Order parameter of a single event

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An order parameter of a single event is introduced. Its advantage in describing dynamic fluctuations of a single event is demonstrated. [S0556-2813(99)01411-9]

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With the increase of incident energy and number of nucleons in relativistic heavy ion collisions, the number of final state particles in an event becomes very large. A recent result from NA49 [1] shows about 1000 charged particles in an interval of c.m. rapidity $|y|\leqslant 1$ at the Super Proton Synchrotron (SPS) Pb beam of 160 GeV per nucleon. This number is expected to be even larger, by factors of about 4 at the Relativistic Heavy Ion Collider and of about 15 at the Large Hadron Collider. For such a high multiplicity event, the dynamic fluctuation analysis of the single event comes true.

For a low multiplicity sample, there are large statistical fluctuations in an event. The analysis can only be done under the average of the whole event sample, where the dynamic fluctuation strength (intermittency indices [2] or multifractal dimensions [3]) is the average strength of whole sample. Bialas and Ziaja were the first to study the intermittency indices of individual events [4]. In addition to the statistical noise due to insufficient number of particles in an event, the dynamic analysis of single events usually still fluctuates from event to event due to the finite resolution of experimental measurement. The width of this fluctuation depends on the quantity used in the analysis. In this paper, we introduce an order parameter of single event which has the most narrow distribution width in all known indices commonly used in characterization of the dynamic fluctuations.

It is well known that the order parameter in thermodynamics is thermal entropy. For multiparticle final state, a perfectly ordered distribution means that all particles fall in the same position of phase space while a completely disordered distribution is all particles falling in all possible positions with equal possibilities. In the description of this system the thermal entropy has been generalized to information entropy [5]:

$$S=-\sum_{i=1}^{M} p_i \ln p_i, \tag{1}$$

where p_i is the probability of particles appearing in the ith cell. If the number N of particles produced in a single event is large enough, the probability of particles falling in the ith cell can be approximately written as

$$p_i \approx \frac{n_i}{N}, \tag{2}$$

where n_i is the number of particles falling in the ith cell. [In the following we will call the approximation using Eq. (2) the particle-number-ratio (PNR) approximation. The lower limit of N for the validity of this approximation will be discussed later.] If the probability in each cell is equal, the information entropy reaches its maximum value: $S_{\max}=-\sum_{i=1}^{M}(1/M)\ln(1/M)=\ln M$, where M is the number of subdivisions. If all the particles fall into a special cell and all the other cells are empty, the information entropy goes to its minimum value: $S_{\min}=0$. Generally, the value of information entropy is lying in the region $0\leqslant S\leqslant \ln M$. If S approaches to its maximum value $S_{\max}$, the corresponding distribution of final state particles in phase space is more uniform, or smooth. In contrast, if S approaches its minimum value $S_{\min}$, the distribution is of more order, fluctuating strongly from cell to cell. So information entropy of final-state particle-distribution describes the order degree of this distribution.

It is clear from the above discussion that the maximum value of entropy is determined by the total number of its subdivisions. If we compare the entropy of particle distribution in a different event, we have to fix to the same interval with exactly the same division pattern. Otherwise, its value is incomparable. In fact, information dimension of entropy which is independent of special subdivision pattern [6] has been introduced:

$$D_I=\lim_{M\to\infty}\frac{S}{\ln M}. \tag{3}$$

This is the speed of the variation of information entropy with the increasing of division number. It does not depend on the size of bin anymore. Its value is located in the region [0,1]. $D_I=0$ means that the order degree of system does not change with scale. It can only happen when all particles fall into the same place of phase space. On the other hand, the larger the value of D_I is, the smoother of particle distribution in phase space is.

To show the applicability of information dimension D_I in dynamic analysis of a single event, we used the Monte Carlo simulation of a random cascading α model as a demonstration. In the model, the M division of rapidity interval ΔY is given out by ν steps. At the first step, it is divided into two equal parts; at the second step, each part in the first step is

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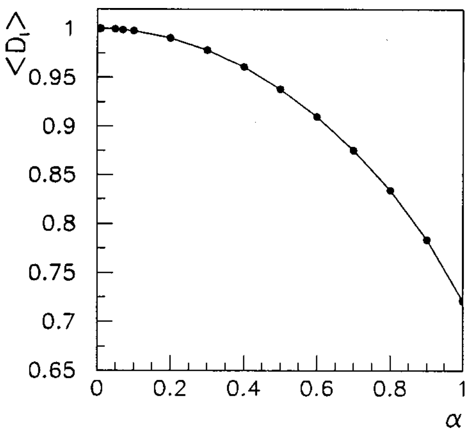

FIG. 1. Relation between average information dimension and α.

further divided into two equal parts, and so on. The steps are repeated until $M=(\Delta Y/\delta y)=2^{\nu}$. How particles are distributed from step-to-step among the two parts of a given rapidity interval is defined by independent random variable $\omega_{\nu j_{\nu}}$, where, j_{ν} is the position of window ($1\leqslant j_{\nu}\leqslant 2^{\nu}$) and ν is the number of steps. It is given by

$$\omega_{\nu,2j-1}=\frac{1}{2}(1+\alpha r), \quad \omega_{\nu,2j}=\frac{1}{2}(1-\alpha r),$$

where r is a random number distributing uniformly in $[-1,1]$. α is a positive number less than unity, which determines the region of random variable ω and describes the strength of dynamic fluctuations in the model. After ν steps, the probability in mth window is $p_m=\omega_{1j_1}\omega_{2j_2},\ldots,\omega_{\nu j_{\nu}}$. Then according to Eq. (1), the information entropy in each division step is calculated. The slope of the line S vs $\ln M$ is the information dimension which is a single-valued function of fluctuation strength α.

We simulated 1 000 000 events by using the above model. The average information dimension $\langle D_I \rangle$ of all events vs the fluctuation parameter α is given in Fig. 1. In each event, the generation of cascading is up to $\nu=6$ and the information dimension is the results from fitting $\nu=1$ to $\nu=6$. It can be seen from the figure that, when α goes from 0 to 1, the corresponding average information dimension changes only from 1 to about 0.7. In particular, when α is small, or $\alpha \leqslant 0.5$, the value of $\langle D_I \rangle$ varies very slowly with α, i.e., $\langle D_I \rangle$ is insensitive to α. So we try to find a single value function of D_I which has a larger region corresponding to the change of α. The relation between $\langle D_I \rangle$ and α can be well fitted by the function $\langle D_I \rangle = 1-\alpha^2/4$. If we define the order parameter as

$$D_O=2\sqrt{1-D_I}, \tag{4}$$

then D_O has almost the same region as α, cf. Fig. 2. Moreover, no matter what kind of model it is, in general, $dD_O=(1/\sqrt{1-D_I})dD_I$, where factor $(1/\sqrt{1-D_I})>1$. This means that for a small variation of α, D_O always has greater variation, or in other words, D_O is more sensitive to the variation of α than D_I. So in comparison with D_I, D_O is a better quantity in describing the order degree of a single event.

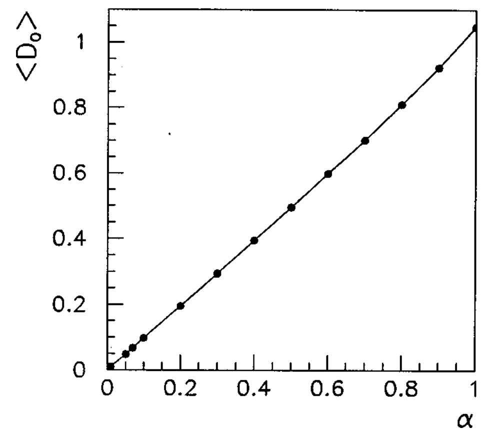

FIG. 2. Relation between average order parameter and α.

In the ideal case, the generation of a random cascading model should be high enough so that the division number $M\to\infty$ or $\delta\to 0$. However, as we know, due to the finite resolution of experimental measurement, a physical quantity cannot be measured in infinitely small cell, or exactly $\delta\to 0$. In present experiments, the available values of M usually equal 40–60 corresponding to $\nu=5-6$ in our model. This makes the order parameter of single event D_O scatter around its average value. In order to estimate the influence of this factor in D_O measurement, the relative standard deviation of D_O, i.e., $\sigma(D_O)/\langle D_O \rangle$, vs α is shown in Fig. 3. With the increase of α, $\sigma(D_O)/\langle D_O \rangle$ has a small increase, but its value is much smaller than unity. Especially, for small α (e.g., $\alpha\leqslant 0.6$), where most present high energy experiments reached, this value is less than 0.1. So for very high multiplicity events, we need not average the order parameter for the whole sample. D_O itself is a good quantity in describing dynamic fluctuation of final state particles in its phase space.

To compare the influence of finite experimental resolution in all known indices, we also calculate other dimensions under the α model. In multifractal theory, information dimen-

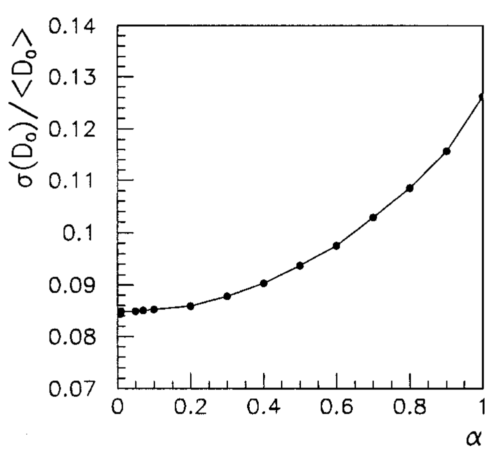

FIG. 3. Relation between relative standard deviation of order parameter and α.

TABLE I. Average multifractal dimensions of order 1 to 3 and their dispersions, for $\alpha = 0.1, 0.3, 0.7, 0.9$ and $\nu = 5,...,10$ cascade steps respectively, multiplied by 10.

		D_1		
	$\alpha=0.1$	$\alpha=0.3$	$\alpha=0.7$	$\alpha=0.9$
Theory	9.976	9.782	8.754	7.835
$\nu=5$	9.976 ± 0.005	9.783 ± 0.047	8.762 ± 0.323	7.829 ± 0.615
$\nu=6$	9.976 ± 0.004	9.783 ± 0.038	8.754 ± 0.257	7.853 ± 0.509
$\nu=7$	9.976 ± 0.003	9.782 ± 0.032	8.749 ± 0.213	7.825 ± 0.431
$\nu=8$	9.976 ± 0.003	9.781 ± 0.025	8.749 ± 0.181	7.856 ± 0.357
$\nu=9$	9.976 ± 0.002	9.782 ± 0.023	8.755 ± 0.156	7.838 ± 0.309
$\nu=10$	9.976 ± 0.002	9.781 ± 0.018	8.755 ± 0.132	7.845 ± 0.270
		D_2		
	$\alpha=0.1$	$\alpha=0.3$	$\alpha=0.7$	$\alpha=0.9$
Theory	9.952	9.574	7.817	6.552
$\nu=5$	9.952 ± 0.010	9.579 ± 0.097	7.881 ± 0.609	6.677 ± 0.982
$\nu=6$	9.952 ± 0.008	9.578 ± 0.080	7.862 ± 0.506	6.703 ± 0.833
$\nu=7$	9.952 ± 0.007	9.575 ± 0.066	7.837 ± 0.435	6.653 ± 0.737
$\nu=8$	9.952 ± 0.006	9.573 ± 0.053	7.842 ± 0.377	6.688 ± 0.630
$\nu=9$	9.952 ± 0.005	9.574 ± 0.049	7.846 ± 0.328	6.646 ± 0.564
$\nu=10$	9.952 ± 0.004	9.574 ± 0.038	7.846 ± 0.291	7.845 ± 0.270
		D_3		
	$\alpha=0.1$	$\alpha=0.3$	$\alpha=0.7$	$\alpha=0.9$
Theory	9.928	9.378	7.123	5.720
$\nu=5$	9.928 ± 0.015	9.391 ± 0.149	7.311 ± 0.782	6.081 ± 1.125
$\nu=6$	9.928 ± 0.013	9.388 ± 0.124	7.283 ± 0.666	6.096 ± 0.967
$\nu=7$	9.929 ± 0.010	9.383 ± 0.104	7.237 ± 0.585	6.038 ± 0.860
$\nu=8$	9.928 ± 0.008	9.381 ± 0.085	7.246 ± 0.512	6.061 ± 0.752
$\nu=9$	9.928 ± 0.007	9.381 ± 0.079	7.241 ± 0.451	6.007 ± 0.679
$\nu=10$	9.928 ± 0.006	9.382 ± 0.062	7.241 ± 0.412	6.009 ± 0.615

sion D_I corresponds to D_1. In Table I, the average multifractal dimensions D_1, D_2, and D_3, and their dispersions are given. They are under $\alpha=0.1,0.3,0.7,0.9$, and the generation of cascading goes up to $\nu=5,6,7,8,9,10$, respectively. The results show that the lowest order dimension $\langle D_1\rangle$ has the smallest standard deviation in all cases. For large dynamic fluctuation or large α, this advantage of D_1 becomes more obvious. The reason is because the moment of probability deviates from its average value when the model is cascaded to finite generation, and this deviation is enlarged by the higher order of moment. So the information dimension is superior to other dynamic indices in single event investigation.

All of the above calculations, of the information dimension as well as the other multifractal dimensions are based directly on the dynamical probabilities p_i obtained from the model. In reality, what can be measured in the experiments is only the distribution of particles in the phase space. Therefore, the particle-number-ratio approximation of Eq. (2) has to be used. Now, we turn to estimate the lower limit of multiplicity for the validity of this approximation.

For this purpose the statistical fluctuation of multiplicity is added to the above-mentioned random cascading model by Bernoulli distribution. In Fig. 4, the average order parameter

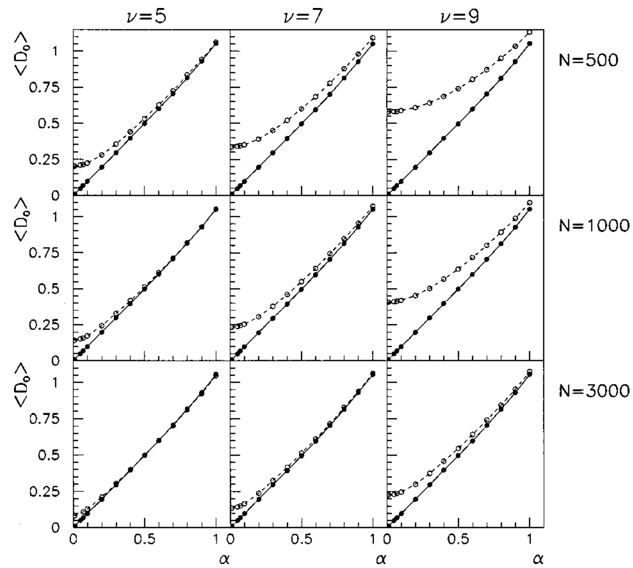

FIG. 4. The average order parameter $\langle D_O\rangle$ vs dynamical-fluctuation parameter α in PNR approximation at different total multiplicity of events, $N=500$, 1000, 3000 (open points connected by dashed lines) are compared to corresponding $\langle D_O\rangle$ from dynamical probability without multiplicity fluctuation (full circles connected by solid lines). The cascading generation is $\nu=5,7,9$, respectively.

$\langle D_O\rangle$ in PNR approximation vs fluctuation parameter α for events with different total multiplicity, $N=500$, 1000, 3000, respectively (open circles connected by dashed lines) are compared to the corresponding $\langle D_O\rangle$ calculated directly from dynamical probability without multiplicity fluctuation (solid lines) as the cascading generation goes up to $\nu=5,7,9$, respectively. It can be concluded from the figure that (i) when generation is up to $\nu=5$, the statistical deviation due to insufficient number of particles is negligible after $N>1000$. (ii) With the increasing of cascading generation, the deviation due to insufficient number of particles becomes larger. (iii) At large dynamic fluctuation ($\alpha\to1$), the deviation due to insufficient number of particles is much smaller than those at small dynamic fluctuation ($\alpha\to0$). This result tells us that for the events with large dynamic fluctuation, the deviation due to insufficient number of particles is largely weakened and D_O can present well the major dynamic fluctuation in such event.

Finally, we discuss the superiority of the information dimension over the other multifractal dimensions from the point of view of elimination of statistical fluctuations. It is well known that the statistical fluctuation can be eliminated by event average factorial moments for moment order $q\geqslant2$. In single event investigation, event factorial moments cannot fully eliminate statistical fluctuations due to the finite number of bins and particles. However, is it still better than PNR approximation in estimating single event multifractal dimensions when multiplicity is pretty high, such as $N\geqslant1000$? To answer this question, the average values of D_1 and D_2 and their relative deviations at different α values for various of multiplicity numbers are given in Fig. 5.

While the information dimension D_1 can be given only by

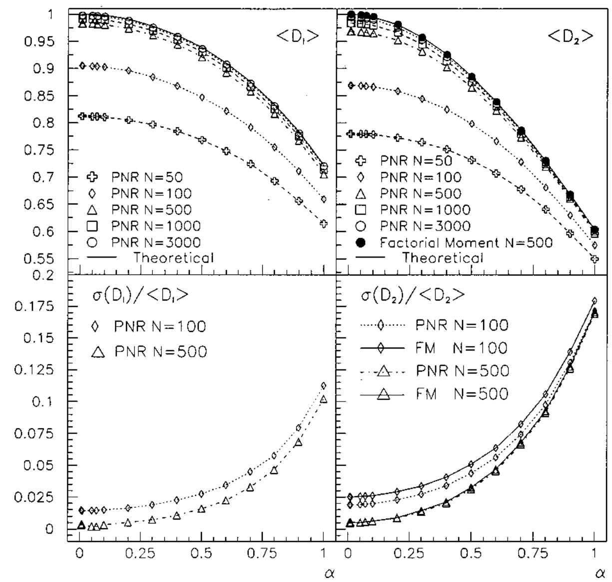

FIG. 5. Average value of D_1 and D_2 (upper figures) and their relative deviation (lower figures) vs dynamical-fluctuation parameter α at different total multiplicity of events, $N=50$, 100, 500, 1000, 3000, using the PNR approximation (dashed line, dotted line, dotted-dashed line, etc.). The full lines are the theoretical results. The results of D_2 from FM method, being only slightly dependent on multiplicity, are shown in the figure (full circles) only for $N=500$.

PNR approximation, the correlation dimension D_2 is given by both PNR approximation and factorial moments (FM) method. The relation between intermittency indices ϕ_q and multifractal dimensions D_q is given by [7]

$$D_q = 1 - \frac{\phi_q}{q-1}.$$

The particle numbers are 50, 100, 500, 1000, and 3000, respectively.

From the results, we can see that, by FM method, average value of D_2 is close to its theoretical value for all chosen numbers of particles; while by PNR approximation, average multidimensions D_1 and D_2 only approach to their theoretical values after multiplicity is higher than 1000. It shows that FM method is still better than the PNR approximation in estimating average D_2 over whole event sample. However, the important point for single event investigation is the behavior of relative deviations of the variables, or the fluctuation of the variables around their average values. They are shown in two lower figures of Fig. 5: (i) When $N \leqslant 500$, the relative deviation of D_2 obtained by the FM method is bigger than that obtained by the PNR approximation, after $N > 500$, the difference of relative deviation of D_2 between the FM method and PNR approximation is negligible. (ii) No matter whether the PNR approximation or FM method is used, the relative deviation of D_2 is always larger than that of D_1. Therefore, when the multiplicity is higher than 1000, the FM method does not help in reducing statistical fluctuations of event multifractal dimensions. The PNR approximation does, as well as the FM method. Information dimension D_1 still contains the least statistical noise in comparison to higher order multifractal dimensions.

In this paper we suggest using the order parameter D_O to do the dynamic fluctuation analysis of single event. It is demonstrated that, when the number of particles in each event is larger than 1000, under the influences of finite multiplicity and finite experimental resolution, order parameter has the best convergence behavior in comparison to higher order multifractal dimensions, or intermittency indices of a single event. So this order parameter describes well the dynamic fluctuation of particle distribution in the phase space. If the particle distribution is completely disordered in the phase space, i.e., if the probabilities of particles falling in each bin are equal, the information dimension $D_I = 1$ and order parameter has its minimum value $D_O = 0$; on the contrary, if all the final state particles fall in the same point of phase space, or the probability in a certain bin is 1 and in all the others is 0, the information dimension $D_I = 0$ and order parameter has its maximum value $D_O = 2$. Thus, the larger the order parameter of single event is, the larger order is the particle distribution in its phase space. This order parameter can be used to classify events with different dynamic fluctuation quantitatively and choose those significant ones from the whole sample for further investigation.

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