Bremsstrahlung in α decay and ''interference of space regions''

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The spectrum of bremsstrahlung in alpha decay is calculated within the framework of quantum and classical electrodynamics. The formulas for the spectra of $E1$ and $E2$ radiation are obtained. The bremsstrahlung is evaluated for the Coulomb barrier and for the spherical symmetry rectangular potential barrier. Experimental results for the nuclei ^{210,214}Po and ^{226}Ra are analyzed. A concept of interference of the space regions in the emission amplitude is discussed. [S0556-2813(99)04910-9]

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Tunneling through a potential barrier is observed experimentally by counting particles, which have gone through the barrier. A charged particle emits photons passing through the space region with nonzero gradient of potential. The barrier is absent for the photons. Thus detection of the γ radiation enables one, in principle, to investigate a particle motion inside a potential barrier.

α decay of atomic nuclei is a well-known example of tunneling. D'Arrigo *et al.* [1] carried out experiments with the nuclei ^{214}Po and ^{226}Ra and detected a continuum γ radiation associated with α decay. The authors explained the experimental results within the framework of a sudden (instant) acceleration model, in which the α particle is accelerated instantaneously to its final kinetic energy. Kasagi *et al.* [2,3] explored the α decay of the nuclei ^{210}Po and ^{244}Cm. They also detected a corresponding γ radiation, and tried to explain a measured spectrum within the framework of the quasiclassical approximation developed for a tunneling charge by Dyakonov and Gornyi [4]. Both groups reached a conclusion concerning the significant contribution of the barrier region to the probability of bremsstrahlung, and have postulated a destructive interference effect between the space regions under and outside the Coulomb barrier.

Recently Papenbrock and Bertsch [5] presented a quantum-mechanical calculation of bremsstrahlung. They explained the experimental results for ^{210}Po partly using a single-particle barrier model for the α-nucleus wave function, and the dipole approximation for the coupling of the photon to the current. They found only a small contribution to the photon emission from the tunneling wave function under the barrier.

The photon emission ''under a barrier'' is a particular case of radiation from the region forbidden for classical motion. Such processes are well known. An example is light emission in atomic transitions. The electron wave functions of bound states in the attractive Coulomb potential have nonzero amplitudes in regions inaccessible to classical motion. In spite of the fast reduction of the wave function in this region, the corresponding contribution to the emission probability is quite real and may easily be evaluated. The wave function and the energy of atomic states are calculated usually by the stationary Schrödinger equation. The photon emission probability is evaluated within the framework of perturbation theory. α decay can be described by a single-particle model with stationary wave function [6]. Accordingly, one can try to calculate photon emission in the α decay by analogy with atomic radiation [5]. It is impossible to locate the space region where the photon was emitted—under or outside the barrier in such an approach. One can only speak about the contributions of different regions to the total integral.

The spectrum of emitted photons is calculated in the center of mass of the α particle and daughter nucleus from Fermi's golden rule $dW_\gamma/d\omega=2\pi|\langle\psi_f|\hat{H}_{int}|\psi_i\rangle|^2\rho_f$, where $\psi_{i,f}$ are the α particle initial and final states, ρ_f is the density of final states: $\rho_f=mK'/(2\pi)^3 d\Omega_\alpha\omega^2/(2\pi)^3 d\Omega_\gamma$, m is the reduced mass, $K'=\sqrt{2mE'}$ is the wave number of the final state with the energy E' at $r\to\infty$, ω is the photon energy (the adopted system of units is $\hbar=c=1$).

The Hamiltonian of the α-particle–photon interaction has the form $\langle\psi_f|\hat{H}_{int}|\psi_i\rangle=\int d^3r\,\mathbf{j}_{fi}(\mathbf{r})\mathbf{A}_\lambda(\mathbf{r})$, where the current is $\mathbf{j}_{fi}(\mathbf{r})=Z_{eff}e/(2mi)\{\psi_f^*(\mathbf{r})\boldsymbol{\nabla}\psi_i(\mathbf{r})-[\boldsymbol{\nabla}\psi_f^*(\mathbf{r})]\psi_i(\mathbf{r})\}$, and $\mathbf{A}_\lambda$ is a photon transverse plane wave with a polarization λ and a moment $\mathbf{q}$. The effective charges are $Z_{eff}^{E1}\approx[2(A-4)-4(Z-2)]/A$, and $Z_{eff}^{E2}\approx Z_\alpha=2$, where A and Z are the atomic number and the charge of the mother nucleus, respectively [7].

We will use the standard expansion of $\mathbf{A}_\lambda$ with electric $\mathbf{A}_{L,m}^E$ and magnetic $\mathbf{A}_{L,m}^M$ multipole fields (see Ref. [7]). The electric multipole field in the Coulomb gauge is $\mathbf{A}_{Lm}^E(\mathbf{r};\omega)=\sqrt{(L+1)/(2L+1)}\,j_{L-1}(\omega r)\mathbf{Y}_{LL-1;m}(\mathbf{n})-\sqrt{L/(2L+1)}\,j_{L+1}(\omega r)\mathbf{Y}_{LL+1;m}(\mathbf{n})$, where $\mathbf{Y}_{JL;M}$ are vector spherical harmonics, and j_L are spherical Bessel functions.

Taking into account boundary conditions for the wave functions at $r=0$ and $r=\infty$, one finds that the matrix element $\langle\psi_f|\hat{H}_{int}|\psi_i\rangle$ is reduced to a calculation of the expression

$$\int d^3r\,\mathbf{A}_{Lm}^E(\mathbf{r})\psi_f^*(\mathbf{r})\boldsymbol{\nabla}\psi_i(\mathbf{r}). \tag{1}$$

We consider here the α decay of even-even nuclei with a monoenergetic α transition from the 0^+ ground state of the mother nucleus to the 0^+ ground state of the daughter nucleus. The initial α particle wave function is the S wave

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$\psi_i(\mathbf{r})=\phi_i(r)Y_{00}(\mathbf{n_r})$, where ϕ_i is a solution of the Schrödinger equation for radial wave function with an orbital quantum number $L=0$.

There are three approaches to calculate the bremsstrahlung $dW_\gamma/d\omega$. The first is a direct calculation of the matrix element in Eq. (1). Using the relation $\nabla\psi_i(\mathbf{r})=-d\phi_i(r)/dr\,\mathbf{Y}_{01;0}(\mathbf{n_r})$, which follows from the gradient formulas [7], one obtains

$$\frac{dW_\gamma^{(E1)}}{d\omega}=\frac{8(Z_{eff}^{E1}e)^2}{3\pi}\frac{K'\omega}{m}\mathcal{F}_1, \tag{2}$$

where the function $\mathcal{F}_1$ and the corresponding radial matrix elements are $\mathcal{F}_1=\{|R_0(\omega)|^2+1/5[R_0(\omega)R_2^*(\omega)+R_0^*(\omega)R_2(\omega)]+3/25|R_2(\omega)|^2\}$, and

$$R_n(\omega)=\int_0^\infty dr\,r^2\,\phi_{f_1}^*(r)j_n(\omega r)\frac{d\phi_i(r)}{dr}. \tag{3}$$

The wave function ϕ_{f_L} in Eq. (3) is a solution of the radial Schrödinger equation for the orbital quantum number L.

The second approach is similar to the Siegert theorem approximation. The following expression is true [7] in the low-energy limit for transverse electric multipoles: $\mathbf{A}_{Lm}^E(\mathbf{r};\omega)\approx 1/\omega\sqrt{(L+1)/L}\nabla j_L(\omega r)Y_{Lm}(\mathbf{n})$. The equation of continuity for the current is $\nabla\mathbf{j}_{fi}(\mathbf{r})=iZ_{eff}e\omega\psi_f^*(\mathbf{r})\psi_i(\mathbf{r})$. Given these two equations, one arrives, by partial integration in Eq. (1), at the expression

$$\int d^3r\,\mathbf{j}_{fi}(\mathbf{r})\mathbf{A}_{Lm}^E(\mathbf{r};\omega)$$
$$\approx -iZ_{eff}e\sqrt{(L+1)/L}\int d^3r\,\psi_f^*(\mathbf{r})\,j_L(\omega r)$$
$$\times Y_{Lm}(\mathbf{n})\,\psi_i(\mathbf{r}).$$

Then, a final formula for the spectrum of the EL bremsstrahlung and for the radial matrix elements takes the form

$$\frac{dW_\gamma^{(EL)}}{d\omega}=\frac{4(2L+1)(L+1)}{L\pi}(Z_{eff}^{EL}e)^2K'm\omega|\overline{R_L}(\omega)|^2, \tag{4}$$

$$\overline{R_L}(\omega)=\int_0^\infty dr\,r^2\phi_{f_L}^*(r)j_L(\omega r)\phi_i(r). \tag{5}$$

Papenbrock and Bertsch employed a third approach. The formula $\psi_f^*(\mathbf{r})\nabla\psi_i(\mathbf{r})=1/\omega\psi_f^*(\mathbf{r})[\nabla V(\mathbf{r})]\psi_i(\mathbf{r})=1/\omega\psi_f^*(\mathbf{r})[-dV(r)/dr]\phi_i(r)\mathbf{Y}_{01;0}(\mathbf{n_r})$ follows from Heisenberg's equation of motion with a spherical symmetry potential $V(\mathbf{r})$ and from the gradient formula. Using this result one obtains for the $E1$ and $E2$ spectra of the bremsstrahlung,

$$\frac{dW_\gamma^{(EL)}}{d\omega}=\frac{8(Z_{eff}^{EL}e)^2}{3\pi}\frac{K'}{m}\frac{1}{\omega}\tilde{\mathcal{F}}_L, \tag{6}$$

where $\tilde{\mathcal{F}}_1=\{|\tilde{R}_0(\omega)|^2+1/5[\tilde{R}_0(\omega)\tilde{R}_2^*(\omega)+\tilde{R}_0^*(\omega)\tilde{R}_2(\omega)]+3/25|\tilde{R}_2(\omega)|^2\}$, $\tilde{\mathcal{F}}_2\approx 9/5|\tilde{R}_1(\omega)|^2$ (the first summand from the expression for the $\mathbf{A}_{2m}^E$ was taken, because the $E2$ spectra were calculated here at low energies only). The radial matrix elements in the $\tilde{\mathcal{F}}_L$ are

$$\tilde{R}_n(\omega)=\int_0^\infty dr\,r^2\,\phi_{f_L}^*(r)j_n(\omega r)\frac{dV(r)}{dr}\phi_i(r). \tag{7}$$

We consider a spherical symmetry potential barrier of two kinds—the rectangular potential barrier $V^{\mathcal{R}}(r)=V_1\theta(R_1-r)+V_2\theta(r-R_1)\theta(R_2-r)$, and the Coulomb barrier $V^C(r)=V_1\theta(R_1-r)+Z_dZ_\alpha e^2/r\,\theta(r-R_1)$, where Z_d is the charge of the daughter nucleus, and the radius $R_1=1.2[(A-4)^{1/3}+4^{1/3}]$ fm.

The external radius of the rectangular barrier R_2 coincides here with the classical turning point for the Coulomb potential $r_E=Z_dZ_\alpha e^2/E$. The height of the barrier V_2 is determined from the equality of quasiclassical transmission coefficients, i.e., from the equation $\int_{R_1}^{R_2}dr\sqrt{2m(V_2-E)}=\int_{R_1}^{r_E}dr\sqrt{2m(Z_dZ_\alpha e^2/r-E)}$.

The radial wave function of the initial state for the rectangular potential barrier is

$$\phi_i^{\mathcal{R}}(r)=\begin{cases} a_0\sin(kr)/r, & 0\leqslant r<R_1\\ b_0^{(1)}\exp[-\kappa(r-R_1)]/r+b_0^{(2)}\exp[\kappa(r-R_1)]/r, & R_1\leqslant r<R_2\\ c_0\exp[iK(r-R_2)]/r, & R_2\leqslant r.\end{cases}$$

Here the wave numbers are $k=\sqrt{2m(E-V_1)}$, $\kappa=\sqrt{2m(V_2-E)}$, and $K=\sqrt{2mE}$ in regions I ($0\leqslant r<R_1$), II ($R_1\leqslant r<R_2$), and III ($R_2\leqslant r$), respectively, E is the α particle energy in the initial state. $c_0=1/\sqrt{v_\alpha}$, where $v_\alpha=K/m$ because the outgoing flux is normalized to unity. The coefficients a_0, $b_0^{(1,2)}$ and the energy E are determined from the conditions of continuity and differentiability of the wave function at the boundaries $r=R_1$ and $r=R_2$.

The initial radial wave function for the Coulomb barrier $\phi_i^C(r)$ is constructed analogously. It coincides up to the coefficient with the $\phi_i^{\mathcal{R}}(r)$ in region I, and has the form $\phi_i^C(r)=\sqrt{1/v_\alpha}[G_0(\eta,\rho)+iF_0(\eta,\rho)]/r$ for $R_1\leqslant r$. The

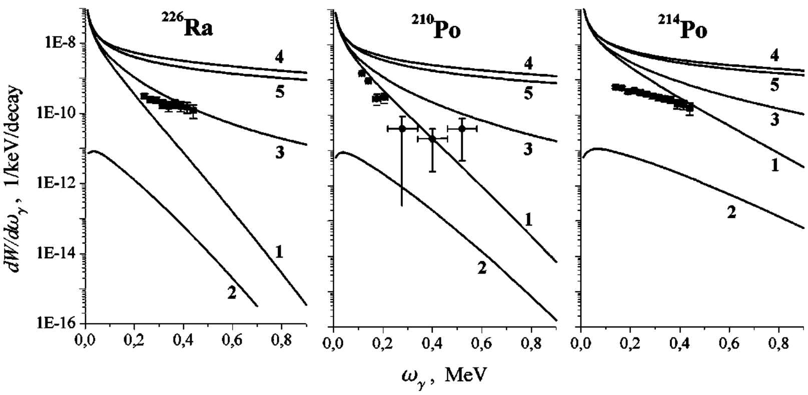

FIG. 1. The experimental data are taken from Refs. [1,9] for ^{226}Ra and ^{214}Po, and from Refs. [2,3] for ^{210}Po. Lines 1 (2), quantum $E1$ ($E2$) spectra for the Coulomb barrier; 3, classical motion in the Coulomb potential from the point r_E to ∞ with acceleration $w=Z_\alpha Z_d e^2/(mr^2)$ (the breaking of the α particle in the process of photon emission can be neglected); 4, quantum $E1$ spectra for the rectangular potential barrier; 5, instant acceleration model.

regular $F_L(\eta,\rho)$ and irregular $G_L(\eta,\rho)$ Coulomb functions are solutions of the Coulomb wave equation [8], $\eta = Z_d Z_\alpha e^2/v_\alpha$.

The wave function of the final state should have the asymptotic behavior of a plane wave plus a convergent spherical wave. The function $\psi_f^{\mathcal{R}}(\mathbf{r}) = 4\pi\Sigma_{L=0}^{\infty}\Sigma_{m=-L}^{L} i^L Y_{Lm}^*(\mathbf{n}_{\mathbf{K}'}) Y_{Lm}(\mathbf{n}_{\mathbf{r}})\phi_{f_L}^{\mathcal{R}}(r)$ with the radial part

$$\phi_{f_L}^{\mathcal{R}}(r)=\begin{cases} A_L j_L(k'r), & 0\leqslant r<R_1 \\ B_L^{(1)} h_L^{(1)}(i\kappa' r)+B_L^{(2)} h_L^{(2)}(i\kappa' r), & R_1\leqslant r<R_2 \\ j_L(K'r)+C_L h_L^{(2)}(K'r), & R_2\leqslant r \end{cases}$$

satisfies this condition for the rectangular potential barrier. Here $h_L^{(2)}$ denotes a spherical Hankel function. The wave numbers are $k'=\sqrt{2m(E'-V_1)}$, $\kappa'=\sqrt{2m(V_2-E')}$. The coefficients A_L, $B_L^{(1,2)}$, and C_L are determined from matching conditions of the wave function.

For the Coulomb barrier one should multiply $\psi_f^{\mathcal{R}}(\mathbf{r})$ by the phase factor $\exp(i\sigma_L)$, where $\sigma_L = \arg\Gamma(L+1+i\eta)$, and use the function $\phi_{f_L}^C(r)=A_L j_L(k'r)$ in region I, and $\phi_{f_L}^C(r) = [F_L(\eta,K'r)+C_L(G_L(\eta,K'r)-iF_L(\eta,K'r))]/(K'r)$ for $R_1\leqslant r$.

Spectra of bremsstrahlung in α decay were calculated by the formulas (6) and (7) with the wave functions ϕ_i^C and $\phi_{f_L}^C$. The $E1$ spectra are in excellent agreement with the result of Papenbrock and Bertsch. A sequence of calculations was described in detail in Ref. [5]. One should note that the calculation of the $E2$ spectrum has a peculiarity. A resonance level with $L=2$ exists and lies not far from the ground state for all values of V_1 except the first (the maximum) value permitted by the α particle energy E. The $E2$ resonances in the spectrum of bremsstrahlung are very wide. That is why a potential well in region I should have the least possible depth. The values $V_1=4.31$ MeV for ^{226}Ra, 4.82 MeV for ^{210}Po, and 7.26 MeV for ^{214}Po were used in this work.

The most reliable experimental data now available are for the nucleus ^{210}Po [2,3]. A comparison of these experimental data with results obtained in this work is displayed in Fig. 1, which shows that there are no longer reasons to discuss a discrepancy between the experimental data and a quantum calculation for the Coulomb barrier. More precise measurements are needed for photon energies $\omega \geqslant 0.4$ MeV.

In [1] an angle of 90° was chosen between the γ detector and the detector of α particles. Experimental date for the nuclei ^{214}Po and ^{226}Ra were displayed without angle averaging. The revised data were provided for this publication by Eremin *et al.* [9]. One can see an appreciable difference between theory and experiment for the nuclei ^{214}Po and ^{226}Ra in Fig. 1.

It is easy to evaluate bremsstrahlung within the framework of classical electrodynamics for the instant acceleration model. According to [10] the formula describing the $E1$ bremsstrahlung is $dW_\gamma^{(E1)}/d\omega = 2/(3\pi\omega)(Z_{eff}^{E1}e)^2|\mathbf{w}_\omega|^2$, where $\mathbf{w}_\omega$ is the Fourier transform of the charge acceleration $\mathbf{w}(t)$. The Fourier transform is $|\mathbf{w}_\omega| = v_\alpha$ for instant acceleration up to the velocity v_α. Corresponding lines in Fig. 1 lie considerably above the experiment. This result contradicts the statement of [1] that the instant acceleration model describes the experiment well.

Let us briefly consider the concept of interference of space regions, which was discussed by Kasagi *et al.* in [2,3]. They explained, in particular, a suppression of the emission

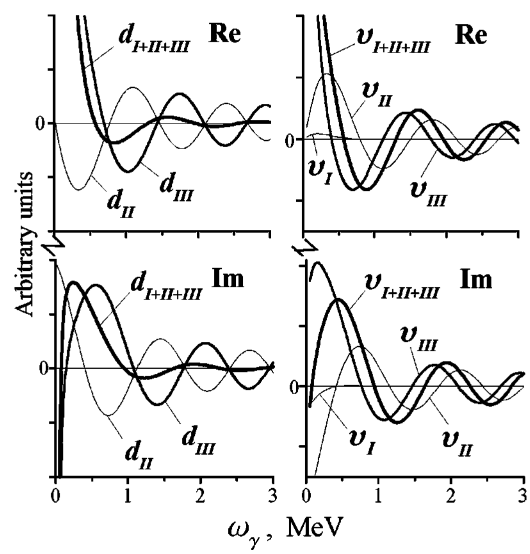

FIG. 2. The real and imaginary parts of the ‘‘dipole moment of transition’’ d and of the matrix element of ‘‘velocity’’ v for the spherical rectangular potential barrier. Subscripts II and III mean the barrier and potential-free regions correspondingly.

probability for $\omega < 0.4$ MeV in comparison with the classical emission probability by the destructive interference of the radiation amplitudes between the inside and the outside of the barrier.

It should be stated that the same amplitude is meant when one discusses the interference of radiation amplitudes between the different space regions. Such interference of space regions should be dealt with in the framework of the model developed here as a contribution of integrals calculated in these regions to the total integral.

The rectangular potential barrier is an illustrative example of this problem. The quantum emission probability calculated with the wave functions $\phi_i^{\mathcal{R}}$ and $\phi_{f_1}^{\mathcal{R}}$ (lines 4 in Fig. 1) exceeds the classical one (lines 5) over a wide photon-energy range $0 \leqslant \omega \leqslant 3$ MeV for the rectangular barrier reproduced the α decay of the ^{210}Po, for example. Constructive interference of the regions inside and outside the barrier should take place according to this concept.

We consider firstly the spectrum of bremsstrahlung from Eqs. (4) and (5). This spectrum is derived from the function $d = Z_{eff}^{E1} e \int_0^\infty dr\, r^2 \phi_{f_1}^{\mathcal{R}*} r \phi_i^{\mathcal{R}}$, which looks like a dipole moment of transition. One can expand this matrix element as a sum of $d_I \propto \int_0^{R_1} dr \cdots$, $d_{II} \propto \int_{R_1}^{R_2} dr \cdots$, and $d_{III} \propto \int_{R_2}^{\infty} dr \cdots$. The dipole moment d_I is negligibly small. The moments d_{II} and d_{III} are shown in Fig. 2. Phases of the d_{II} and d_{III} differ by π approximately. They have mostly different signs and largely compensate each other. Destructive interference takes place contrary to the initial hypothesis.

The spectrum of bremsstrahlung from Eqs. (2) and (3) is derived from the matrix element of the velocity operator $\hat{\mathbf{v}} = -i\boldsymbol{\nabla}/m$. One can evaluate the radial matrix element $v \propto \int_0^\infty dr\, r^2 \phi_{f_1}^{\mathcal{R}*} d\phi_i^{\mathcal{R}}/dr$ in regions I, II, and III by analogy with the dipole matrix element *d*. The result is shown in Fig. 2. Phases of the v_{II} and v_{III} differ by $\pi/2$, approximately. Both constructive and destructive interferences take place in the energy range $0 \leqslant \omega \leqslant 3$ MeV.

A further strange conclusion follows from an analysis of Eqs. (6) and (7). The dV/dr operator is localized at the boundary of regions II and III, i.e., at the point $r = R_2$ where the particle is accelerated. The contributions of the barrier region II and the potential-free region III to the emission probability are equal to zero. That is why the interference of the regions cannot explain the spectrum of bremsstrahlung within the framework of Eqs. (6) and (7).

These three examples, then, show that the concept suggested in [2] has a very limited range of applicability.

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