Fermionic symmetries: Extension of the two to one relationship between the spectra of even-even and neighboring odd mass nuclei

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In the single j shell there is a two to one relationship between the spectra of certain even-even and neighboring odd mass nuclei; e.g., the calculated energy levels of $J=0^+$ states in ^{44}Ti are at twice the energies of corresponding levels in ^{43}Ti(^{43}Sc) with $J=j=7/2$. Here an approximate extension of the relationship is made by adopting a truncated seniority scheme; i.e., for ^{46}Ti and ^{45}Sc we get the relationship if we do not allow the seniority $v=4$ states to mix with the $v=0$ and $v=2$ states. Better than that, we get *very* close to the two to one relationship if seniority $v=4$ states are admixed perturbatively. In addition, it is shown that for the $J=0$ $T=3$ state in ^{46}Ti and for the $J=j$ $T=5/2$ state in ^{45}Sc (i.e., the states of higher isospin) there are no admixtures in which the neutrons have seniority 4. [S0556-2813(99)03811-X]

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I. INTRODUCTION

Single j-shell ($j=7/2$ in particular) configurations are not only simple but also offer ideal situations for realizing a wide variety of relationships in the otherwise complex spectra. Although the semimagic Ca isotopes are quite extensively studied [1], the existence of such relations for open-shell nuclei is doubtful due to the presence of the proton-neutron interaction. However, it was noted by McCullen, Bayman, and Zamick (MBZ) [2] that in a single j-shell ($j=f_{7/2}$) calculation for nuclei with open shells of both neutrons and protons (e.g., scandium and titanium isotopes) there were in some cases striking relations in the calculated spectra of even-even nuclei and neighboring odd-even nuclei. For example, the excitation energies of the $J=0^+$ states in ^{44}Ti were at twice the energies of $J=j$ states in ^{43}Sc (or ^{43}Ti). It was further shown in MBZ's technical report that the wave functions for the even-even and even-odd nuclei bear a striking visual relationship. The wave functions for Ti were written as

$$\psi^{J_\alpha}=\sum_{L_pL_nv_pv_n} D^{J_\alpha}(v_pL_pv_nL_n)|(j)^2v_pL_p;(j)^nv_nL_n;J\rangle \tag{1}$$

and those of Sc as

$$\psi^{J_\beta}=\sum_{v_nL_n} C^{J_\beta}(v_nL_n)|j^1v_p=1L_p=j;(j)^nv_nL_n;J\rangle. \tag{2}$$

In the above equation $D^{J_\alpha}(v_PL_pv_nL_n)$ is the probability amplitude that in a state α with total angular momentum J the protons couple to angular momentum L_p with seniority v_p and the neutrons to L_n with seniority v_n; a similar definition holds for $C(v_nL_n)$.

In the case of ^{43}Sc ($J=j=7/2$) and ^{44}Ti ($J=0$) the relationship is

$$D^0(vLvL)=C^j(vL), \tag{3}$$

for a state in ^{44}Ti which is at twice the excitation energy of the corresponding state in ^{43}Sc.

This relationship also holds for other pairs as well, e.g., (^{48}Ti, ^{49}Ti), (^{52}Fe, ^{53}Fe).

Note that the dimensions of the column vectors in the two cases is the same. For ^{43}Sc $J=j$ and the possible values of L_n are 0, 2, 4, and 6. For ^{44}Ti, $J=0$ and the allowed $[L_pL_n]$ states are [0,0], [2,2], [4,4], and [6,6]. In both cases the dimensions are the same, i.e., four and four. This is necessary in order to have an *exact* two to one relationship.

A comparison between theory and experiment was carried out by Zamick and Zheng [3], focusing on the excitation energies of high isospin states. For example, for ^{48}Ti the *experimental* $T=4$ $J=0$ excitation energy is 17.379 MeV, while in ^{49}Ti the excitation energy of the $T=7/2$ $J=j=7/2^-$ state is 8.724 (the ground states have isospins $T=2$ and $T=5/2$, respectively). The deviation from a two to one relation is -0.40%. For ^{52}Fe and ^{53}Fe the excitation energies of the $T=2$ and $T=3/2$ states are, respectively, 8.559 and 4.25 MeV and the percent deviation is 0.69%. Other cases are considered where there is no exact two to one ratio in the theory, e.g., ^{46}Ti and ^{47}Ti, where the percent deviation is -1.54% and ^{44}Ti and ^{45}Ti where the percent deviation is -0.98%. The agreement with the two to one relation is surprisingly good for these pairs. However, in these nuclei the two to one ratio is not expected to hold for states of lower isospin.

In the following section we show the exactness of the two to one relationship in neighboring pairs of nuclei with two and one particles (or holes), respectively, in the $j=7/2$ shell. The approximate extension of the relation to certain other pairs of nuclei with more valence particles/holes is discussed in Sec. III and the absence of seniority 4 contributions to the higher isospin states in these nuclei is shown in Sec. IV. In Sec. V higher seniority component admixtures are treated perturbatively. Finally, some additional remarks are made in Sec. VI.

II. EXACT TWO TO ONE RELATION FOR [^{44}Ti, ^{43}Sc] (AND [^{48}Ti, ^{49}Ti], AND [^{52}Fe, ^{53}Fe])

The amplitudes of the wave function of Eq. (1) for the titanium isotopes satisfy the normalization condition

$$\sum_{L_pL_nv_pv_n} |D^{J\alpha}(v_pL_pv_nL_n)|^2=1 \qquad (4)$$

and the completeness relation

$$\sum_{\alpha} |D^{J\alpha}(v_pL_pv_nL_n)|^2=1. \qquad (5)$$

The coefficients can be regarded as parts of a column vector representing the wave function such that $|D^{J\alpha}(v_pL_pv_nL_n)|^2$ is the probability that in a given state α with angular momentum J, the protons couple to v_p, L_p and the neutrons to v_n, L_n. However, from now on we drop the seniority and other labels (their significance will be discussed explicitly wherever necessary) and adopt a simplified notation based on angular momentum labels.

To obtain the wave function we have to diagonalize the Hamiltonian matrix, a typical matrix element of which is

$$\langle [L_p'L_n']^J|V|[L_pL_n]^J\rangle.$$

Normally what one tries to do, and this was indeed done by MBZ [2], is to reduce this to sums over two particle matrix elements of the form $\langle (j^2)^J|V|(j^2)^J\rangle$ $J=0,1,\ldots,7$. However, we will not follow that procedure. Rather we will first consider the matrix element of the even-even nucleus ^{44}Ti and we will manipulate the expression so that we can get rid of the coordinates of one of the particles and thus establish a relationship with ^{43}Sc (and its mirror ^{43}Ti). Even though we focus our attention exclusively on the pair (^{44}Ti,^{43}Sc), the discussion here is more general as it is applicable to several other pairs as mentioned in the section heading. We will assume charge symmetry $V_{nn}=V_{pp}$.

The matrix element of an even-even Ti is written as

$$M(\mathrm{Ti})=\langle [(j^2)^{L_p'}L_n']^J|V|[(j^2)^{L_p}L_n]^J\rangle.$$

We can break this into (a) an interaction between the protons, (b) an interaction between the neutrons, and (c) an interaction between neutrons and protons.

For (a) and (b) we get

$$[\langle L_p|V|L_p\rangle+\langle L_n|V|L_n\rangle]\delta_{L_p'L_p}\delta_{L_n'L_n}.$$

For the interaction between neutrons and protons we have

$$V_{\text{proton-neutron}}=\langle [(j^2)^{L_p'}L_n']^J|V(p;\text{neutrons})|[(j^2)^{L_p}L_n]^J\rangle.$$

We can use the Racah algebra to couple the second proton to the neutrons. It is convenient to use the unitary Racah coefficients defined by

$$[[ab]j_{ab}c]^J=\sum_{J_{bc}} U(abJc;j_{ab}j_{bc})[a[bc]j_{bc}]^J.$$

They are related to the more familiar $6j$ symbols by

TABLE I. Two to one relation in ^{44}Ti,^{43}Sc.

Eigenvalues and wave functions for $J=0$ levels in ^{44}Ti					
Energy		0.0	6.5007	8.3449	10.8567
L_p	L_n		$T=2$		
0	0	−0.7608	0.4006	−0.5000	0.1037
2	2	−0.6090	−0.6995	0.3727	0.0317
4	4	−0.2093	0.4156	0.5000	−0.7304
6	6	−0.0812	0.4213	0.6009	0.6744

$$U(abcd;ef)=(-1)^{a+b+c+d}\sqrt{(2e+1)(2f+1)} \times \begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}.$$

We get

$$V_{\text{proton-neutron}}=\{1+(-1)^{L'-L}\} \times\sum_{I_X} U(jjJL_n';L_p'I_X)U(jjJL_n;L_pI_X) \times\langle [j_p[j_pL_n']^{I_X}]^J|V|[j_p[j_pL_n]^{I_X}]^J\rangle. \qquad (6)$$

Since L and L' are even for two protons in a single j shell, the factor $\{1+(-1)^{L-L'}\}=2$.

We now specialize to $J=0$ states of ^{44}Ti for which L_p and L_n are equal. From the unitarity condition,

$$U(jj0L_n';L_p'I_X)=\delta_{I_Xj}.$$

Thus

$$V_{\text{proton-neutron}}=2\langle [jL_n']^j|V|[jL_n]^j\rangle.$$

Invoking charge symmetry we find that the proton-proton+neutron-neutron interaction equals $2\langle L_n|V|L_n\rangle\delta_{L_nL_n'}\delta_{L_pL_n}$. But this is just twice the corresponding matrix element between two neutrons in ^{43}Sc. We thus see that a given matrix element for the $J=0$ state of ^{44}Ti is twice that of the corresponding matrix element for the $J=j$ state in ^{43}Sc. Thus the column vectors will have identical numbers and there will be a two to one ratio for the energy levels.

We show the energies and column vectors for ^{44}Ti in Table I, as they were originally calculated by MBZ and published in their technical report [2]. This table can be easily adjusted to give the calculated spectrum of $J=7/2$ states in ^{43}Sc. For such an adjusted table, in the column for L_p, all entries take the value $j=7/2$; the energies in Table I are to be divided by 2 and the isospin $T=2$ changes to the value of 3/2.

It should be emphasized that even if we do not have charge independence but still have charge symmetry, the two to one relation will hold. There will be isospin mixing, of course, but the fact that $V_{\text{proton-neutron}}$ is a factor of 2 larger for

TABLE II. Approximate two to one relation in ^{46}Ti,^{45}Sc.

		Eigenvalues and wave functions for $J=0$ levels in ^{46}Ti						
Energy		0.0	5.1973	7.1207	9.2493	11.4350	12.9491	
L_p	L_n						$T=3$	
0	0	0.8224	−0.3982	0.1527	−0.0724	0.1913	−0.3162	
2	2	0.5420	0.5245	−0.1105	0.3756	−0.3333	0.4082	
4	4	0.0861	−0.4461	−0.2342	−0.5244	−0.4046	0.5477	
6	6	−0.0127	−0.1454	0.3367	0.1686	0.6353	0.6583	
2	2*	0.0563	0.4309	0.6819	−0.5783	−0.1082	0.0000	
4	4*	−0.1383	−0.4006	0.5755	0.4645	−0.5228	0.0000	
		Eigenvalues and wave functions for $J=7/2$ levels in ^{45}Sc						
Energy		0.0	2.6204	3.2255	4.9559	5.5225	6.4779	6.6443
L_p	L_n						$T=5/2$	
7/2	0	0.8210	−0.4154	0.0811	−0.0536	0.2068	−0.3162	0.0343
7/2	2	0.5434	0.5555	0.1042	0.1420	−0.4362	0.4082	0.0904
7/2	4	0.0846	−0.4740	−0.4599	−0.0533	−0.2079	0.5477	−0.4588
7/2	6	−0.0130	−0.1496	0.3570	−0.0695	0.5429	0.6583	0.3422
7/2	2*	0.0428	0.2197	0.1706	−0.9142	0.0160	0.0000	−0.2912
7/2	4*	−0.1462	−0.4540	0.6329	−0.0354	−0.6030	0.0000	0.0850
7/2	5	−0.0120	−0.1319	−0.4625	−0.3638	−0.2554	0.0000	0.7556

^{44}Ti ($J=0^+$) than it is for ^{43}Sc ($J=j$), still holds. Of course, in Ref. [2] we used charge-independent matrix elements.

III. APPROXIMATE TWO TO ONE RELATIONSHIP FOR ^{46}Ti AND ^{45}Sc

As an example consider the pair ^{45}Sc,^{46}Ti. The basis states for ^{45}Sc with $J=j=7/2$ consist of a single proton with $L_p=j$ and four neutrons with angular momenta $L_n=0,2,4,6,2^*,4^*,5^*$, where the states 2,4,6 have seniority 2 and the states 2^*, 4^*, and 5^* have seniority 4. The $J=0$ basis states for ^{46}Ti are [0,0], [2,2], [4,4], [6,6], [2,2*], and [4,4*].

The dimension being 7 for ^{45}Sc and 6 for ^{46}Ti, it does not appear to give rise to any obvious exact two to one relationship.

Suppose, however, we make the approximation that for the lowest lying states we can omit the seniority 4 admixtures. The dimensions then become 4 and 4, so there is hope for getting a two to one relationship. In the following paragraphs we will show that this hope is realized.

The wave functions and energy levels for ^{46}Ti and ^{45}Sc, as calculated by MBZ [2], are shown in Table II.

For $J=0$ states in ^{46}Ti the wave functions are of the form

$$|\psi^{0\alpha}\rangle=\sum_{L,v,v'} D^{0\alpha}(L,v,L,v')|[LL]^0\rangle;$$

i.e., the angular momentum of the two protons must equal the angular momentum of the four neutrons. As mentioned before there are two $L=2$ and $L=4$ states corresponding to seniorities $v=2$ and $v=4$.

Just as in Eq. (6) in the previous section, the Hamiltonian matrix is of the form

$$\begin{aligned}\langle[L'L_n']^0|H|[LL_n]^0\rangle&=V_{pp}^{L'}\delta_{L'L}+V(f_{7/2}^4)^{L_n}\delta_{L_nL_n'}\\&\quad+2U(jj0L';L'j)U(jj0L;Lj)\\&\quad\times\langle[j[jL']^j]^0|V_{pn}|[j[jL]^j]^0\rangle.\quad(7)\end{aligned}$$

The last factor is equal to $\langle[jL']^j|V_{pn}|[jL]^j\rangle$, i.e., the proton-neutron interaction in ^{45}Sc. The unitary Racah coefficients are both equal to unity because of the zero on the left side of the semicolon. Once again it should be noted that just as in the previous section the discussion here could be relevant for several other pairs of nuclei, with minor modifications, even though we focus on ^{46}Ti-^{45}Sc pair.

Consider the interaction between the neutrons. At first glance it does not seem possible that the interaction between four neutrons could equal that of two protons. But there is the remarkable result discussed in De Shalit and Talmi [4] and in Talmi's more recent work [5] that with any two-body effective interaction between identical particles in the $f_{7/2}$ shell, seniority will be conserved and the resulting spectrum which depends only on seniority and angular momentum and not on the number of particles. Therefore if we limit ourselves to seniority $v=0$ and $v=2$ states we have

$$V(f_{7/2}^4)_n^L=V(f_{7/2}^2)_n^L+\text{const.}$$

A consequence of this result is that the seniority two states in all the isotopes described by $(f_{7/2})^n$ configurations have nearly the same spectra. Therefore, if we truncate to seniority 0 and seniority 2 states, we find

$$\langle [LL]^0|H|[L'L']^0\rangle = (V^L_{pp}+V^L_{nn})\delta_{LL'} + 2\langle [jL']^j|H|[jL]^j\rangle + \text{const.}$$

By charge symmetry $V^L_{nn}=V^L_{pp}$ and so the spectrum of ^{46}Ti will be double that of ^{45}Sc provided we limit ourselves to $v=0$ and $v=2$. This approximation should be quite good for the first few states of the two nuclei. We shall see in the next section that the situation is even better for states of higher isospin—they do not have components in which the four neutrons couple to seniority $v=4$.

IV. HIGHER ISOSPIN STATES

In this section we will assume that charge independence (as well as charge symmetry) holds so that the states have definite values of isospin.

In the single j shell all but one of the $J=0$ states in ^{46}Ti have isospin $T=1$. The other state has isospin $T=3$. If we compare the wave function of this state with the higher isospin state in ^{45}Sc with $J=j=7/2$, we see that the numbers in the column vectors are the same. Furthermore, there are no seniority 4 neutron state components in the wave functions.

We can explain the result as follows. ^{46}Ti$(T=3)J=0^+$ is the double analog of the $J=0^+$ ground state of ^{46}Ca, a nucleus with only valence neutrons. Thus the amplitudes $D(v_pL_pv_nL_n)^{J=0\,T=3}$ should be two-particle fractional parentage coefficients:

$$\begin{aligned}|^{46}\mathrm{Ca}^{(J=0)}\rangle &= \sum_{I_0,v,v'} \langle (j^4)^{I_0}v(j^2)^{I_0}v'|\}j^6J=0v=0\rangle| \\ &\quad\times[(j^4)^{I_0}v(j^2)^{I_0}v']^{J=0v=0}\rangle \\ &= \sum_{I_0,v}\langle (j^4)^{I_0}vj|\}j^5J=jv'=1\rangle \\ &\quad\times\langle (j^5)^jv'=1j|\}j^6J=0v=0\rangle \\ &\quad\times U(I_0j(J=0)j;jI_0)|[(j^4)^{I_0}v(j^2)^{I_0}v']^{J=0}\rangle. \end{aligned} \tag{8}$$

Here the one-particle coefficient of fractional parentage (cfp) $\langle (j^5)^jv'=1j|\}j^6J=0v=0\rangle$ is equal to 1 as the coupling of five particles to the sixth particle to give angular momentum zero and a seniority zero state is unique and the other one-particle cfp has nonzero values for seniority $v=0$ and $v=2$ only. Once again, since $J=0$, the U coefficient is equal to 1. Hence the two-particle cfp $\langle (j^4)^{I_0}v(j^2)^{I_0}v'|\}j^6J=0v=0\rangle$ is equal to the one-particle cfp $\langle (j^4)^{I_0}vj|\}j^5J=jv'=1\rangle$. Therefore the nonzero numbers in the column vectors (or wave functions) for the $T=3$ state in ^{46}Ti and $T=5/2$ (which correspond to the $v'=1$, $J=j$ with five particles in the $j=7/2$ shell) state in ^{45}Sc are the same and they correspond to the nonzero values of the cfp's and they can be analytically calculated [5,6] as (it should be noted that the cfp's can be calculated to within an overall phase),

$$\langle (j^{n-1})^0v=0j|\}j^nJ=jv'=1\rangle = \sqrt{\frac{(2j+2-n)}{(n)(2j+1)}},$$

$$\langle (j^{n-1})^{I_0}v=2j|\}j^nJ=jv'=1\rangle = -\sqrt{\frac{2(n-1)(2I_0+1)}{(n)(2j+1)(2j-1)}}.$$

However, it should be noted that although the numbers in the column vectors for the higher isospin states in ^{45}Sc ($J=j$) and ^{46}Ti ($J=0$) are exactly the same, it does not mean that the excitation energies should be exactly in a two to one ratio, and indeed they are not. The excitation energy of the $T=3$ state in ^{46}Ti is calculated to be 12.9491 MeV while twice the excitation energy of the $T=5/2$ state in ^{45}Sc is 12.9558 MeV. The reason for the discrepancy is that the excitation energy is the difference of the energies of the higher isospin state and the ground states. The coefficients for the ground states of the two nuclei are slightly different and it is this fact that causes a small but real deviation from the two to one ratio.

V. HIGHER SENIORITY ADMIXTURES IN PERTURBATION THEORY

The approximate two to one relationship for ^{46}Ti and ^{45}Sc also applies to the cross conjugate pair in which protons and neutron holes are interchanged as well as neutrons and proton holes. The pair in question is ^{50}Cr and ^{51}Cr. If we examine the Nuclear Data Sheets [7], we find that there is not sufficient data for the pair [^{46}Ti,^{45}Sc]; i.e., even though the $T=3$ 0^+ state in ^{46}Ti is observed at 14.153 MeV, the corresponding $T=5/2,7/2^-$ state in ^{45}Sc is still missing, but there is for [^{50}Cr,^{51}Cr]. The $T=3-T=1$ splitting in ^{50}Cr is 13.222 MeV and the $T=5/2-T=3/2$ splitting is 6.611 MeV. This is amazing; the two to one relationship holds to four significant digits.

The closeness of the results leads us to ask if we have gone as far as one can go in the previous sections. The answer is no. From Table II we can evaluate the calculated percent admixtures of $v=4$ components in the ground states of ^{46}Ti and ^{45}Sc. The respective values are 2.232% and 2.335%. They are almost the same. The results are not so good for higher excited states.

Let us therefore consider seniority 4 admixtures in perturbation theory. Suppose we have obtained approximate ground states for ^{46}Ti and ^{45}Sc by not allowing $v=4$ admixtures. The approximate wave functions will be

$$^{46}\mathrm{Ti:}\quad |\phi_0\rangle = \sum_{L;v=0,2}\tilde{D}(vLvL)|[LL]^0\rangle,$$

$$^{45}\mathrm{Sc:}\quad |\phi_j\rangle = \sum_{L;v=0,2}\tilde{C}(vL)|[jL]^j\rangle.$$

Let us now consider the matrix element which couples seniority 4 admixtures in ^{46}Ti:

$$M=\sum_{L'v=0,2}\tilde{D}(vL'vL')\langle [L'L']^0|V|[L(Lv=4)]^0\rangle.$$

There will be no contribution from the proton-proton interaction because of the orthogonality of the neutron wave functions $\langle L'v\neq 4|Lv=4\rangle=0$. There will be no contribution from the neutron-neutron interaction because $\langle L'v\neq 4|V|Lv=4\rangle=0$; i.e., as mentioned before seniority is a good quantum number for particles of one kind in the $f_{7/2}$ shell [5].

The only contribution is from the proton-neutron interaction. Using the same techniques (which are independent of seniority) as in previous sections we obtain

$$M=2\sum_{L'v=0,2}\tilde{D}(vL'vL')\langle[jL']^j|V|[j(Lv=4)]^j\rangle.$$

This is exactly twice the corresponding mixing matrix element for ^{45}Sc, except for the fact that for ^{45}Sc one can have $L=5$, $v=4$, but not in ^{46}Ti.

In perturbation theory the energy shift is given by

$$\Delta E^{\alpha}=\sum_i \frac{|\langle iv=4|V|\phi_0^{\alpha}\rangle|^2}{(E_\alpha-E_i)},$$

where i is a state in which the four neutrons have seniority four. We shall see that we have to consider the energy denominators with considerable care.

It should be noted that the matrix element $\langle[L_p=0L_n=0v=0]^0|V|[L_pL_nv=4]^0\rangle$ vanishes; i.e., there is no direct coupling from a state in which the neutrons have seniority 4 to one in which the neutrons have seniority 0. This is due to the fact that the coefficients of fractional parentage $\langle(7/2)^3k=7/2;7/2|\}7/2^4Lv=4\rangle$ vanish for all $Lv=4$ states. Thus the nonvanishing of the matrix element $\langle i,v=4|V|\psi_0^{\alpha}\rangle$ comes from the presence of $v=2$ admixtures in $|\phi_0^{\alpha}\rangle$, the most important component in the ground state being $L=2$, $v=2$.

Let us next look at the four-neutron excitation energies, i.e., the calculated spectrum of ^{44}Ca. The results in MeV are as follows:

J	$v=2$	J	$v=4$
2	1.509	2*	3.853
4	2.998	4*	2.463
6	3.400	5*	4.117
		8*	5.709

Concerning the energy denominator $E_\alpha-E_i$, if it were a factor of 2 larger in ^{46}Ti than it is in ^{45}Sc, then the energy shift ΔE^{α} would also be a factor of 2 larger in ^{46}Ti than in ^{45}Sc. However, this is not precisely true. This can be seen in Eq. (7). The neutron-proton interaction is increased by a factor of 2—this is good. But consider the diagonal nn and pp interactions.

In ^{46}Ti the expression is

$$V_{pp}^L+V(f_{7/2}^4)^L,$$

whereas in ^{45}Sc it is just $V(f_{7/2}^4)^L$. As mentioned before for seniority 2 states,

$$V(f_{7/2}^4)^L=V(f_{7/2}^2)^L+\text{const.}$$

However, this is not the case for $v=4$ states. The dominant "$v=4$" admixture in the ground state of ^{46}Ti is $(L_p=4L_n=4^*]$ with amplitude of -0.1383; in ^{45}Sc it is $C[jL_n=4^*]$ with an amplitude of -0.1462.

The closeness of the results can be explained by the accidental fact that the $L=4$, $v=2$ and $L=4$, $v=4$ states are rather close in energy, 2.998 MeV and 2.463 MeV, respectively. This means that indeed the energy denominator in ^{46}Ti is almost twice that in ^{45}Sc and hence the energy shift ΔE^{α} is almost a factor of 2 larger.

Our explanation above is rather detailed and not so pretty, but it is accurate. It relies on being able to use perturbation theory, and this can only be justified when the dominant $v=2$ admixture has $L=2$ and not $L=4$, i.e., for the ground states of the two nuclei. There will be no energy shifts for the states of higher isospin because they have no $v=4$ admixtures.

The spectroscopic strength for the proton pickup reaction, e.g., $^{46}\text{Ti}^{J=0\alpha}(d,^3\text{He})^{45}\text{Sc}^{J=j\beta}$, can be calculated as twice the square of the overlaps of the corresponding wave functions given in Table II. We find that the strength is large for several α,β combinations. For example, from the ground state of ^{46}Ti to the ground state of ^{45}Sc one exhausts 99.96% of the total strength. From the first excited state in ^{46}Ti to the first excited $J=j$ state the value is 93.5%. From the $T=3$ double analog state in ^{46}Ti to the $T=5/2$ single analog state in ^{45}Sc the value is 100%. There is one more pair with a rather large strength of 86.0%. However, for the other combinations where the $L=5$ neutron component admixtures are large in ^{45}Sc (which are absent in the $J=0$ states of ^{46}Ti), the spectroscopic strengths are rather small. Therefore spectroscopic strength analysis not only corroborates our perturbative analysis for the ground state but also extends its scope to other excited states.

Thus for selected states the two to one ratio holds better than we would expect from merely truncating in seniority—it holds when higher seniority states are admixed in perturbation theory.

VI. ADDITIONAL REMARKS

As mentioned in [3], Sherr [8] noted that a simple interaction $a+bt_1t_2$, where a and b are constants, will lead to a two to one ratio for excitations of states of higher isospin, not only in the nuclei covered thus far but also for the pairs ^{44}Ti,^{45}Ti, and ^{46}Ti,^{47}Ti. Indeed the percent deviation for these nuclei is small, -0.98% and -1.54%, respectively. However, these nuclei do not have any two to one relationship predicted for the states of lower isospin and the counting of states is quite different. In ^{44}Ti there are 4 $J=0$ states in the $f_{7/2}$ shell while in ^{45}Ti there are 17 $J=j$ states. The corresponding values for ^{46}Ti and ^{47}Ti are 6 and 17.

There is one comment worth making about the seniority content of the $J=0_1^+$ state in ^{46}Ti and the $j=7/2^-$ state in ^{45}Sc. While the $L_n=2$, $v=2$ probability in the states is much larger than the $L_n=2$, $v=4$, we find that for $L_n=4$, the $v=4$ probability is somewhat larger than $v=2$. This can be understood in terms of boson models. Roughly speaking, the $L_n=2$, $v=2$ state corresponds to a single *d* boson whereas the $L_n=2$, $v=4$ state corresponds to two *d* bosons coupled to $L_n=2$. It is not surprising that one-*d*-boson admixture in the ground states should be larger than the two-*d*-boson admixture.

For $L_n=4$ the $v=2$ state corresponds to one *g* boson while the $v=4$ state corresponds to two *d* bosons [6]. The *g* boson is at about twice the energy of the *d* boson and this fact causes the admixture of two *d* bosons to be comparable to the amount of one *g* boson in the ground state.

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