained by transforming the potential at fixed energies. Our conclusions are presented in Sec. V.

II. GENERALIZED SUPERSYMMETRIC FORMALISM

In Refs. [30,31] (see also Refs. [33]), Darboux transformations for a Sturm-Liouville equation were presented. These include also the case of the radial Schrödinger equation

$$\left[-\frac{d^2}{dr^2}+V_0(E,r)\right]\Psi_0(E,r)=E\Psi_0(E,r), \tag{1}$$

with a linearly energy-dependent potential

$$V_0(E,r)=V_0(0,r)+EV_E(r), \tag{2}$$

where E is the center-of-mass energy (we choose $\hbar=2\mu=1$, where μ is the reduced mass of the colliding particles).

It has been known for a long time [20] that a Darboux transformation can be formulated in terms of a supersymmetric transformation, i.e., as a factorization of the Schrödinger equation. Here, we generalize this result to radial Schrödinger equations with a linearly energy-dependent potential [Eq. (2)]. For this purpose, we introduce $t(r)$ by

$$V_E(r)=1-\frac{1}{t^2(r)}. \tag{3}$$

It can be verified that all the formulas below are independent of the sign of $t(r)$ and that its possible singularities do not create any problem. Multiplying Eq. (1) on the left with $t(r)$ leads to

$$\tilde{D}_0\tilde{\Psi}_0(E,r)=E\tilde{\Psi}_0(E,r), \tag{4}$$

with

$$\tilde{D}_0=t(r)\left[-\frac{d^2}{dr^2}+V_0(0,r)\right]t(r) \tag{5}$$

and

$$\tilde{\Psi}_0(E,r)=\frac{\Psi_0(E,r)}{t(r)}. \tag{6}$$

It should be remarked that the modified wave function $\tilde{\Psi}_0(E,r)$ is well defined if we assume $V_E(r)$ to be bounded in the whole domain $(0\leqslant r<\infty)$.

The new differential operator $\tilde{D}_0$ does not depend on energy [compare with Eq. (1)]. It can be factorized in the form

$$\tilde{D}_0=A_0^+A_0^-+E_0, \tag{7}$$

where E_0 is called the factorization energy. The mutually adjoint operators A_0^+ and A_0^- are given by

$$A_0^-=\left[-\frac{d}{dr}+U_0(r)\right]t(r) \tag{8}$$

and

$$A_0^+=(A_0^-)^\dagger=t(r)\left[\frac{d}{dr}+U_0(r)\right], \tag{9}$$

where

$$U_0(r)=\Sigma_0(E_0,r)^{-1}\frac{d}{dr}\Sigma_0(E_0,r) \tag{10}$$

is the so-called superpotential. It can be verified that Eq. (7) is satisfied when the function $\Sigma_0(E_0,r)$ is a solution of the Schrödinger equation (1) at energy E_0, i.e., when it satisfies

$$\left[-\frac{d^2}{dr^2}+V_0(E_0,r)\right]\Sigma_0(E_0,r)=E_0\Sigma_0(E_0,r), \quad (11)$$

where the potential is calculated at the factorization energy. The solution $\Sigma_0(E_0,r)$ is called the factorization solution.

Applying operator A_0^- to Eq. (4) leads to a new equation

$$\tilde{D}_1\tilde{\Psi}_1(E,r)\equiv[A_0^-A_0^++E_0]\tilde{\Psi}_1(E,r)=E\tilde{\Psi}_1(E,r), \quad (12)$$

with

$$\tilde{\Psi}_1(E,r)=A_0^-\tilde{\Psi}_0(E,r). \quad (13)$$

Dividing this equation on the left by $t(r)$ and introducing $\Psi_1(E,r)=t(r)\tilde{\Psi}_1(E,r)$ in analogy with Eq. (6) leads to

$$\left[-\frac{d^2}{dr^2}+V_1(E,r)\right]\Psi_1(E,r)=E\Psi_1(E,r), \quad (14)$$

with a new potential $V_1(E,r)$. This potential has the same energy dependence as $V_0(E,r)$, i.e.,

$$V_1(E,r)=V_1(0,r)+EV_E(r), \quad (15)$$

but has a new energy-independent part given by

$$V_1(0,r)=V_0(0,r)-\frac{2}{t(r)}\frac{d}{dr}\left[t(r)\frac{d}{dr}\ln\frac{\Sigma_0(E_0,r)}{\sqrt{t(r)}}\right]. \quad (16)$$

This expression is equivalent to the results obtained in Refs. [30,31], but here it has been established via the techniques of supersymmetric quantum mechanics. For $t(r)=1$ [or more generally $t(r)=\text{const}$] Eq. (16) reduces to the relationship of the supersymmetric partners V_0 and V_1 in the case of energy-independent potentials [18]. It should be emphasized that there arise no difficulties with singularities of $t(r)$ as one might suppose from the compact form of Eq. (16). However, the superpotential $U_0(r)$ and the transformed potential $V_1(0,r)$ will exhibit singularities for $r>0$ if the factorization energy E_0 is greater than the energy of the ground state of Eq. (12), i.e., if the operator $A_0^+A_0^-$ is not positive semidefinite [see Eq. (7)]. Similarly to Ref. [23] this defect vanishes if iterations for the construction of phase-equivalent potentials are performed with the same factorization energy.

The supersymmetric transformation thus allows the construction of a new potential V_1 from an initial potential V_0. The subscripts refer to the number of transformations. According to Eq. (13), the solutions of the new equation are related to those of the initial equation by

$$\Psi_1(E,r)=t(r)\left[-\frac{d}{dr}+U_0(r)\right]\Psi_0(E,r) \quad (17)$$

$$=-t(r)\Sigma_0(E_0,r)^{-1}W[\Sigma_0(E_0,r),\Psi_0(E,r)], \quad (18)$$

where

$$W[\Sigma_0(E_0,r),\Psi_0(E,r)]=\Sigma_0(E_0,r)\left(\frac{d}{dr}\Psi_0(E,r)\right)-\left(\frac{d}{dr}\Sigma_0(E_0,r)\right)\Psi_0(E,r) \quad (19)$$

is the Wronskian of $\Sigma_0(E_0,r)$ and $\Psi_0(E,r)$ which satisfies

$$\frac{d}{dr}W[\Sigma_0(E_0,r),\Psi_0(E,r)]=\frac{E_0-E}{t^2(r)}\Sigma_0(E_0,r)\Psi_0(E,r). \quad (20)$$

Because of Eq. (20) the Wronskian can be expressed via an integral. Specifically, for $E=E_0$ the Wronskian is constant and the solution $\Psi_1(E_0,r)$ of Eq. (12) simplifies to

$$\Psi_1(E_0,r)=t(r)\Sigma_0(E_0,r)^{-1} \quad (21)$$

up to a multiplicative constant. The most general solution of Eq. (12) at $E=E_0$ can be cast into the form

$$\Psi_1(E_0,r)=t(r)\Sigma_0(E_0,r)^{-1}\left[B+\int_{r_0}^{r}\Sigma_0(E_0,u)^2\frac{du}{t^2(u)}C\right], \quad (22)$$

where B, C, and r_0 are arbitrary constants. This can be verified by substituting this expression into Eq. (14).

III. PHASE-EQUIVALENT BOUND-STATE REMOVAL

Depending on the behavior of the chosen factorization solution $\Sigma_0(E_0,r)$ at the boundaries ($r=0$ and $r\to\infty$) [18], supersymmetric transformations may modify the discrete spectrum of the Hamiltonian. This result remains valid for the generalized transformations presented above [34]. If the factorization energy E_0 is chosen as the ground-state energy of the initial Hamiltonian with V_0 and if the factorization solution $\Sigma_0(E_0,r)$ is chosen to be the ground-state wave function, the transformed Hamiltonian with potential V_1 has the same discrete spectrum as the original one except for the ground state at E_0 which is removed.

From Eq. (17) it is seen that for $E\neq E_0$, $\Psi_1(E,r)$ and $\Psi_0(E,r)$ have the same characteristic behavior at the boundaries. For instance, if $\Psi_0(E,r)$ is a bound-state wave function vanishing at the origin and at infinity, so is $\Psi_1(E,r)$. If $\Psi_0(E,r)$ is a scattering wave function vanishing at the origin and oscillating at infinity, so is $\Psi_1(E,r)$. The relationship between the phase shifts (or additional phase shifts in the presence of a Coulomb interaction) of these scattering wave functions can be extracted from Eq. (17) and is given by

$$\delta_1(E)=\delta_0(E)\pm\arctan(\sqrt{|E/E_0|}). \quad (23)$$

The plus sign is associated with a factorization solution diverging at infinity, while the minus sign results for factorization solutions vanishing at infinity. Hence, the modification of the phase shift only depends on the factorization energy and the asymptotic behavior of the factorization solution

[35]. Equation (23) shows that V_1 is not phase equivalent to V_0 but that the iteration of two successive transformations at the same factorization energy and with factorization solutions behaving differently at infinity leads to a potential V_2 exactly phase equivalent to V_0 [22,24].

Here, we focus on the removal of a bound state while maintaining the phase equivalence. Therefore, in the first step we choose $\Sigma_0(E_0,r)$ to be the wave function of the ground state. Then the potential V_1 does not sustain a bound state at E_0. According to Eq. (22) the function

$$\Sigma_1(E_0,r)=t(r)\Sigma_0(E_0,r)^{-1}\int_0^r \Sigma_0(E_0,u)^2\frac{du}{t^2(u)} \quad (24)$$

is a specific solution of the Hamiltonian with V_1 ($B=0$, $C=1$, and $r_0=0$). This solution vanishes at the origin as can be seen by a series expansion. Furthermore, the integral in Eq. (24) is bounded because $\Sigma_0(E_0,r)$ is normalizable and $V_E(r)$ is assumed to be bounded; hence $\Sigma_1(E_0,r)$ diverges at infinity. The existence of such a solution implies that V_1 does not sustain a bound state at E_0; if it had such a bound state, a vanishing solution at the origin would also vanish at infinity.

If we perform a second supersymmetric transformation at the factorization energy E_0 using $\Sigma_1(E_0,r)$ as the factorization solution we obtain by iteration of Eq. (16) the potential

$$V_2(0,r)=V_0(0,r)-2\sqrt{1-V_E(r)}\,\frac{d}{dr}\times\left\{\frac{1}{\sqrt{1-V_E(r)}}\frac{d}{dr}\ln\int_0^r\Sigma_0(E_0,u)^2\times[1-V_E(u)]du\right\}, \quad (25)$$

where Eq. (3) has been used. According to the discussion of the sign in Eq. (23), this potential is exactly phase equivalent to V_0. We must now prove that the Hamiltonian with V_2 has no bound state at E_0. According to Eq. (21) the function $t(r)\Sigma_1(E_0,r)^{-1}$ is a solution of this Hamiltonian at E_0. Since this solution vanishes at infinity but is singular at the origin, there can be no bound state at E_0; if there were such a bound state, the solution vanishing at infinity would also vanish at the origin.

In analogy to supersymmetric transformations of energy-independent potentials the phase-equivalent potential V_2 must exhibit a singularity at the origin. Using a series expansion of Eq. (25) yields the behavior

$$V_2(r)\underset{r\to 0}{\sim}\frac{(\nu_0+2)(\nu_0+3)}{r^2}, \quad (26)$$

where we have assumed that the original potential behaves as

$$V_0(r)\underset{r\to 0}{\sim}\frac{\nu_0(\nu_0+1)}{r^2}. \quad (27)$$

Hence, the construction of a phase-equivalent potential which does not sustain the ground state leads to an increase of 2 of the singularity parameter ν.

The solutions of V_2 bounded at infinity are related to those of V_0 by

$$\Psi_2(E,r)=\Psi_0(E,r)+\Sigma_0(E_0,r)\times\frac{\int_r^\infty\Psi_0(E,u)\Sigma_0(E_0,u)[1-V_E(u)]du}{\int_0^r\Sigma_0(E_0,u)^2[1-V_E(u)]du}, \quad (28)$$

as obtained by using the integral form of Eq. (18) for the first transformation and Eq. (17) for the second transformation. This equation shows that the solutions $\Psi_0(E,r)$ and $\Psi_2(E,r)$ have the same asymptotic behavior; when these solutions are scattering states, this confirms the phase equivalence of V_0 and V_2.

IV. APPLICATION TO $\alpha+^{16}$O POTENTIALS

In Ref. [32], an $\alpha+^{16}$O optical potential is reported which fits the elastic scattering data over a large energy range, namely, 32 MeV $<E_\alpha<$ 146 MeV, where E_α is the energy of the α particle in the laboratory frame. This energy is related to the c.m. energy E by

$$E_\alpha=\frac{5}{4}E. \quad (29)$$

This potential is deep and angular-momentum independent. Its number of forbidden bound states for a given angular momentum l is given by

$$n=\frac{N_c-l}{2}, \quad (30)$$

as recommended by microscopic models [9]. In this equation, N_c is a critical number which can be estimated in the shell-model frame, for instance. For the $\alpha+^{16}$O system, one has $N_c=8$ for even partial waves and $N_c=9$ for odd partial waves [9,32]. This potential has two energy-dependent parameters: the radius of the imaginary part, which has a smooth but nonlinear energy dependence, and the depth of the real part, which has a linear behavior.

Since the method presented above can only deal with linear energy dependences, and to avoid the problem of calculating normalizable solutions of complex energy-dependent potentials (which occur at complex energies, see Ref. [27]), we restrict ourselves to the real part of the potential. Moreover, since we just want to study the principle of the method, we only treat the $l=0$ partial wave for which the number of forbidden bound states is the highest. The same method could be applied to other partial waves and the final potential would be l dependent. The initial potential we start with is

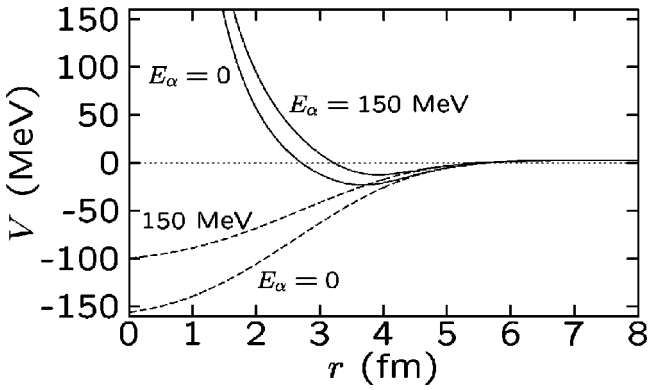

FIG. 1. Real part V_0 of the deep $\alpha+{}^{16}$O potential of Ref. [32], for $l=0$ at $E_\alpha=0$ and 150 MeV [Eq. (31), dashed lines]; and phase-equivalent supersymmetric partner V_8 obtained by removal of the four forbidden states with Eq. (25), at the same energies (full lines).

$$V_0(E,r) = -38\,\frac{1+(3.625-0.0105E_\alpha)\exp[-(r/4.5)^2]}{\{1+\exp[(r-4.3)/1.2]\}^2} + V_C(r), \tag{31}$$

where energies are expressed in MeV, distances in fm, and $V_C(r)$ is the Coulomb potential of a uniformly charged sphere of radius $1.3\times 16^{1/3}$. This potential at $E_\alpha=0$ and 150 MeV is drawn with dashed lines in Fig. 1, where it can be seen that the energy dependence is not negligible.

We have determined the bound states of this energy-dependent potential. In Fig. 2 the bound-state energies $E^{(i)}$ are displayed as a function of the E_α value for which the potential is calculated. It can be seen that the number of bound states varies with E_α. A ''true'' bound state occurs when a bound-state energy equals the energy at which the potential is calculated, i.e., when $E^{(i)}(E_\alpha)=\frac{4}{5}E_\alpha$. If the variation of $E^{(i)}$ with respect to E_α were too strong, this equality could be impossible to satisfy and the potential would not have any bound state.

In the present case, the energy-dependent potential has five bound states; their energies are given in the first column of Table I. As explained above, the four lowest bound states are forbidden states. The fifth one is physical and roughly describes the ^{20}Ne ground state, the experimental energy of which is -4.73 MeV.[1] The four Pauli forbidden states can be removed by applying Eq. (25) to each bound state successively, which provides potentials V_2, V_4, V_6, and V_8. The potential $V_8(E,r)$ is represented by full lines in Fig. 1 for $E_\alpha=0$ and 150 MeV. One verifies that the initial and final potentials have the same energy dependence. Moreover, the transformed potential exhibits a repulsive core at the origin with

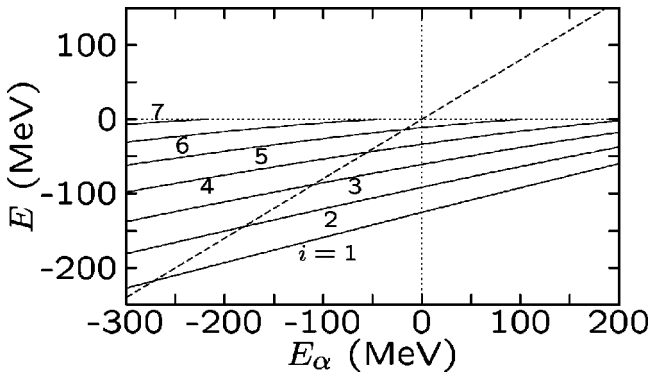

FIG. 2. Bound-state energies $E^{(i)}$ $(i=1,2,\ldots)$ (full lines) and center-of-mass energy (dashed line) as a function of the energy E_α at which the energy-dependent $\alpha+{}^{16}$O potential of Ref. [32] is calculated. Intersections are ''true'' bound states of the energy-dependent potential.

$$V_8(0,r) \underset{r\to 0}{\sim} \frac{16\times 17}{r^2} \tag{32}$$

according to the iteration of Eq. (26).

A more complicated way of transforming an energy-dependent potential is to calculate it at each concerned energy E and to remove its bound states as if it were an energy-independent potential, i.e., with [22]

$$V_2(E,r) = V_0(E,r) - 2\frac{d^2}{dr^2}\ln\int_0^r \Sigma_0(E_0,u)^2\,du. \tag{33}$$

Let us notice that this formula has the same structure as Eq. (25) but with $V_E(r)\equiv 0$. The factorization solution $\Sigma_0(E_0,r)$ has here a different meaning: it is now a bound-state wave function of the *fixed-energy* potential

$$\left[-\frac{d^2}{dr^2}+V_0(E,r)\right]\Sigma_0(E_0,r) = E_0\Sigma_0(E_0,r) \tag{34}$$

[compare with Eq. (11)]. This procedure has the advantage that it is valid for any energy dependence of the potential (not only linear ones). But it has the big disadvantage that the calculation has to be performed at each energy, whereas in the method presented in this article the potential has to be

TABLE I. Bound-state energies of the energy-dependent $\alpha+{}^{16}$O potential and of the fixed-energy potential at $E_\alpha=0$ and 150 MeV.

$E^{(i)}(E_\alpha=\frac{5}{4}E^{(i)})$ (MeV)	$E^{(i)}(E_\alpha=0)$ (MeV)	$E^{(i)}(E_\alpha=150$ MeV) (MeV)
-13.63	-11.32	
-44.48	-33.66	-8.61
-87.94	-60.88	-27.44
-144.81	-91.64	-50.25
-217.66	-125.37	-75.92

[1]In Ref. [32], it is shown that the experimental rotational bands of ^{20}Ne are qualitatively well reproduced with the potential given in Eq. (31) calculated at $E_\alpha=32.2$ MeV. In Ref. [36], potentials are constructed which provide a more precise reproduction of the ^{20}Ne spectrum and of the $\alpha+{}^{16}$O low-energy scattering data.

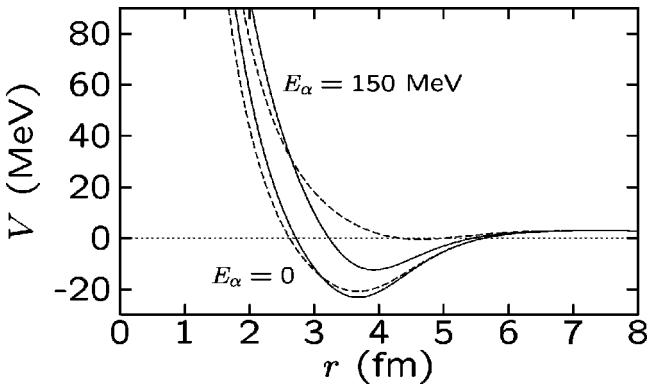

FIG. 3. Supersymmetric partner of the energy-dependent potential, calculated at $E_\alpha = 0$ and 150 MeV (full lines, see Fig. 1); supersymmetric partners of the $E_\alpha = 0$ and 150 MeV fixed-energy potentials (dashed lines).

calculated only once provided the bound-state energies of the energy-dependent potential have been found.

Let us now compare the results of both methods. For this purpose, we apply the fixed-energy method to the 0 and 150 MeV $\alpha + {}^{16}$O potentials. The bound-state energies of these fixed-energy potentials are given in the second and third columns of Table I and in Fig. 2. The number of forbidden states is taken as four, as for the energy-dependent potential. The energy-independent transformed potentials are drawn in Fig. 3 as dashed lines and the shallow potentials of Fig. 1 are represented as full lines for comparison. At $E_\alpha = 0$, both potentials are very close to one another, whereas at $E_\alpha = 150$ MeV the difference is important. The energy dependence of the potential is stronger with the second method.

V. CONCLUSION

Supersymmetric transformations have been generalized to potentials with a linear dependence on energy. They have been shown to be equivalent to the generalized Darboux transformations [30,31] for such potentials. The extended supersymmetric transformations of the present paper offer an elegant way to remove a bound state from a Hamiltonian with a linearly energy-dependent potential provided the wave function of this bound state has been calculated. A second supersymmetric transformation is then needed to recover the phase shifts of the initial potential. The potential resulting from the pair of transformations is thus phase equivalent to the initial one but has one bound state less. Compact formulas have been established for the new potential and for its solutions, as in the energy-independent case. The method is not equivalent to the removal of a bound state from the potential considered at fixed energy: (i) it is more elegant since the calculation has to be performed only once, (ii) the potentials obtained by both methods are different.

As a nuclear-physics example, we have removed the forbidden bound states from the real part of a deep $\alpha + {}^{16}$O potential for $l = 0$. The obtained potential has an r^{-2} repulsive core at the origin and the same energy dependence as the initial potential. This energy dependence is weaker than the one obtained when removing the bound states from the potential considered at different fixed energies. The off-shell properties of the obtained phase-equivalent deep and shallow potentials might be compared in calculations of the $\alpha + {}^{16}$O bremsstrahlung, for instance.

The example presented here mainly aims at testing the principle of our method: more natural applications would concern complex potentials as a whole (not only their real part), for which linear energy dependences of the imaginary part are very common. Applications to complex potentials raise, however, a technical problem: their normalizable solutions occur at complex energies and are more complicated to calculate. This problem has been solved (see Ref. [27], and references therein) in the case of fixed-energy potentials but has not been addressed yet in the energy-dependent case. The iterative bound-state-calculation method used in the present article should be generalized to complex-energy normalizable solutions; the supersymmetric formalism presented above could then be directly used in the frequently encountered case of optical potentials with a linearly dependent imaginary part.

ACKNOWLEDGMENTS

J.-M.S. thanks the National Fund for Scientific Research, Belgium. This text presents research results of the Belgian program on interuniversity attraction poles initiated by the Belgian-state Federal Services for Scientific, Technical and Cultural Affairs.

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