Neutron matter model

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The Bertsch, nonparametric model of neutron matter is analyzed and strong indications are found that, in the infinite system limit, the ground state is a Fermi liquid with an effective mass, except for a set of measure zero. [S0556-2813(99)05510-7]

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As a challenge, Bertsch [1] proposed the following model, inspired as a parameter-free model of neutron matter at subnuclear density. The model is described as follows:

What are the ground state properties of the many-body system composed of spin-1/2 fermions interacting via a zero-range, infinite scattering-length contact interaction.

It may be assumed that the interaction has no two-body bound states. Also, the zero range is approached with finite-ranged forces and finite particle number by first taking the range to zero and then the particle number to infinity.

This problem is tricky in the following sense: if one reverses the limit order and takes the particle number to infinity before the range goes to zero, one obtains the well-known nuclear collapse result [2] where the whole system collapses into a region of the order of the range of the potential in size. Likewise, if the particles were bosons, collapse would occur.

To solve this problem as stated, I employed the formalism in [3]. Here we continue to use the summations over states as described therein, before the limit as the particle number tends to infinity converts those sums into integrals. This method involves series expansions in powers of the Fermi momentum or alternatively in powers of the strength of the potential.

For ease of illustration, I will use the following potential:

$$V(r)=\begin{cases}-V_0 & \text{if } r<c\\ 0 & \text{if } r>c.\end{cases} \tag{1}$$

In point of fact, the shape of the potential will not matter, as we will see later. It is useful to note the dimensionless strength or well-depth parameter of this potential is [2]

$$s=\frac{4}{\pi^2}\frac{MV_0}{\hbar^2}c^2. \tag{2}$$

The strength is defined in such a way that $s=1$ corresponds exactly to the case of an infinite scattering length, as called for in this model. The range of this potential is just c, of course. For the limit as $c\to 0$ one may compute, using the standard equations [4], that the phase shift vanishes for all $l>0$. That is to say, this interaction is only effective in S states. This result should not be surprising as the angular-momentum barrier excludes the wave function from the origin where the potential is effective. For the current case, we may put two neutrons in the lowest state (one with spin up and the other with spin down). This creates a potential for a possible third neutron of strength $s=2$; however, it must go into the first excited S state, which requires a strength of $s=9$ to bind. [See Eq. (13) below.] If we put (somehow) two neutrons in this state, we get a potential well of $s=4$ to attract a fifth neutron, but now we need a strength of $s=25$ to put it in the third excited S state. And so the argument goes that an $l=0$ only potential of strength $s=1$ cannot bind fermions into a collapsed state, as would be the case if the interaction were effective in all angular momentum states.

It is worthwhile to mention that this type of model is closely related to the theory of Feshbach resonance scattering. This theory is very relevant to the study of atomic Bose-Einstein condensates [5].

The next useful step is to compute the Fourier transform of $V(r)$. We will express things in terms of the dimensionless variables $\vec{\rho}=\vec{r}/c$ and $\vec{\kappa}=c\vec{k}$. We will write this expression as an intergal, but remember that, for the time being, it should really have been a sum because of the finite-sized box in which the finite number of particles are confined:

$$\begin{aligned}\tilde{V}(\vec{k})&=\frac{1}{(2\pi)^3}\int d\vec{r}\,V(r)\exp(-i\vec{k}\cdot\vec{r})\\ &=\frac{c^3}{(2\pi)^3}\int_{|\vec{\rho}|\leqslant 1} d\vec{\rho}\,V(c\rho)\exp(-i\vec{\kappa}\cdot\vec{\rho})=c^3\tilde{v}(\kappa).\end{aligned} \tag{3}$$

Evaluating the integrals we obtain

$$\tilde{v}(\kappa)=-\frac{1}{2\pi^2\kappa^3}[\sin\kappa-\kappa\cos\kappa]\underset{\kappa\to 0}{\to}-\frac{1}{6\pi^2}. \tag{4}$$

Baker [3] [Eq. (4.21)] gives the following expansion for the many-body energy:

$$\frac{E}{N}=\frac{3\hbar^2k_F^2}{10M}+\frac{\pi^2\hbar^2}{4Mc^2}A_1s+\frac{\pi^4\hbar^2}{16Mc^2}A_2s^2+\cdots, \tag{5}$$

as an expansion in the strength of the potential, where k_F is the Fermi momentum. The first term for neutrons is

$$\begin{aligned}A_1=&\frac{3}{4\pi k_F^3V_0}\int_{|\vec{m}|\leqslant k_F,\,|\vec{n}|\leqslant k_F}\\ &\times d\vec{m}\,d\vec{n}\left[\tilde{V}(0)-\frac{1}{2}\tilde{V}(|\vec{m}-\vec{n}|)\right].\end{aligned} \tag{6}$$

This equation may be rewritten as

$$A_1=\frac{3}{4\pi\kappa_F^3 V_0}\int_{|\vec{\mu}|\le\kappa_F,\,|\vec{\nu}|\le\kappa_F}\times d\vec{\mu}\,d\vec{\nu}\left[\tilde{v}(0)-\frac{1}{2}\tilde{v}(|\vec{\mu}-\vec{\nu}|)\right]. \tag{7}$$

Since we are concerned with the limit as $c\to 0$, and finite Fermi momentum, $\kappa_F\to 0$, thus we only need the value of $\tilde{v}(\lambda)$ at $\lambda=0$. Hence, for this case we get

$$A_1=-\frac{\pi}{3}\kappa_F^3. \tag{8}$$

Thus, substituting into Eq. (5) we get

$$\frac{E}{N}=\frac{3}{10}\frac{\hbar^2k_F^2}{M}-\frac{\pi^3\hbar^2k_F^2}{12M}\kappa_F s+\frac{\pi^4\hbar^2}{16Mc^2}A_2 s^2+\cdots. \tag{9}$$

The next step is to examine the higher-order terms of this expansion. The basic organizing principle will be the number of holes in the Fermi sea created for each term. The point is that for each new hole there is a factor $d^3\vec{k}$ and a denominator proportional to k^2, which works out to an additional overall factor of k_F for the term. Note that here there is a major distinction between a finite and an infinite system. In a finite system of N particles, there can never be more that N such holes in the Fermi sea, whereas there is no such limitation for an infinite system. We start with all the two hole-line (in the diagrammatic representation) terms. These are just the so-called ladder graphs as only the filled-state–filled-state interactions do not generate additional holes in the Fermi sea. Baker [3] has carried forward this analysis and also considered the ladder-type insertions in the other terms, and he gives (spin-$\frac{1}{2}$ fermions) for the expansion in powers of the Fermi momentum the results [3] [Eq. (4.88)],

$$\begin{aligned}\frac{EM}{N\hbar^2}=k_F^2\Bigg[&\frac{3}{10}+\frac{1}{3\pi}k_Fa+0.055\,661(k_Fa)^2\\&+\frac{1}{20\pi}(k_Fa)^2(k_Fr_0)+\frac{9}{10\pi}A_1(0)k_F^3\\&+0.009\,14(k_Fa)^3+0.024\,631(k_Fa)^3(r_0a)\\&+0.015\,680(k_Fa)k_F^3A_0''(0)-0.018\,604(k_Fa)^4+\cdots\Bigg],\end{aligned} \tag{10}$$

where a is the scattering length, r_0 is the effective range, and some numerical infelicities that have been noticed since publication are corrected. To this order for spin-$\frac{1}{2}$ and a pure isospin state, terms like $k_F^6a^4\log k_F$ do not occur. Such terms arise from true three-body scattering, when the number of spin and isospin states is three or more. The Pauli exclusion principle prevents their occurrence here. The other quantities are, for the square-well case [3],

$$A_1(0)=\frac{1}{3}c^3\left[1+3\frac{\theta\cot\theta-1}{\theta^2}\right]=\frac{1}{3}c^3\left[1+\frac{3a}{\theta^2(c-a)}\right], \tag{11}$$

and

$$\begin{aligned}A_0''(0)&=\frac{1}{3}c^3\left\{\left[\frac{3}{\theta}-\frac{6}{\theta^3}\right]\tan\theta-1+\frac{6}{\theta^2}\right\}\\&=\frac{1}{3}c^3\left\{\left[3-\frac{6}{\theta^2}\right]\left(1-\frac{a}{c}\right)-1+\frac{6}{\theta^2}\right\},\end{aligned} \tag{12}$$

where use has been made of the equations for θ,

$$\theta=\frac{\pi}{2}\sqrt{s},\quad \frac{\tan\theta}{\theta}=1-\frac{a}{c}. \tag{13}$$

For the time being we will hold the scattering length fixed and finite, and let $c\to 0$. Thus, $r_0,\ A_1(0),\ A_0''(0)\to 0$. What remains is

$$\begin{aligned}\frac{EM}{N\hbar^2}=k_F^2\Bigg[&\frac{3}{10}+\frac{1}{3\pi}k_Fa+0.055\,661(k_Fa)^2+0.00\,914(k_Fa)^3\\&-0.018\,604(k_Fa)^4+\cdots\Bigg],\end{aligned} \tag{14}$$

which just depends on the scattering length a and not on the shape of the potential.

It is to be noticed that to extract the desired result from Eq. (14), we need to sum the series for an infinite value of the argument. Before attempting to do this, we will take some guidance from the ladder approximation. Unfortunately, it is well known that for an attractive potential, the ladder approximation is beset with Emery singularities [6]. It has been argued that these singularities may be the result of an unfortunate choice of a summation method for an asymptotic series. We will instead use the R-matrix formulation [3], where the Emery singularities do not occur. I will use the usual approximations in solving for the R matrix, involving angular averaging and center of mass averaging. For this application we will not need to consider the R matrix in the presence of an excited Fermi sea. The R-matrix equation is very similar to the usual K-matrix equation in ladder approximation, except it has been regularized at the Fermi surface to avoid the Emery singularities. It is

$$R_l(k)=\frac{2}{\pi}\int_0^\infty j_l(kr)V(r)\tilde{u}(r)r^2\,dr, \tag{15}$$

$$\tilde{u}_{k,l}(r)=j_l(kr)-\frac{2}{\pi}\int_0^\infty \tilde{G}_{k,l}(r,r')V(r')\tilde{u}_{k,l}(r')r'^2\,dr', \tag{16}$$

$$\tilde{G}_{k,l}(r,r') = \int_0^\infty \frac{[k''^2 j_l(k''r) j_l(k''r') - k^2 j_l(kr) j_l(kr')]}{k''^2 - k^2} \times F(p,k'')dk'' + \tilde{a}(p,k) j_l(kr) j_l(kr'), \tag{17}$$

$$F(p,k'') = \begin{cases} 1, & \frac{1}{2}p - k_F > k'' \\ 0, & (k''^2 = \frac{1}{4}p^2)^{1/2} < k_F \\ 1, & k'' - \frac{1}{2}p \geq k_F \\ (k''^2 + \frac{1}{4}p^2 - k_F^2)/k''p, & \text{otherwise,} \end{cases} \tag{18}$$

$$\tilde{a} = k^2/k_F, \tag{19}$$

$$\frac{1}{4}\overline{p^2} = \frac{3}{5}k_F^2\left(1 - \frac{k}{k_F}\right) \times \begin{cases} \left[\left(1 + \frac{1}{2}\frac{k}{k_F} + \frac{1}{6}\frac{k^2}{k_F^2}\right) \Big/ \left(1 + \frac{1}{2}\frac{k}{k_F}\right)\right], & k < k_F \\ 0, & k > k_F, \end{cases} \tag{20}$$

where $\overline{p}$ is used to replace p, in order to change the dependence on (k,p) to that on k alone. The single particle energies are then given by

$$E(m) = \frac{1}{2}m^2 + 4\left[\int_0^{(k_F-m)/2} 2k^2\,dk\,I + \int_{(k_F-m)/2}^{(k_F+m)/2} \times \left(1 - \frac{m^2 + 4k^2 - k_F^2}{4km}\right)k^2\,dk\,I\right] m < k_F$$

$$= \frac{1}{2}m^2 + 4\left[\int_{(k_F-m)/2}^{(k_F+m)/2}\left(1 - \frac{m^2 + 4k^2 - k_F^2}{4km}\right)k^2\,dk\,I\right] \times m \geq k_F, \tag{21}$$

where

$$I = \sum_l (2l+1)\begin{pmatrix} \nu - 1, & l \text{ even} \\ \nu + 1, & l \text{ odd} \end{pmatrix} R_l(k), \tag{22}$$

where ν is the number of spin and isospin states and is just 2 in the case of neutron matter. The contribution of the R-matrix terms to the total energy is then given by

$$E_R = \frac{3}{2k_F^3}\int_0^{k_F} [E(m) - \tfrac{1}{2}m^2]m^2\,dm. \tag{23}$$

For the potential under current consideration there is the considerable simplification that V acts only in the $l=0$ states so that $R_l = 0 \quad \forall l > 0$.

The difference between the K matrix in ladder approximation and the R matrix is in the Green's function. For the K matrix the Green's function is

$$G_{k,l}(r,r') = \int_0^\infty \frac{k''^2 j_l(k''r) j_l(k''r')}{k''^2 - k^2} F(p,k''). \tag{24}$$

It has been shown [3] that

$$K_l(k) = \frac{R_l(k)}{1 + (\frac{1}{2}\tau_1 - \tilde{a})R_l(k)}, \tag{25}$$

where

$$\tau_1 = (k_F p)^{-1}\left\{(k^2 + \tfrac{1}{4}p^2 - k_F^2)\ln[(k_F^2 + k_F p + \tfrac{1}{4}p^2 - k^2)/(k_F^2 - \tfrac{1}{4}p^2 - k^2)] + \left(1 - \frac{p^2}{4k_F^2}\right)\ln[(k_F + \tfrac{1}{2}p)/(k_F - \tfrac{1}{2}p)]\right\} + \left(\frac{k}{k_F}\right)\ln[(k_F + \tfrac{1}{2}p + k)/(k_F + \tfrac{1}{2}p - k)]. \tag{26}$$

The quantity τ_1 is lower semibounded, but diverges logarithmically to $+\infty$. Thus by Eq. (25), any time that R is negative there is a singularity in K. One consequence of this result is that although it is expected that the radius of convergence of R in powers of the strength s is unity, the radius of convergence of the K-matrix series is zero. As a further point in this regard, if Eq. (25) is expanded in powers of R, since the divergence of τ_1 is only logarithmic, the intergals of τ_1^j all exist. However, the values increase like $j!$ so that this series is at best an asymptotic one. In the numerical solution of the R-matrix equations, I have found that 59 mesh points in r, 10 mesh points in k, and 20 mesh points in k'' are sufficient for a few percent accuracy, which is in turn sufficient for our present needs. The computer code I have used is an adaptation of that of Ref. [7]. Our numerical results for several strengths are plotted in Fig. 1. It is to be noticed that in line with Eq. (27), the slope at the origin is negative and is increasing rapidly in size with increasing well depth. However, outside a small initial region the curve for $E_R M/N\hbar^2 k_F^2$ is relatively flat. This behavior is strongly suggestive of the idea that, in the limit as the strength of the potential goes to unity and the scattering length goes to infinity, this curve is discontinuous at the origin, and is some reasonable function of k_F for $k_F > 0$.

We cannot use the above method for the complete energy, but there are two other methods that we can use, and are also available for the R-matrix energy.

An examination of the structure of the terms generated in the expansion of the R-matrix expansion in powers of the potential strength shows that they are all proportional to moments of a distribution. For the case of the ladder approximation to the K matrix the Green's function is a positive definite operator, but this happy feature is not necessarily true for the Green's function for the R matrix. Thus even for the current case of a single-signed potential (for background

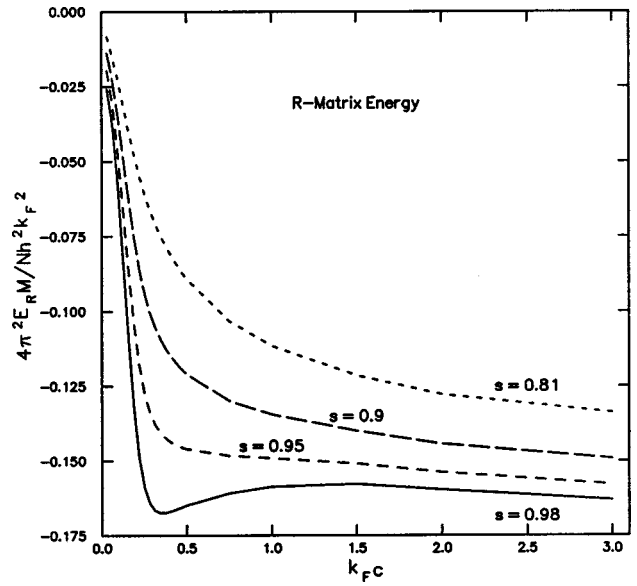

FIG. 1. The numerical evaluation of the *R*-matrix energy. The short dashed curve is for $s=0.81$, the long dashed curve is for $s=0.9$, the dashed curve is for $s=0.95$, and the solid curve is for $s=0.98$.

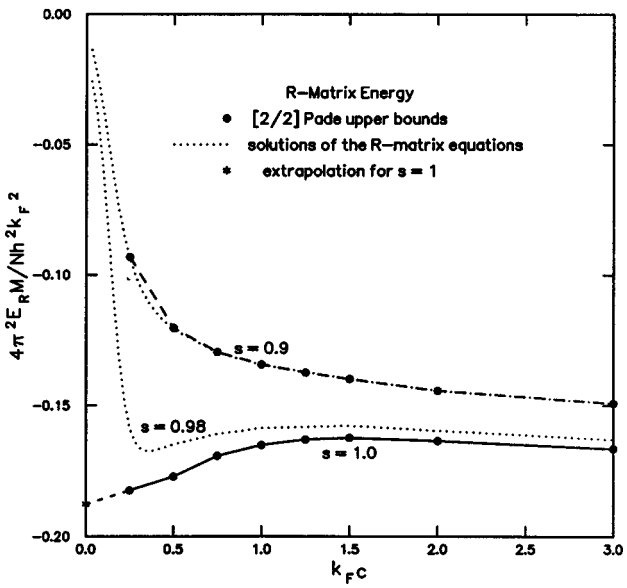

FIG. 2. The Padé approximant upper bounds on the *R*-matrix-approximation energy divided by $\hbar^2 k_F^2/M$ for various potential strengths. Some of the numerical solutions of the *R*-matrix equation are included for reference.

see [8]), the distribution can be over a range with two signs. Consequently we have what is called the Hamburger moment problem [9] with a finite radius of convergence. In this case by means of a linear fractional transformation the problem can be converted into $-s$ times a series of Stieltjes. Such a series is the moment generating function for a moment problem whose distribution has a range that has just a single sign [9]. It has been shown [9] that in our case of an attractive potential whose strength is less than the radius of convergence of the *R*-matrix expansion, all the Padé approximants form upper bounds to the actual *R*-matrix energy [9]. We have deduced the necessary coefficients by using a tenth-order finite difference method (equivalent to fitting with a tenth-order polynomial) at several values of $k_F c$ on the numerical solution of the *R*-matrix equation. We have used a spacing of $1/(4\pi^2)$ in the strength for this computation. Our results are shown in Fig. 2 and some of the numerical solutions of the *R*-matrix equation are shown for comparison. The result we seek is for a zero range force, $c\to 0$ which is extrapolated in this figure. It is about $-0.18\hbar^2 k_F^2/M$.

The series analogous to that of Eq. (14) is also available [3] for the ladder energy. It is, for the $c=0$ limiting case,

$$\frac{\Delta E_L M}{N\hbar^2}=k_F^2\left[\frac{1}{3\pi}k_F a+0.055\,661(k_F a)^2+0.032\,031(k_F a)^3 +0.019\,156(k_F a)^4+\cdots\right]. \tag{27}$$

In this equation the numbers were supplied by numerical integration of some double intergals, and Monte Carlo evaluation was not required, as was the case in Eq. (14). The coefficients are just the g_j's of Ref. [3]. By the solution of Eq. (25) for *R* as a function of *K* we find that *R*, in this limit as $c\to 0$, must also be expandable in powers of $k_F a$. To derive the coefficients in this expansion, we would have needed to use similar analytical methods [3] to those employed in the case of the *K* matrix. We can, however, see at least part of the picture by computing for a series of small values of $k_F c$ the results for several fixed values of $k_F a$. The determination of the appropriate values of the strength parameters needed for this exercise is accomplished by expanding Eq. (13) for a/c in powers of θ^2. Then this series is reverted to give θ^2 in powers of a/c. A [3/3] Padé approximant with the built in asymptotic value of $\pi^2/4$ is found to be quite accurate and to give good results for $\theta^2=\pi s/2$. I have again used a tenth-order differencing scheme on the numerical solution of the *R*-matrix equations. We have chosen a mesh spacing of $k_F c=0.05$. The results of the extrapolation of the *R*-matrix energies to $k_F c=0$ are displayed in Fig. 3. These values are not inconsistent with the asymptotic extrapolation shown in Fig. 2 for $s=1$ for the *R*-matrix energy.

Since the leading coefficient here should also be $1/(3\pi)$ as in Eq. (27), we find that our extrapolation in Fig. 3 is about 2% low for this term. Our numerics are insufficient to give good values of the rest of the terms. Merely for reference, we display in Fig. 4 several Padé approximant [9] estimates from the series [Eq. (27)] for the Ladder energy approximation. We are interested, as we will be in the analysis of Eq. (14), in the asymptotic behavior as $a\to\infty$. The asymptote for the [2/2] Padé approximant is about $0.24\hbar^2 k_F^2/M$, which is not vastly different from our estimates for the *R*-matrix energy, and also corresponds to no negative energy ground state.

For the case of the complete energy for this model, I illustrate this behavior in Fig. 5. Again I use the method of Padé approximants [9] in order to sum the series in $k_F a$.

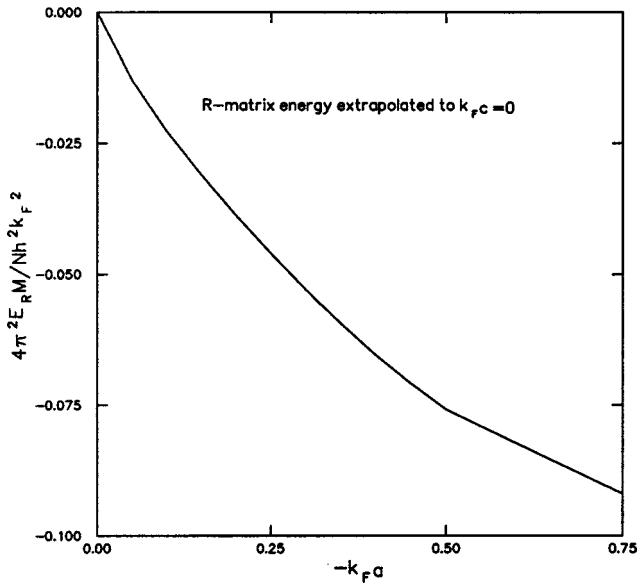

FIG. 3. The extrapolation of the R-matrix energy to $k_F c=0$ as a function of $k_F a$.

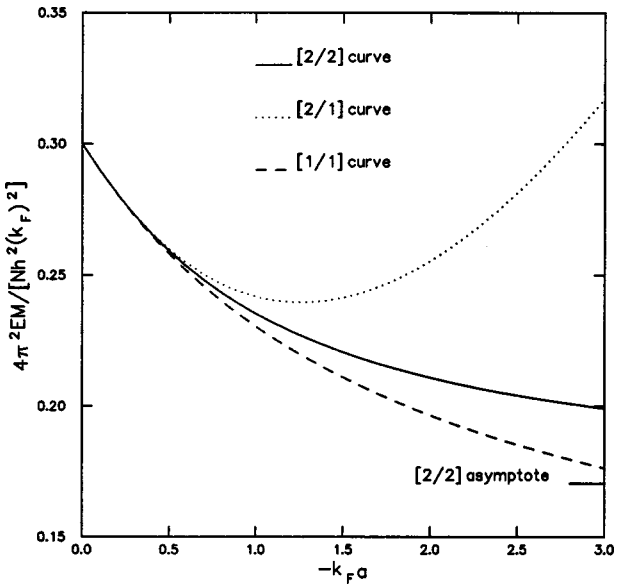

FIG. 5. The ratio of the many-body energy per particle to $\hbar^2 k_F^2/M$ vs $-ak_F$. For the case of interest, $a\rightarrow-\infty$ is expected.

There is an upper limit of 0.3 for the ratio $EM/[N\hbar^2 k_F^2]$, because the interaction is purely attractive; so the energy cannot exceed the kinetic energy of the ideal Fermi gas. All the estimates shown agree well to at least $k_F a\approx -0.5$. The [2/1] Padé approximant exceeds the upper bound for $-k_F a \geqslant 2.5$, and so does not give a satisfactory value for the case of interest, $a\rightarrow-\infty$. This brings us to the [1/1] and the [2/2] Padé approximant. They both have finite asymptotes, which are of the right order of magnitude. That for the [2/2] is shown in Fig. 5. Numerically, the asymptotes are $0.0977\hbar^2 k_F^2/M$ for the [1/1] and $0.1705\hbar^2 k_F^2/M$ for the [2/2]. The latter corresponds to a shift in the complete energy from the ideal gas energy of $\Delta E=-0.1295\hbar^2 k_F^2/M$.

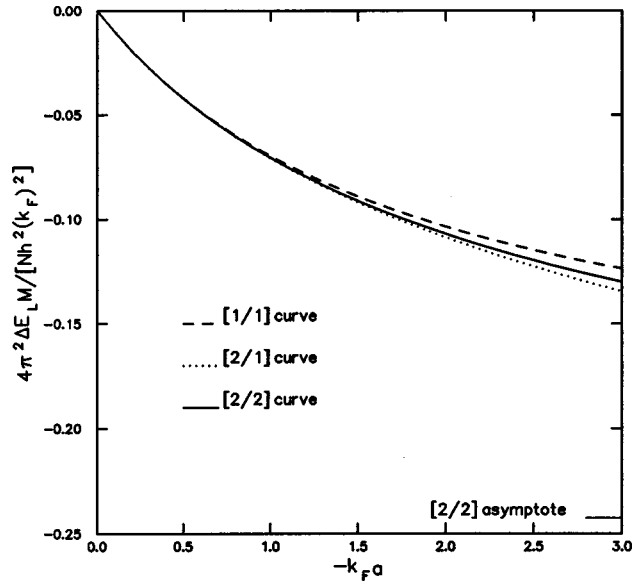

FIG. 4. The ratio of the ladder terms contribution to the energy per particle divided by $\hbar^2 k_F^2/M$.

These results correspond to the effective masses $M^*=3.07M$ and $M^*=1.76M$, respectively. Manifestly, neutron matter is unbound for this potential.

In addition I have analyzed the series in Eq. (9) for various densities, using the data given in Ref. [3]. The [3/1] Padé approximants are the best behaved ones in this case and I have plotted them in Fig. 6. The extrapolated asymptotic value is about $-0.17\hbar^2 k_F^2/M$ vs the value of about $-0.13\hbar^2 k_F^2/M$ just quoted for the Fermi momentum series value just quoted above. In light of Fig. 2, I estimate a value of $(-0.17\pm0.04)\hbar^2 k_F^2/M$, which corresponds to an effective

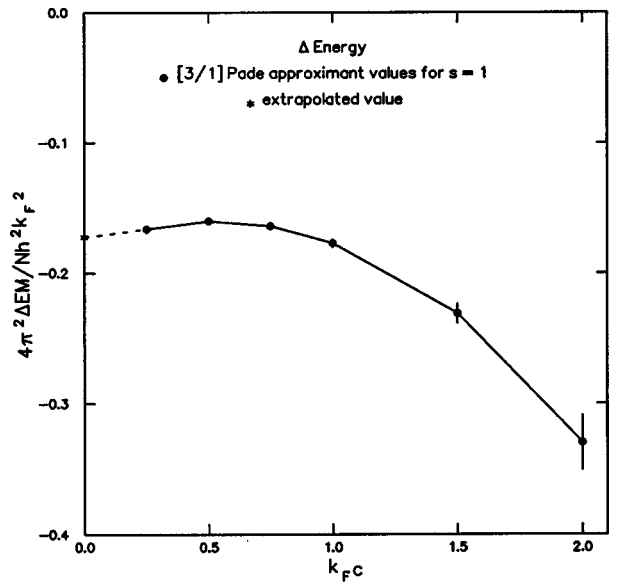

FIG. 6. The estimates of the many-body energy per particle based on the series expansions in potential strength. The extrapolation to $k_F c=0$ is also shown. The error bars reflect only the coefficient uncertainty.

mass of about 2.3 ± 0.5.

These results for the ground-state energy mean that the system is a Fermi liquid, with an effective mass. The wave function is expected to correspond to that structure, aside from a set of exceptional points where $\vec{r}_i = \vec{r}_j$, the origins of the set of relative coordinates between all the pairs. These points, however, only constitute a set of measure zero.

I would mention that relatively simple models of neutron matter with a potential consisting of a repulsive core and an attractive part beyond have been studied [10] with reasonable results. The potential used there gives a pretty good representation of the low to medium energy scattering data.

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