Poincaré invariant exchange model of pion-nucleon scattering

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(Received 12 April 1999; published 26 August 1999)

Pion-nucleon scattering up to a pion laboratory kinetic energy of 700 MeV is described by a Poincaré invariant, instant form model. The model is constructed in a space spanned by single-baryon states $|B\rangle$, where B is the nucleon, or any resonance that contributes in the energy range considered; and by meson-baryon states $|\mu B\rangle$, where $|\mu B\rangle=|\pi N\rangle$, $|\pi\Delta\rangle$, or $|\eta N\rangle$. The model specifies a mass operator in the form $M=M_0+U$, where M_0 is a noninteracting mass operator and U contains the interactions. The $\langle\pi N|U|\pi N\rangle$ potentials are derived from N, Δ, σ, and ρ exchange processes. The vertex interactions $\langle\mu B|U|B'\rangle$ are derived from field theory interaction Hamiltonians. Coupling to the inelastic channels, $|\pi\Delta\rangle$ and $|\eta N\rangle$, is provided by a $\langle\pi N|U|\pi\Delta\rangle$ transition potential due to nucleon exchange; and by interactions of the form $\mu B\Leftrightarrow B''\Leftrightarrow\mu' B'$. [S0556-2813(99)02809-5]

PACS number(s): 13.75.Gx, 11.80.−m, 24.10.Jv

I. INTRODUCTION

Particle exchange models have been successful in describing the nucleon-nucleon interaction [1], electroweak interaction currents [2,3], meson-meson scattering [4], and meson-nucleon scattering [5–15]. Here we are concerned only with pion-nucleon scattering. In the historically important Chew-Low model for pion-nucleon scattering [5], the only interaction included was the $\pi N\Leftrightarrow N$ vertex. In spite of the simplicity of this model, it was able to reproduce the $\Delta(1232)$ resonance. The discovery of other mesons, in particular, the rho meson (ρ) [16], led to the inclusion of other vertices such as $\rho\pi\Leftrightarrow\pi$ and $\rho N\Leftrightarrow N$. The development of the quark model and QCD have also had an important impact on exchange models of the pion-nucleon system. In particular, it is now widely believed that baryons such as the $\Delta(1232)$ resonance are just as elementary as the nucleon. This change in viewpoint quickly led to the development of effective field theories in which, for example, the π, N, and Δ are the quanta of the fields [17].

An effective Lagrangian for the pion-nucleon system does not lead to unique predictions for pion-nucleon scattering, as there is no exact procedure for deriving few-particle equations from a quantum field theory. A common approach for deriving integral equations for the pion-nucleon system is to use a three-dimensional reduction of the ladder, Bethe-Salpeter equation [6,7,13–15]. Even within this framework, there is a lack of uniqueness due to the freedom in choosing the reduction scheme. Another approach is to use a straightforward relativistic generalization of the Lippmann-Schwinger equation, in which the role of the potentials is played by amplitudes derived using time-ordered perturbation theory [9–12].

Alternatively, it is possible to derive an effective Hamiltonian from the model Lagrangian. The Tamm-Dancoff method [18] can be used to accomplish this, however, it has the disadvantage of leading to an energy dependent Hamiltonian. An energy independent Hamiltonian can be obtained by developing a unitary transformation that block diagonalizes the quantum field theory Hamiltonian in its Fock space. The effective Hamiltonian acts in the subspace of Fock space that is relevant to the few-particle system of interest. A number of workers have contributed to the development of this approach [19–21]. Here we will use Okubo's formulation [20]. The unitary transformation approach has proved to be very useful in investigating nuclear electromagnetic currents [22–25], and in the construction of nuclear interactions [26–30]. Sato and Lee [8] have used this method to construct a model for πN scattering, and the reaction $\gamma N\to\pi N$.

Fuda and Zhang have developed a variation of this approach, in which an effective mass operator, rather than an effective Hamiltonian, is derived [31–35]. The mass operator so determined serves as the essential ingredient in a Bakamjian-Thomas construction [36,37] of a Poincaré invariant model. Such a model specifies a complete set of Poincaré generators which satisfy exactly the algebra of the Poincaré group, the so-called Poincaré algebra. Moreover, with such a model, there exists a set of unitary operators which provide a representation of the Poincaré group, and can be used to map the quantum mechanical state vectors from one inertial frame to another. This approach is three dimensional in character, and is not manifestly covariant; however, it can be shown that the S-matrix elements obtained transform properly from one inertial frame to another, i.e., probabilities are invariant. A general proof of this claim has been given by Coester and Polyzou [38]. A less general, but more transparent, proof has been given [39] by Fuda. The mass operator–Okubo approach was first developed in the context of the front form of relativistic quantum mechanics, and used to develop simple models of the πN and NN systems [31,32]. The method was subsequently used to develop a realistic, front form, one-boson exchange model of the NN system [33]. The methodology was extended to the instant form in the context of a simple model of the πN system [34], and then used to obtain an instant form, one-boson exchange model of the NN system [35].

These earliest applications of the mass operator–Okubo method only dealt with elastic scattering. The extension to coupled channels and inelastic scattering was made in the context of a front form model for πN scattering [40,41]. This model was constructed in a space spanned by single-baryon states $|B\rangle$, where B is the nucleon, or any resonance that contributes in the energy range considered; and by the meson-baryon states $|\pi N\rangle$, $|\pi\Delta\rangle$, and $|\eta N\rangle$. The model specifies a mass-square operator in the form $M^2 = M_0^2 + V$, where M_0 is the noninteracting mass operator, and V contains the interactions. The purely phenomenological interactions consist of vertex interactions, $\langle \mu B|V|B'\rangle$, and two-particle potentials, $\langle \mu B|V|\mu' B'\rangle$, where the potentials are assumed to be separable. This model gives a remarkably good fit to the πN elastic scattering amplitudes up to a pion laboratory kinetic energy of 1.0 GeV [41].

The original coupled channel formalism [40] has recently been extended to include a photon-nucleon channel [42]. This extension satisfies not only Poincaré invariance, but also gauge invariance, and provides a framework for calculating meson photoproduction from the nucleon.

The success of our purely phenomenological model for πN scattering has naturally led us to develop a physically more satisfying model, in which the interactions are derived from an effective Lagrangian. Here we report the results of such a model. The model specifies a mass operator in the form $M = M_0 + U$, where M_0 is the mass operator for the system without interactions, and U contains the interactions. The model space consists of single-baryon states $|B\rangle$, and meson-baryon states $|\mu B\rangle$. The single-baryon states include the nucleon N, and the following resonances: $\Delta = P_{33}(1232)$, $R = P_{11}(1440)$, $D_{13}(1520)$, $S_{11}(1535)$, and $S_{31}(1620)$; while for the meson-baryon states, we have $|\mu B\rangle = |\pi N\rangle, |\pi\Delta\rangle, |\eta N\rangle$. The model accounts for πN-elastic scattering up to a pion laboratory kinetic energy of 700 MeV, which corresponds to a total c.m. energy of $W = 1574$ MeV. Our single-baryon states include all of the resonances in this range, as well as the $S_{31}(1620)$ resonance, which lies somewhat above this range, but turns out to have a non-negligible effect. Its width is 150 MeV. The two-particle, inelastic channels which have thresholds below $W = 1574$ MeV are the $\pi\Delta$ and ηN channels, with thresholds at $W = 1372$ MeV and $W = 1486$ MeV, respectively. The $\pi\pi N$ channel, which we assume to be approximated by the $\pi\Delta$ channel, has its threshold at $W = 1218$ MeV.

We have chosen to develop our mass operator in the framework of the instant form of relativistic quantum mechanics, since with this form it is straightforward to construct mass operator matrix elements from the effective, few-particle Hamiltonian obtained with the Okubo method [20]. This is a simple consequence of the fact that in a c.m. frame, the action of the mass operator and the Hamiltonian is the same. In modeling the $\langle \pi N|U|\pi N\rangle$ matrix elements, we have made the rather common assumption that this interaction is due to N, Δ, ρ, and σ exchange. We have also included a $\langle \pi N|U|\pi\Delta\rangle$ transition potential due to N exchange. The other interactions that are included in the model are the vertex interactions $\langle \mu B|U|B'\rangle$, where we allow $B' = N$, $P_{11}(1440)$, $D_{13}(1520)$, and $S_{31}(1620)$ to couple to the $\mu B = \pi N$ and $\pi\Delta$ channels, $B' = \Delta$ to couple to the $\mu B = \pi N$ channel, and $S_{11}(1535)$ to couple to the $\mu B = \pi N$ and ηN channels.

The outline of the paper is as follows. In Sec. II a brief description of the Bakamjian-Thomas method [36,37] for constructing Poincaré invariant models is given, and the restrictions on a mass operator necessary to ensure Poincaré invariance are stated. The essential relation for constructing effective interactions with the Okubo method [20] is presented at the beginning of Sec. III. The application of the method to the pion-nucleon system is illustrated by a derivation of the effective interactions that arise from the well known $\pi N \Leftrightarrow N$ vertex. In particular, the $\langle \pi N|U|N\rangle$ and $\langle \pi N|U|\pi N\rangle$ matrix elements that arise from this vertex are constructed. A brief presentation of the $\langle \pi N|U|\Delta\rangle$ and $\langle \pi N|U|\pi N\rangle$ contributions to U arising from the $\pi N \Leftrightarrow \Delta$ vertex is also given, as well as the basic formula for obtaining the $\langle \pi N|U|\pi\Delta\rangle$ transition potential due to N exchange. The method for constructing the $\langle \pi N|U|\pi N\rangle$ potentials due to σ and ρ exchange is not given here as it has been given previously in Ref. [34].

The derivation of the Lippmann-Schwinger equations that arise from the type of mass operator developed here has also been given previously [40], so the basic equations are simply stated in Sec. IV. Here some useful identities for carrying out a partial wave analysis of the Lippmann-Schwinger equations are also given. The results obtained for the parameters in our model, by fitting to an experimental phase shift analysis, are given in Sec. V. A discussion of the results and future prospects for the model is presented in Sec. VI. The interaction Lagrangian densities and interaction matrix elements not given in Sec. III are presented in the Appendix.

Throughout we work in units in which $\hbar = c = 1$.

II. GENERAL BACKGROUND

A Poincaré transformation is a linear, inhomogeneous transformation that maps the components of a space-time vector x associated with one inertial frame to the components of a vector x' associated with another inertial frame according to the relation

$$x' = ax + b, \tag{2.1}$$

where b is a vector and a is a Lorentz transformation, which for proper transformations can be parametrized in the form

$$a = \exp[i(\boldsymbol{\omega}\cdot\mathbf{k} + \boldsymbol{\theta}\cdot\mathbf{j})]. \tag{2.2}$$

Here $\mathbf{j}$ is the generator of three-rotations, $\mathbf{k}$ is the generator of rotationless boosts, and $\boldsymbol{\omega}$ and $\boldsymbol{\theta}$ are three-vectors whose components provide the necessary parameters. In a satisfactory relativistic model, there exists a unitary operator $U(a,b)$, corresponding to the Poincaré transformation (a,b), that maps a quantum mechanical state vector $|\psi\rangle$ associated with the x frame to the vector $|\psi'\rangle$ associated with the x' frame according to

$$|\psi'\rangle = U(a,b)|\psi\rangle, \tag{2.3}$$

where for proper transformations the unitary operator can be parametrized in the form

$$U(a,b) = \exp(ib\cdot P)\exp[i(\boldsymbol{\omega}\cdot\mathbf{K} + \boldsymbol{\omega}\cdot\mathbf{J})], \tag{2.4}$$

$$P = (H, \mathbf{P}). \tag{2.5}$$

Here **K** is a boost operator, **J** is the angular momentum operator, H is the Hamiltonian of the system, and **P** is the three-momentum operator. Since the law of combination for the Poincaré transformations is $(a',b')\circ(a,b)=(a'a,a'b+b')$, the unitary operators must combine according to

$$U(a',b')U(a,b)=U(a'a,a'b+b') \tag{2.6}$$

so as to provide a representation of the Poincaré group. This implies a set of commutation rules for the generators $\{H,\mathbf{P},\mathbf{K},\mathbf{J}\}$, which is commonly referred to as the Poincaré algebra [37].

In constructing the ten generators $\{H,\mathbf{P},\mathbf{K},\mathbf{J}\}$, it is convenient to work with another set of ten Hermitian operators, i.e., $\{M,\mathbf{P},\mathcal{J},\mathbf{X}\}$, where M is the mass operator, $\mathcal{J}$ is a spin operator, and **X** is the so-called Newton-Wigner position operator. This second set of operators satisfies a much simpler set of commutation rules than the Poincaré algebra; in fact the only nonzero commutators are

$$[P^m,X_n]=-i\delta_{mn}, \quad [\mathcal{J}^l,\mathcal{J}^m]=i\varepsilon_{lmn}\mathcal{J}^n. \tag{2.7}$$

The three-momentum operator **P** is common to both sets, while the other generators can be expressed in terms of the operators of the second set by the relations [36,37]

$$H=(\mathbf{P}^2+M^2)^{1/2}, \tag{2.8a}$$

$$\mathbf{J}=\mathbf{X}\times\mathbf{P}+\mathcal{J}, \tag{2.8b}$$

$$\mathbf{K}=-\frac{1}{2}(\mathbf{X}H+H\mathbf{X})-\frac{\mathbf{P}\times\mathcal{J}}{M+H}. \tag{2.8c}$$

It can be shown that if the commutators of the set $\{M,\mathbf{P},\mathcal{J},\mathbf{X}\}$ are zero, except for those given by Eq. (2.7), then the generators given by Eq. (2.8), in combination with **P**, satisfy the Poincaré algebra.

In the Bakamjian-Thomas construction [36,37] of the set $\{M,\mathbf{P},\mathcal{J},\mathbf{X}\}$, the operators **P**, $\mathcal{J}$, and **X** are chosen to be the same as those for the system of particles without interactions, while the mass operator M contains interactions. The commutation rules for **P**, $\mathcal{J}$, and **X** are then automatically satisfied, and it is only necessary to ensure that

$$[M,\mathbf{P}]=[M,\mathcal{J}]=[M,\mathbf{X}]=0. \tag{2.9}$$

III. CONSTRUCTING A MASS OPERATOR

Here we deduce a mass operator for the pion-nucleon system by using the Okubo method [20] for constructing effective interactions, starting from a quantum field theory Hamiltonian. The Fock space of the field theory is divided into a subspace consisting of various single-baryon states, i.e., $|N\rangle,|\Delta\rangle,|R\rangle, \ldots$; and various meson-baryon states, i.e., $|\pi N\rangle,|\pi\Delta\rangle,|\eta N\rangle, \ldots$. We denote the projection operator onto this subspace by Π and onto the complementary, orthogonal subspace by Λ, so that

$$\Pi+\Lambda=1. \tag{3.1}$$

The quantum field theory Hamiltonian is divided into the noninteracting part H_0 and the interacting part H_1, according to

$$H_{\mathrm{QFT}}=H_0+H_1, \tag{3.2}$$

where the eigenstates of H_0, designated here by $|\zeta\rangle$, are assumed known, and satisfy

$$H_0|\zeta\rangle=E(\zeta)|\zeta\rangle. \tag{3.3}$$

The effective Hamiltonian in the Π subspace, denoted by H^{Π}, is given to second order by [20]

$$\langle\zeta|H^{\Pi}|\zeta'\rangle=\left\langle\zeta\left|H_{\mathrm{QFT}}+\frac{1}{2}H_1\left[\frac{\Lambda}{E(\zeta)-H_0}+\frac{\Lambda}{E(\zeta')-H_0}\right]H_1\right|\zeta'\right\rangle+\cdots. \tag{3.4}$$

We now illustrate the method for constructing the matrix elements of the mass operator by considering a well known πNN interaction. The nucleon and pion fields are given by

$$N_i(x)=\int\frac{d^3p}{(2\pi)^3 2\varepsilon_N(\mathbf{p})}\sum_h[b_i(\mathbf{p},h)u(p,h)e^{-ip\cdot x}+d_i^{\dagger}(\mathbf{p},h)v(p,h)e^{ip\cdot x}], \tag{3.5a}$$

$$\pi_t(x)=\int\frac{d^3k}{(2\pi)^3 2\omega_\pi(\mathbf{k})}[(-1)^t a_{-t}(\mathbf{k})e^{-ik\cdot x}+a_t^{\dagger}(\mathbf{k})e^{ik\cdot x}], \tag{3.5b}$$

$$\varepsilon_N(\mathbf{p})=(\mathbf{p}^2+m_N^2)^{1/2}, \quad \omega_\pi(\mathbf{k})=(\mathbf{k}^2+m_\pi^2)^{1/2}, \tag{3.6}$$

$$\overline{u}(p,h)u(p,h')=-\overline{v}(p,h)v(p,h')=2m_N\delta_{hh'}. \tag{3.7}$$

Here $b_i^{\dagger}(\mathbf{p},h)$ creates a nucleon of three-momentum **p**, energy $\varepsilon_N(\mathbf{p})$, z component of isospin i, and z component of spin h; $a_t^{\dagger}(\mathbf{k})$ creates a pion of three-momentum **k**, energy $\omega_\pi(\mathbf{k})$, and z component of isospin t. The Dirac spinors $u(p,h)$ and $v(p,h)$ are the same as those of Bjorken and Drell [43] except for the normalization (3.7). The nonzero commutators and anticommutators are given by

$$\{b_i(\mathbf{p},h),b_{i'}^{\dagger}(\mathbf{p}',h')\}=\{d_i(\mathbf{p},h),d_{i'}^{\dagger}(\mathbf{p}',h')\}=(2\pi)^3 2\varepsilon_N(\mathbf{p})\delta^3(\mathbf{p}-\mathbf{p}')\delta_{ii'}\delta_{hh'}, \tag{3.8}$$

$$[a_t(\mathbf{k}),a_{t'}^{\dagger}(\mathbf{k}')]=(2\pi)^3 2\omega_\pi(\mathbf{k})\delta^3(\mathbf{k}-\mathbf{k}')\delta_{tt'}.$$

The πNN-interaction Lagrangian density and interaction Hamiltonian are given by

$$\mathcal{L}_{\pi NN}(x)=-ig_{\pi NN}\overline{N}(x)[\Gamma(i\partial)\boldsymbol{\tau}\cdot\boldsymbol{\pi}(x)]N(x), \tag{3.9}$$

$$\Gamma(q)=\left(\lambda+\frac{1-\lambda}{2m_N}\not{q}\right)\gamma_5, \tag{3.10}$$

$$H_{\pi NN} = -\int d^3x\, \mathcal{L}_{\pi NN}(x)|_{t=0} + \cdots, \tag{3.11}$$

where the ellipsis indicates a term due to derivative coupling. Such terms do not contribute here. This interaction involves a mixture of pseudoscalar and pseudovector coupling, with the mix determined by λ [13,14].

In our model, the Π subspace includes the single-nucleon and pion-nucleon states defined by

$$|\mathbf{p}ih\rangle = b_i^\dagger(\mathbf{p},h)|0\rangle, \tag{3.12}$$

$$|\mathbf{k}t,\mathbf{p}ih\rangle = a_t^\dagger(\mathbf{k}) b_i^\dagger(\mathbf{p},h)|0\rangle. \tag{3.13}$$

Obviously the inner product for the single-nucleon states is given by

$$\langle \mathbf{p}ih|\mathbf{p}'i'h'\rangle = (2\pi)^3 2\varepsilon_N(\mathbf{p})\,\delta^3(\mathbf{p}-\mathbf{p}')\,\delta_{ii'}\,\delta_{hh'}\,. \tag{3.14}$$

In carrying out the Bakamjian-Thomas construction, it is important to work with a basis in which $\mathbf{P}$, $\mathcal{J}$, and $\mathbf{X}$ have simple representatives. The single-particle states (3.12) are perfectly satisfactory in this regard, however, the two-particle states (3.13) are not. In order to improve on them, we begin by replacing the labels $\mathbf{k}$ and $\mathbf{p}$ by the total three-momentum $\mathbf{Q}$ and relative three-momentum $\mathbf{q}$ defined by

$$Q = (E_{\pi N}, \mathbf{Q}) = k + p = (\omega_\pi(\mathbf{k}) + \varepsilon_N(\mathbf{p}), \mathbf{k}+\mathbf{p}),$$

$$\mathbf{q} = \mathbf{k}_{\text{c.m.}} = -\mathbf{p}_{\text{c.m.}}, \tag{3.15}$$

where we see that $\mathbf{q}$ is the three-momentum of the pion in the pion-nucleon, c.m. frame. The invariant mass of the pion-nucleon state $|\mathbf{k}t,\mathbf{p}ih\rangle$ is given by

$$W_{\pi N}(\mathbf{q}) \equiv +(Q\cdot Q)^{1/2} = \omega_\pi(\mathbf{q}) + \varepsilon_N(\mathbf{q}), \tag{3.16}$$

while its energy is given by

$$E_{\pi N}(\mathbf{Q},\mathbf{q}) = [\mathbf{Q}^2 + W_{\pi N}^2(\mathbf{q})]^{1/2}. \tag{3.17}$$

A satisfactory two-particle basis state is obtained by boosting a rest frame state according to [34,35,37]

$$\begin{aligned} |\mathbf{qQ}tih\rangle &\equiv \exp(-i\gamma\,\hat{\mathbf{Q}}\cdot\mathbf{K})|\mathbf{q}t,(-\mathbf{q})ih\rangle \\ &= \exp(-i\gamma\,\hat{\mathbf{Q}}\cdot\mathbf{K})\, a_t^\dagger(\mathbf{q}) b_i^\dagger(-\mathbf{q},h)|0\rangle, \end{aligned} \tag{3.18a}$$

$$\gamma = \tanh^{-1}[|\mathbf{Q}|/E_{\pi N}(\mathbf{Q},\mathbf{q})]. \tag{3.18b}$$

It can be shown that the inner product for the basis states (3.18) that follows from Eq. (3.8) is given by

$$\begin{aligned} \langle \mathbf{qQ}tih|\mathbf{q}'\mathbf{Q}'t'i'h'\rangle &= (2\pi)^3 2E_{\pi N}(\mathbf{Q},\mathbf{q})\,\delta^3(\mathbf{Q}-\mathbf{Q}')\Delta_{\pi N}(\mathbf{q}) \\ &\quad\times \delta^3(\mathbf{q}-\mathbf{q}')\,\delta_{tt'}\,\delta_{ii'}\,\delta_{hh'}\,, \end{aligned} \tag{3.19a}$$

$$\Delta_{\pi N}(\mathbf{q}) = (2\pi)^3 2\,\omega_\pi(\mathbf{q})\varepsilon_N(\mathbf{q})/W_{\pi N}(\mathbf{q}). \tag{3.19b}$$

In the single-particle basis defined by Eq. (3.13), the representatives of $\mathbf{P}$, $\mathcal{J}$, and $\mathbf{X}$ are given by

$$\langle \mathbf{p}ih|\mathbf{P} = \mathbf{p}\langle \mathbf{p}ih|, \tag{3.20a}$$

$$\langle \mathbf{p}ih|\mathcal{J} = \sum_{h'} (\boldsymbol{\sigma}/2)_{hh'}\langle \mathbf{p}ih'|, \tag{3.20b}$$

$$\langle \mathbf{p}ih|\mathbf{X} = \left[i\nabla_{\mathbf{p}} - i\frac{\mathbf{p}}{2\varepsilon_N^2(\mathbf{p})}\right]\langle \mathbf{p}ih|, \tag{3.20c}$$

while in the two-particle basis defined by Eq. (3.18) they are given by

$$\langle \mathbf{qQ}tih|\mathbf{P} = \mathbf{Q}\langle \mathbf{qQ}tih|, \tag{3.21a}$$

$$\langle \mathbf{qQ}tih|\mathcal{J} = \sum_{h'} [\mathcal{J}(\mathbf{q})]_{hh'}\langle \mathbf{qQ}tih'|,\quad \mathcal{J}(\mathbf{q}) = i\nabla_{\mathbf{q}}\times\mathbf{q} + \boldsymbol{\sigma}/2, \tag{3.21b}$$

$$\langle \mathbf{qQ}tih|\mathbf{X} = \left[i\nabla_{\mathbf{Q}} - i\frac{\mathbf{Q}}{2E_{\pi N}^2(\mathbf{Q},\mathbf{q})}\right]\langle \mathbf{qQ}tih|. \tag{3.21c}$$

The second terms in the representatives of $\mathbf{X}$ are needed to ensure that it is Hermitian with respect to the inner products implied by Eqs. (3.14) and (3.19).

In our model, the mass operator is assumed to be of the form

$$M = M_0 + U, \tag{3.22}$$

where M_0 is the mass operator for the noninteracting system, and U contains the interactions. The noninteracting mass operator is defined in the $|N\rangle - |\pi N\rangle$ sector by the eigenvalue equations

$$M_0|\mathbf{p}ih\rangle = m_N|\mathbf{p}ih\rangle, \tag{3.23a}$$

$$M_0|\mathbf{qQ}tih\rangle = W_{\pi N}(\mathbf{q})|\mathbf{qQ}tih\rangle. \tag{3.23b}$$

We recall that in the Bakamjian-Thomas construction [36,37], the operators $\mathbf{P}$, $\mathcal{J}$, and $\mathbf{X}$ are the same for the interacting system as for the noninteracting system, therefore M_0 and U must commute separately with them in order to satisfy Eq. (2.9), the necessary condition for Poincaré invariance. The commutativity of M_0 follows trivially from Eqs. (3.20), (3.21), and (3.23). In order for U to commute with $\mathbf{P}$, $\mathcal{J}$, and $\mathbf{X}$ its matrix elements must have the forms

$$\langle \mathbf{p}ih|U|\mathbf{p}'i'h'\rangle = (2\pi)^3 2\varepsilon_N(\mathbf{p})\,\delta^3(\mathbf{p}-\mathbf{p}')\,\delta_{ii'}\,\delta_{hh'}\,U_{NN}\,, \tag{3.24a}$$

$$\begin{aligned} \langle \mathbf{qQ}tih|U|\mathbf{p}'i'h'\rangle &= (2\pi)^3 2[E_{\pi N}(\mathbf{Q},\mathbf{q})\varepsilon_N(\mathbf{p}')]^{1/2}\delta^3(\mathbf{Q}-\mathbf{p}') \\ &\quad\times \left\langle tih\left| \frac{U_{\pi N,N}(\mathbf{q})}{2[W_{\pi N}(\mathbf{q})m_N]^{1/2}}\right| i'h'\right\rangle, \end{aligned} \tag{3.24b}$$

$$\langle \mathbf{qQ}tih|U|\mathbf{q}'\mathbf{Q}'t'i'h'\rangle = (2\pi)^3 2[E_{\pi N}(\mathbf{Q},\mathbf{q})E_{\pi N}(\mathbf{Q}',\mathbf{q}')]^{1/2}\delta^3(\mathbf{Q}-\mathbf{Q}') \times\left\langle tih\left|\frac{U_{\pi N,\pi N}(\mathbf{q},\mathbf{q}')}{2[W_{\pi N}(\mathbf{q})W_{\pi N}(\mathbf{q}')]^{1/2}}\right|t'i'h'\right\rangle. \quad (3.24c)$$

The Dirac δ functions result from the commutativity of $\mathbf{P}$ and U, while the fact that U_{NN}, $U_{\pi N,N}(\mathbf{q})$, and $U_{\pi N,\pi N}(\mathbf{q},\mathbf{q}')$ do not depend on the total three-momentum is a consequence of the commutativity of $\mathbf{X}$ and U. In order to ensure that U commutes with the spin operator $\mathcal{J}$, $U_{\pi N,N}(\mathbf{q})$, and $U_{\pi N,\pi N}(\mathbf{q},\mathbf{q}')$ must be rotationally invariant functions of the $\mathbf{q}$'s and $\boldsymbol{\sigma}$. The denominators on the right hand sides of Eqs. (3.24b) and (3.24c) are put in simply for convenience.

In order to construct a model for the vertex function $U_{\pi N,N}(\mathbf{q})$, we evaluate $H_{\pi NN}$ between a pion-nucleon state and a nucleon state and obtain

$$\langle \mathbf{k}t,\mathbf{p}ih|H_{\pi NN}|\mathbf{p}'i'h'\rangle = (2\pi)^3\delta^3(\mathbf{Q}-\mathbf{p}')ig_{\pi NN}(\boldsymbol{\varepsilon}_t^*\cdot\boldsymbol{\tau})_{ii'}\overline{u}(p,h) \times\Gamma(-k)u(p',h'), \quad (3.25)$$

where we have introduced the complex unit vectors defined by

$$\boldsymbol{\varepsilon}_{\pm} = \mp(1/\sqrt{2})(1,\pm i,0), \quad \boldsymbol{\varepsilon}_0 = (0,0,1). \quad (3.26)$$

According to Eq. (2.8a) the Hamiltonian and the mass operator yield the same result when acting on a state whose total three-momentum is zero. Also, it follows from Eq. (3.18) that $|\mathbf{qQ}tih\rangle = |\mathbf{q}t,(-\mathbf{q})ih\rangle$ when $\mathbf{Q}=\mathbf{0}$. As a result of these observations, we find upon choosing $\mathbf{p}'=0$ in Eqs. (3.24b) and (3.25) that the πNN vertex function is given by

$$\langle tih|U_{\pi N,N}(\mathbf{q})|i'h'\rangle = ig_{\pi NN}(\boldsymbol{\varepsilon}_t^*\cdot\boldsymbol{\tau})_{ii'}\overline{u}(p,h)\Gamma(-k)u(p',h'), \qquad \mathbf{k}+\mathbf{p}=\mathbf{p}'=0. \quad (3.27)$$

In the c.m. frame, the nucleon's Dirac spinor is given by

$$u(p,h) = [\varepsilon_N(\mathbf{q})+m_N]^{1/2}\begin{bmatrix}\chi_h\\ -\boldsymbol{\sigma}\cdot\mathbf{x}_N\,\chi_h\end{bmatrix}, \quad p = (\varepsilon_N(\mathbf{q}),-\mathbf{q}), \quad (3.28a)$$

$$\mathbf{x}_N = \frac{\mathbf{q}}{\varepsilon_N(\mathbf{q})+m_N}, \quad (3.28b)$$

which when used in Eq. (3.27), along with Eq. (3.10), leads to

$$\langle tih|U_{\pi N,N}(\mathbf{q})|i'h'\rangle = ig_{\pi NN}(\boldsymbol{\varepsilon}_t^*\cdot\boldsymbol{\tau})_{ii'}[\varepsilon_N(\mathbf{q})+m_N]^{1/2}(2m_N)^{1/2} \times\left[\lambda+(1-\lambda)\frac{W_{\pi N}(\mathbf{q})+m_N}{2m_N}\right](\boldsymbol{\sigma}\cdot\mathbf{x}_N)_{hh'}. \quad (3.29)$$

We note that $U_{\pi N,N}(\mathbf{q})$ is a rotationally invariant function of $\boldsymbol{\sigma}$ and $\mathbf{q}$, which is necessary for the commutativity of U and the spin operator $\mathcal{J}$.

We now develop a model for $U_{\pi N,\pi N}(\mathbf{q},\mathbf{q}')$ based on nucleon exchange. This comes from the second term on the right hand side of Eq. (3.4), where we note that the projection operator Λ excludes the single-nucleon intermediate states. We find

$$\left\langle \mathbf{k}t,\mathbf{p}ih\left|H_{\pi NN}\frac{\Lambda}{E-H_0}H_{\pi NN}\right|\mathbf{k}'t',\mathbf{p}'i'h'\right\rangle = \sum_{i''h''}\int\langle\mathbf{p}ih|H_{\pi NN}|\mathbf{k}'t',\mathbf{p}''i''h''\rangle \times\frac{d^3p''}{(2\pi)^3 2\varepsilon_N(\mathbf{p}'')[E-\varepsilon_N(\mathbf{p}'')-\omega_\pi(\mathbf{k})-\omega_\pi(\mathbf{k}')]} \times\langle\mathbf{k}t,\mathbf{p}''i''h''|H_{\pi NN}|\mathbf{p}'i'h'\rangle+\cdots, \quad (3.30)$$

where the ellipsis indicates terms that do not contribute to nucleon exchange. Upon inserting Eq. (3.25) and comparing the result to Eq. (3.24c) with $\mathbf{Q}=\mathbf{0}$, we find that the single-nucleon exchange contribution to the $\pi N,\pi N$ potential is given by

$$\begin{aligned}U^N_{\pi N,\pi N}(\mathbf{q},\mathbf{q}') = {} & g^2_{\pi NN}(-P^{\pi N}_{1/2}+2P^{\pi N}_{3/2})(\varepsilon_N+m_N)^{1/2}\\ & \times(\varepsilon'_N+m_N)^{1/2}\{(\varepsilon''_N-m_N)BB'\\ & -(\varepsilon_N-m_N)AB'-(\varepsilon'_N-m_N)BA'\\ & +(\boldsymbol{\sigma}\cdot\mathbf{x}_N)(\boldsymbol{\sigma}\cdot\mathbf{x}'_N)[(\varepsilon''_N+m_N)AA'\\ & -(\varepsilon_N+m_N)BA'-(\varepsilon'_N+m_N)AB']\}\\ & \times\frac{1}{4\varepsilon''_N}\left(\frac{1}{\varepsilon_N-\varepsilon''_N-\omega'_\pi}+\frac{1}{\varepsilon'_N-\varepsilon''_N-\omega_\pi}\right),\end{aligned} \quad (3.31)$$

$$\varepsilon_N=\varepsilon_N(\mathbf{q}), \quad \varepsilon'_N=\varepsilon_N(\mathbf{q}'), \quad \varepsilon''_N=\varepsilon_N(\mathbf{q}+\mathbf{q}'),$$

$$\omega_\pi=\omega_\pi(\mathbf{q}), \quad \omega'_\pi=\omega_\pi(\mathbf{q}'),$$

$$A=A(\mathbf{q},\mathbf{q}';m_N)=1+\frac{1-\lambda}{2m_N}(\varepsilon_N-\varepsilon''_N-\omega'_\pi),$$

$$B=A(\mathbf{q},\mathbf{q}';-m_N), \quad A'=A(\mathbf{q}',\mathbf{q};m_N),$$

$$B'=A(\mathbf{q}',\mathbf{q};-m_N).$$

Here $P^{\pi N}_I$ is a projection operator onto a state of the pion-nucleon system with total isospin I, and $\mathbf{x}_N$ and $\mathbf{x}'_N$ are defined by Eq. (3.28b). We note that this potential is a rotationally invariant function of $\boldsymbol{\sigma}$, $\mathbf{q}$, and $\mathbf{q}'$.

We now consider interactions involving the $\Delta(1232)$, which we treat as an elementary particle. The $\pi N,\Delta$ vertex function can be calculated from the matrix element

$$\langle \mathbf{k}t,(\mathbf{p}ih)_N|H_{\pi N\Delta}|(\mathbf{p}'i'h')_\Delta\rangle$$
$$=(2\pi)^3\delta^3(\mathbf{Q}-\mathbf{p}')i(g_{\pi N\Delta}/m_\pi)$$
$$\times(\boldsymbol{\varepsilon}_t^*\cdot\mathbf{T}_{N\Delta})_{ii'}\overline{u(p,h)}k_\mu u_\Delta^\mu(p',h'), \tag{3.32}$$

which is obtained from the Lagrangian density (A4). The result is given by Eq. (A16). Here $\mathbf{T}_{N\Delta}$ is an isospin transition operator. In general our spin and isospin transition operators, $\mathbf{S}_{BB'}$ and $\mathbf{T}_{BB'}$, are defined by

$$\mathcal{T}_{BB'}=\sum_{mnn'}\boldsymbol{\varepsilon}_m|Bn\rangle\langle 1Bmn|B'n'\rangle\langle B'n'|,\quad \mathcal{T}=\mathbf{S},\mathbf{T}, \tag{3.33}$$

where B stands for a baryon and its spin or isospin, and $\langle 1Bmn|B'n'\rangle$ is a Clebsch-Gordan coefficient. The Rarita-Schwinger spinor for the Δ is given by

$$u_\Delta^\mu(p,h)=\sum_{mh'}l^\mu{}_\nu(p)(0,\boldsymbol{\varepsilon}_m)^\nu u_\Delta(p,h')\langle 1,1/2,m,h'|3/2,h\rangle$$
$$=\sum_{h'}u_\Delta(p,h')\left[\frac{\mathbf{p}\cdot\mathbf{S}_{N\Delta}}{m_\Delta},\mathbf{S}_{N\Delta}+\frac{(\mathbf{p}\cdot\mathbf{S}_{N\Delta})\mathbf{p}}{m_\Delta(\varepsilon_\Delta+m_\Delta)}\right]^\mu_{h'h}, \tag{3.34}$$

where $l(p)$ is a rotationless boost from the rest frame four-momentum $(m_\Delta,\mathbf{0})$ to $p=(\varepsilon_\Delta,\mathbf{p})$, and $u_\Delta(p,h)$ is a Dirac spinor defined as in Eq. (3.28), but with $m_N\rightarrow m_\Delta$.

The matrix element (3.32) can also be used to calculate the $\pi N,\pi\Delta$ transition potential due to nucleon exchange, using the formula

$$\left\langle \mathbf{k}t,(\mathbf{p}ih)_N\middle|H_{\pi NN}\frac{\Lambda}{E-H_0}H_{\pi N\Delta}\middle|\mathbf{k}'t',(\mathbf{p}'i'h')_\Delta\right\rangle$$
$$=\sum_{i''h''}\int\langle(\mathbf{p}ih)_N|H_{\pi NN}|\mathbf{k}'t',(\mathbf{p}''i''h'')_N\rangle$$
$$\times\frac{d^3p''}{(2\pi)^3 2\varepsilon_N(\mathbf{p}'')[E-\varepsilon_N(\mathbf{p}'')-\omega_\pi(\mathbf{k})-\omega_\pi(\mathbf{k}')]}$$
$$\times\langle\mathbf{k}t,(\mathbf{p}''i''h'')_N|H_{\pi N\Delta}|(\mathbf{p}'i'h')_\Delta\rangle+\cdots, \tag{3.35}$$

$$E=\omega_\pi(\mathbf{k})+\varepsilon_N(\mathbf{p}).$$

Note that here in applying Eq. (3.4) we only use $E=\omega_\pi(\mathbf{k})+\varepsilon_N(\mathbf{p})$ without the factor of 1/2. The term with $E'=\omega_\pi(\mathbf{k}')+\varepsilon_\Delta(\mathbf{p}')$ causes a problem, since the denominator $E'-\varepsilon_N(\mathbf{p}'')-\omega_\pi(\mathbf{k})-\omega_\pi(\mathbf{k}')$ can vanish. This is related to the fact that $\Delta\rightarrow\pi+N$ is not just a virtual process; it can actually occur. This difficulty can be circumvented by expanding the few-particle subspace to include $|\pi\pi N\rangle$ states. This, however, would lead to a three-particle model, which is beyond the scope of the present work. The final result for the transition potential is given by Eqs. (A17) and (A18).

The $\pi N,\pi N$ potential due to Δ exchange can be calculated from the matrix element

$$\langle \mathbf{k}t,(\mathbf{p}ih)_\Delta|H_{\pi N\Delta}|(\mathbf{p}'i'h')_N\rangle$$
$$=(2\pi)^3\delta^3(\mathbf{Q}-\mathbf{p}')i(g_{\pi N\Delta}/m_\pi)$$
$$\times(\boldsymbol{\varepsilon}_t^*\cdot\mathbf{T}_{N\Delta}^\dagger)_{ii'}\overline{u_\Delta^\mu}(p,h)k_\mu u(p',h'), \tag{3.36}$$

and the formula

$$\left\langle \mathbf{k}t,(\mathbf{p}ih)_N\middle|H_{\pi N\Delta}\frac{\Lambda}{E-H_0}H_{\pi N\Delta}\middle|\mathbf{k}'t',(\mathbf{p}'i'h')_N\right\rangle$$
$$=\sum_{i''h''}\int\langle(\mathbf{p}ih)_N|H_{\pi N\Delta}|\mathbf{k}'t',(\mathbf{p}''i''h'')_\Delta\rangle$$
$$\times\frac{d^3p''}{(2\pi)^3 2\varepsilon_\Delta(\mathbf{p}'')[E-\varepsilon_\Delta(\mathbf{p}'')-\omega_\pi(\mathbf{k})-\omega_\pi(\mathbf{k}')]}$$
$$\times\langle\mathbf{k}t,(\mathbf{p}''i''h'')_\Delta|H_{\pi N\Delta}|(\mathbf{p}'i'h')_N\rangle+\cdots, \tag{3.37a}$$

$$\varepsilon_\Delta(\mathbf{p})=(\mathbf{p}^2+m_\Delta^2)^{1/2}. \tag{3.37b}$$

The final result is given by Eq. (A19).

As mentioned in Sec. I, other interactions that we include are those associated with various single-baryon intermediate states, where the baryons are, in order of increasing energy, the following resonances: $R=P_{11}(1440)$, $D=D_{13}(1520)$, $S=S_{11}(1535)$, and $S'=S_{31}(1620)$. We have not included the $\langle\pi N|U|\pi N\rangle$ interactions due to the exchange of these resonances. They only contribute through the processes $\mu B\Leftrightarrow B'\Leftrightarrow\mu''B''$. The $\mu B,B'$ vertex functions associated with these resonances are given in the Appendix.

In order to take account of the composite nature of our particles, matrix elements such as Eqs. (3.26) and (3.29) must be modified by the introduction of form factors. For vertices of the form $\mu+B\Leftrightarrow B'$, we follow the procedure used in Aaron, Amado, and Young's $\pi\pi N$ model [44], and multiply $\langle(kp)_B|H_{\rm int}|(p')_{B'}\rangle$ matrix elements by the form factor given by

$$g_{\mu BB'}(k,p)=\left[\frac{\Lambda_{\mu BB'}^2+\Omega_{\rm pole}}{\Lambda_{\mu BB'}^2+\Omega(k,p)}\right]^n, \tag{3.38a}$$

$$\Omega(k,p)=\frac{(k\cdot p)^2-m_\mu^2m_B^2}{(k+p)^2}, \tag{3.38b}$$

$$\Omega_{\rm pole}=[(m_\mu^2+m_B^2-m_{B'}^2)^2-4m_\mu^2m_B^2]/(2m_{B'})^2, \tag{3.38c}$$

where $\Lambda_{\mu BB'}$ is a cutoff mass, and $\Omega_{\rm pole}$ is the value of $\Omega(k,p)$ when $(k+p)^2=m_{B'}^2$. In the μB c.m. frame $\Omega(k,p)=\mathbf{q}^2$, where $\mathbf{q}$ is the c.m. three-momentum of meson μ. With the introduction of these form factors, a matrix element such as Eq. (3.32) is replaced according to $\langle\mathbf{k}t,(\mathbf{p}ih)_N|H_{\pi N\Delta}|(\mathbf{p}'i'h')_\Delta\rangle$

$\rightarrow\langle \mathbf{k}t,(\mathbf{p}ih)_N|H_{\pi N\Delta}|(\mathbf{p}'i'h')_\Delta\rangle g_{\pi N\Delta}(k,p)$. This type of replacement is also made for matrix elements such as $\langle \mathbf{k}t,(\mathbf{p}''i''h'')_N|H_{\pi N\Delta}|(\mathbf{p}'i'h')_\Delta\rangle$, which appears in Eq. (3.35). When expressing Eq. (3.38b) in terms of the **q**'s it should be kept in mind that all of our four-momenta are on the mass shell.

The $\pi N,\pi N$ potentials due to σ and ρ exchange can be calculated using the method described in Ref. [34], which extracts the potentials from Feynman-like amplitudes. For example, the σ-exchange potential is obtained from the formula

$$\langle tih|U^{\sigma}_{\pi N,\pi N}(\mathbf{q},\mathbf{q}')|t'i'h'\rangle$$
$$=\delta_{tt'}\delta_{ii'}\overline{u(p,h)}u(p',h')\frac{g_{\sigma NN}}{2m_\pi}[g_{\sigma\pi\pi}m_\pi^2$$
$$+\tilde{g}_{\sigma\pi\pi}(k\cdot k')]\left[\frac{1}{(k-k')^2-m_\sigma^2}+\frac{1}{(p-p')^2-m_\sigma^2}\right], \tag{3.39}$$

$$\mathbf{p}+\mathbf{k}=\mathbf{p}'+\mathbf{k}'=\mathbf{0},$$

with the final result given by Eq. (A34). For the σ and ρ exchange potentials we introduce the same type of form factor as used in Ref. [34], i.e.,

$$f(t^2;m,\Lambda,n)=\left[\frac{(\Lambda^2-m^2)^2+\Lambda^4}{(\Lambda^2-t^2)^2+\Lambda^4}\right]^n, \tag{3.40}$$

where t is a momentum transfer, m is the mass of the exchanged particle (σ or ρ), and Λ is a cutoff mass. With the introduction of this form factor, a potential such as Eq. (3.38) is mutiplied by

$$f[(k-k')^2;m_\sigma,\Lambda_{\sigma\pi\pi},n_{\sigma\pi\pi}]f[(p-p')^2;m_\sigma,\Lambda_{\sigma NN},n_{\sigma NN}]. \tag{3.41}$$

IV. LIPPMANN-SCHWINGER EQUATIONS

The scattering amplitudes that our mass operator $M=M_0+U$ gives rise to are obtained by solving the Lippmann-Schwinger equation

$$T(z)=U+U\frac{1}{z-M_0}T(z), \tag{4.1}$$

where z is a complex parameter, which for pion-nucleon scattering is given by $z=W_{\pi N}+i\varepsilon$. In writing out the representation of this equation in our few-particle subspace, we encounter both single-baryon and meson-baryon states. In Ref. [40], it is shown that it is possible to eliminate explicit reference to the single-baryon states, and thereby obtain an effective potential $V_{22}(z)$ that acts only in the subspace of meson-baryon states. The resulting equations are

$$T_{22}(z)=V_{22}(z)+V_{22}(z)\frac{1}{z-M_0}T_{22}(z), \tag{4.2}$$

$$V_{22}(z)=U_{22}+\sum_{Bi_Bh_B}\int U_{21}|\mathbf{p}i_Bh_B\rangle$$
$$\times\frac{d^3p}{(2\pi)^3 2\varepsilon_B(\mathbf{p})(z-m_{0B})}\langle \mathbf{p}i_Bh_B|U_{12}, \tag{4.3}$$

where the subscript 2 refers to the subspace of meson-baryon states, and the 1 refers to the subspace of single-baryon states. The second term on the right hand side of Eq. (4.3) is the effective potential that arises from the elimination of the single-baryon channels, and contains the bare masses of the various baryons. These arise as a result of expressing the constants that appear in matrix elements such as Eq. (3.24a) in the form

$$U_{BB'}=[m_B^{(0)}-m_B]\delta_{BB'}. \tag{4.4}$$

Following Eq. (3.24c), we write our T-matrix elements in the form

$$\langle \mathbf{q}\mathbf{Q}t_\mu i_Bh_B|T(z)|\mathbf{q}'\mathbf{Q}'t'_{\mu'},i'_{B'},h'_{B'}\rangle$$
$$=(2\pi)^3 2[E_{\mu B}(\mathbf{Q},\mathbf{q})E_{\mu' B'}(\mathbf{Q}',\mathbf{q}')]^{1/2}\delta^3(\mathbf{Q}-\mathbf{Q}')$$
$$\times\left\langle t_\mu i_Bh_B\left|\frac{T_{\mu B,\mu' B'}(\mathbf{q},\mathbf{q}';z)}{2[W_{\mu B}(\mathbf{q})W_{\mu' B'}(\mathbf{q}')]^{1/2}}\right|t'_{\mu'},i'_{B'},h'_{B'}\right\rangle, \tag{4.5}$$

as well as a similar equation with $T\rightarrow U$. Following Eq. (3.24b), we write the matrix element for an arbitrary $\mu B,B'$ vertex in the form

$$\langle \mathbf{q}\mathbf{Q}t_\mu i_Bh_B|U|\mathbf{p}'i'_{B'},h'_{B'}\rangle$$
$$=(2\pi)^3 2[E_{\mu B}(\mathbf{Q},\mathbf{q})\varepsilon_{B'}(\mathbf{p}')]^{1/2}\delta^3(\mathbf{Q}-\mathbf{p}')$$
$$\times\left\langle t_\mu i_Bh_B\left|\frac{U_{\mu B,B'}(\mathbf{q})}{2[W_{\mu B}(\mathbf{q})m_{B'}]^{1/2}}\right|i'_{B'},h'_{B'}\right\rangle. \tag{4.6}$$

Using these expressions, as well as the completeness relation implied by inner products such as Eq. (3.19), we find the coupled integral equations

$$T_{\mu B,\mu' B'}(\mathbf{q},\mathbf{q}';z)$$
$$=V_{\mu B,\mu' B'}(\mathbf{q},\mathbf{q}';z)+\sum_{\mu'' B''}\int V_{\mu B,\mu'' B''}(\mathbf{q},\mathbf{q}'';z)$$
$$\times\frac{d^3q''}{\Delta_{\mu'' B''}(\mathbf{q}'')}\frac{T_{\mu'' B'',\mu' B'}(\mathbf{q}'',\mathbf{q}';z)}{2W_{\mu'' B''}(\mathbf{q}'')[z-W_{\mu'' B''}(\mathbf{q}'')]}, \tag{4.7}$$

where the effective μB-$\mu' B'$ potentials are given by

TABLE I. Partial waves and contributing states.

Partial wave	Baryon and meson-baryon states	$m_\Delta^{\text{threshold}}$ (MeV)
S_{11}	$S_{11}(1535), \pi N, \pi\Delta(l_{\pi\Delta}=2), \eta N(l_{\eta N}=0)$	1181.1
S_{31}	$S_{31}(1620), \pi N, \pi\Delta(l_{\pi\Delta}=2)$	1336.9
P_{11}	$N, P_{11}(1440), \pi N, \pi\Delta(l_{\pi\Delta}=1)$	1209.2
P_{31}	$\pi N, \pi\Delta(l_{\pi\Delta}=1)$	1500.0
P_{13}	$\pi N, \pi\Delta(l_{\pi\Delta}=1,3)$	1372.6
P_{33}	$P_{33}(1232), \pi N, \pi\Delta(l_{\pi\Delta}=1,3)$	1172.7
D_{13}	$D_{13}(1520), \pi N, \pi\Delta(l_{\pi\Delta}=0,2)$	1205.7

$$V_{\mu B,\mu' B'}(\mathbf{q},\mathbf{q}';z)=U_{\mu B,\mu' B'}(\mathbf{q},\mathbf{q}')+\sum_{B''}\frac{U_{\mu B,B''}(\mathbf{q})U_{B'',\mu' B'}(\mathbf{q}')}{2m_{B''}[z-m_{B''}^{(0)}]}. \tag{4.8}$$

Since the pi and eta mesons are spinless and our two-particle channels are $|\pi N\rangle$, $|\pi\Delta\rangle$, and $|\eta N\rangle$, the angular momentum eigenstates we need to carry out a partial wave analysis of Eq. (4.7) are the ones defined by

$$\mathcal{Y}_{lsj}^{m}(\hat{\mathbf{q}})=\sum_{m_1 m_2} Y_l^{m_1}(\hat{\mathbf{q}})|sm_2\rangle\langle lsm_1m_2|jm\rangle, \quad s=1/2,3/2, \tag{4.9}$$

where s is the spin of the baryon. The partial wave analysis of the potentials is greatly facilitated by the identity

$$(\hat{\mathbf{q}}\cdot\mathbf{S}_{us})\mathcal{Y}_{lsj}^{m}(\hat{\mathbf{q}})=(-1)^{j-u-2s}\sum_L \mathcal{Y}_{Luj}^{m}(\hat{\mathbf{q}})\sqrt{(2l+1)(2s+1)(2L+1)}\times\begin{pmatrix} l & 1 & L\\ 0 & 0 & 0\end{pmatrix}\begin{Bmatrix} u & L & j\\ l & s & 1\end{Bmatrix}, \tag{4.10}$$

where () and {} are a 3-j and 6-j symbol, respectively. The 3-j symbol restricts the sum on L to $L=1$ for $l=0$ and $L=l\mp 1$ for $l\geqslant 1$. This identity can be derived by combining Eqs. (3.33) and (4.9), and using standard angular momentum recoupling techniques. An important special case is $\mathbf{S}_{1/2,1/2}=\mathbf{S}_{NN}=-\boldsymbol{\sigma}/\sqrt{3}$, where Eq. (4.10) leads to the well known result

$$(\hat{\mathbf{q}}\cdot\boldsymbol{\sigma})\mathcal{Y}_{l,1/2,j}^{m}(\hat{\mathbf{q}})=-\mathcal{Y}_{2j-l,1/2,j}^{m}(\hat{\mathbf{q}}). \tag{4.11}$$

The spin factors in the potentials can be rearranged by using the identity

$$(\hat{\mathbf{q}}'\cdot\mathbf{S}_{sv})(\hat{\mathbf{q}}\cdot\mathbf{S}_{vu})=\sum_w (-1)^{v+w+1-2u}\sqrt{(2v+1)(2w+1)}\times\begin{Bmatrix} 1 & u & w\\ 1 & s & v\end{Bmatrix}(\hat{\mathbf{q}}\cdot\mathbf{S}_{sw})(\hat{\mathbf{q}}'\cdot\mathbf{S}_{wu}), \tag{4.12}$$

which when used in conjunction with Eq. (4.10) makes it possible to work out the partial wave matrix elements in a relatively straightforward manner. The partial wave, πN-πN elastic scattering amplitudes are given by

TABLE II. Vertex parameters. The particles are designated according to $P_{11}(938)-N$, $P_{33}(1232)-\Delta$, $P_{11}(1440)-R$, $D_{13}(1520)-D$, $S_{11}(1535)-S$, and $S_{31}(1620)-S'$.

Vertices	Parameters. Particle masses m and cutoff masses Λ are in MeV. $m_\pi=139.57,\ m_\eta=547.45.$
$\pi N\Leftrightarrow N$, $\pi\Delta\Leftrightarrow N$	$g_{\pi NN}^{(0)}/\sqrt{4\pi}=6.5069,\ m_N^{(0)}=777.78,\ m_N=938.92,\ \lambda_{NN}=0.0,\ \Lambda_{\pi NN}^{(0)}=1529.5,$ $g_{\pi N\Delta}^{(0)}/\sqrt{4\pi}=0.39835,\ \Lambda_{\pi\Delta N}^{(0)}=2349.8$
$\pi N\Leftrightarrow\Delta$	$g_{\pi N\Delta}^{(0)}/\sqrt{4\pi}=0.39835,\ m_\Delta^{(0)}=1367.3,\ m_\Delta=1232.0,\ \Lambda_{\pi N\Delta}^{(0)}=2304.6$
$\pi N\Leftrightarrow R$, $\pi\Delta\Leftrightarrow R$	$g_{\pi NR}^{(0)}/\sqrt{4\pi}=1.3501,\ m_R^{(0)}=1304.4,\ m_R=1440.0,\ \lambda_{NR}=0.0,\ \Lambda_{\pi NR}^{(0)}=4003.2,$ $g_{\pi R\Delta}^{(0)}/\sqrt{4\pi}=-1.3219,\ \Lambda_{\pi\Delta R}^{(0)}=918.44$
$\pi N\Leftrightarrow D$, $\pi\Delta\Leftrightarrow D$	$g_{\pi ND}^{(0)}/\sqrt{4\pi}=0.26510,\ m_D^{(0)}=1451.2,\ m_D=1520.0,\ \Lambda_{\pi ND}^{(0)}=5029.3,$ $g_{\pi\Delta D}^{(0)}/\sqrt{4\pi}=-0.72369,\ \Lambda_{\pi\Delta D}^{(0)}=600.0$
$\pi N\Leftrightarrow S$, $\eta N\Leftrightarrow S$	$g_{\pi NS}^{(0)}/\sqrt{4\pi}=0.16071,\ m_S^{(0)}=1874.3,\ m_S=1535.0,\ \lambda_{\pi NS}=0.0,\ \Lambda_{\pi NS}^{(0)}=4886.3,$ $g_{\eta NS}^{(0)}/\sqrt{4\pi}=0.77541,\ \lambda_{\eta NS}=0.0,\ \Lambda_{\eta NS}^{(0)}=2583.4$
$\pi N\Leftrightarrow S'$, $\pi\Delta\Leftrightarrow S'$	$g_{\pi NS'}^{(0)}/\sqrt{4\pi}=3.6812,\ m_{S'}^{(0)}=1302.0,\ m_{S'}=1620.0,\ \lambda_{\pi NS'}=0.0,\ \Lambda_{\pi NS'}^{(0)}=2581.4$ $g_{\pi\Delta S'}^{(0)}/\sqrt{4\pi}=18.277,\ \Lambda_{\pi\Delta S'}^{(0)}=800.0$

TABLE III. Exchange potential parameters.

Exchange potentials	Parameters. Particle masses m and cutoff masses Λ are in MeV.
N exchange, $\pi N \Leftrightarrow \pi N$	$g_{\pi NN}/\sqrt{4\pi}=3.7815$, $\lambda_{NN}=0.0$, $\Lambda_{\pi NN}=2483.4$
N exchange, $\pi N \Leftrightarrow \pi\Delta$	$g_{\pi NN}/\sqrt{4\pi}=3.7815$, $g_{\pi N\Delta}/\sqrt{4\pi}=0.40707$, $\lambda_{NN}=0.0$, $\Lambda_{\pi NN}=2483.4$, $\Lambda_{\pi N\Delta}=4167.3$
Δ exchange, $\pi N \Leftrightarrow \pi N$	$g_{\pi N\Delta}/\sqrt{4\pi}=0.40707$, $\Lambda_{\pi\Delta N}=2361.8$
σ exchange, $\pi N \Leftrightarrow \pi N$	$g_{\sigma\pi\pi}g_{\sigma NN}/4\pi=256.63$, $\tilde{g}_{\sigma\pi\pi}g_{\sigma NN}/4\pi=-168.44$, $m_\sigma=1011.2$, $\Lambda_{\sigma\pi\pi}=2641.5$, $\Lambda_{\sigma NN}=2013.6$
ρ exchange, $\pi N \Leftrightarrow \pi N$	$g_{\rho\pi\pi}g_{\rho NN}/4\pi=6.7264$, $\kappa_\rho=7.8540$, $m_\rho=769.0$, $\Lambda_{\rho\pi\pi}=4975.4$, $\Lambda_{\rho NN}=2447.0$

$$T^I_{LJ}(W)=\int d\Omega_{\mathbf{q}}\, d\Omega_{\mathbf{q}'}\, \mathcal{Y}^{M\dagger}_{L,1/2,J}(\hat{\mathbf{q}})\langle (IM')_{\pi N}| \times T_{\pi N,\pi N}(\mathbf{q},\mathbf{q}';W+i\varepsilon)|(IM')_{\pi N}\rangle \mathcal{Y}^{M}_{L,1/2,J}(\hat{\mathbf{q}}') = -(32\pi^2 W/2iq)\{\eta^I_{LJ}(W)\exp[2i\delta^I_{LJ}(W)]-1\},$$

$$W=W_{\pi N}(\mathbf{q})=W_{\pi N}(\mathbf{q}'), \tag{4.13}$$

where I is the total isospin of the πN system, and δ^I_{LJ} and η^I_{LJ} are the phase shift and inelasticity, respectively, for the (L,I,J) partial wave.

V. RESULTS

The parameters in our model were determined by a least squares fit to the phase shifts, δ^I_{LJ}, and inelasticities, η^I_{LJ}, of the SAID-SP95 analysis of the pion-nucleon scattering data [45]. The partial waves included in the fit are given in Table I, along with the single-baryon and meson-baryon states that contribute in each partial wave. The parameters in parentheses, i.e., $l_{\eta N}$ and $l_{\pi\Delta}$, are the orbital angular momentum quantum numbers for the inelastic channels. In carrying out the fits we have allowed the $\pi\Delta$ threshold energy to vary with the partial wave. This is partly justified by the fact that the $\Delta=P_{33}(1232)$ has a width of 120 MeV, and by the related fact that the $\pi\Delta$ channel mocks up the effect of the $\pi\pi N$ channel. The Δ threshold masses are given in Table I. The model parameters that result from the fit are given in Tables II and III, while comparisons of the theoretical and experimental phase shifts and inelasticities are shown in Figs. 1–14.

The parameters associated with the $\langle \mu B|U|B'\rangle$ or vertex interactions are given in Table II. For each of the baryons there is a bare mass $m_B^{(0)}$, which appears in the effective μB-$\mu' B'$ potentials given by Eq. (4.8), as well as a bare coupling constant for each meson-baryon channel that the baryon couples to. For some of the interactions $(\pi N \Leftrightarrow N,R,S,S';\eta N \Leftrightarrow S)$ there is also a parameter, called λ, that determines the mix of nonderivative and derivative coupling; where $\lambda=1$ and $\lambda=0$ correspond to pure nonderivative and pure derivative coupling, respectively. Remarkably, we found that in every case the derivative coupling is preferred. With each vertex there is also associated a bare cutoff mass that appears in the form factor given by Eq. (3.38). As is shown clearly by the analysis given in Sec. IV of Ref. [40], the bare coupling constants, masses, and vertex functions are dressed by the interactions, just as in a full-blown field theory. The N-exchange and Δ-exchange coupling constants and cutoff masses given in Table III are interpreted as dressed parameters. The parameters for the σ- and ρ-exchange potentials include products of coupling constants, the masses of the exchanged particles, and different cutoff masses for each vertex. The ρ-exchange potential also

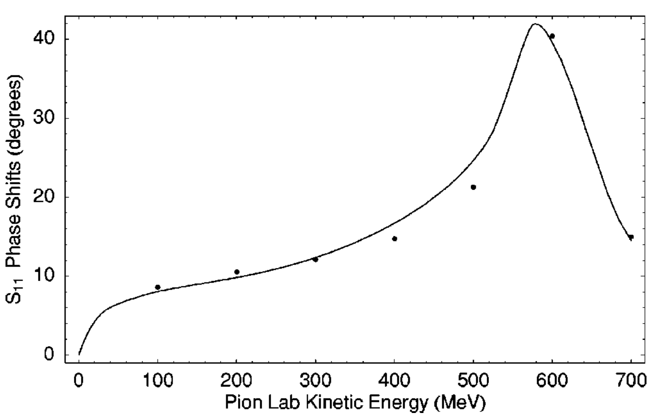

FIG. 1. Fit of the S_{11} phase shifts to the SAID-SP95 analysis.

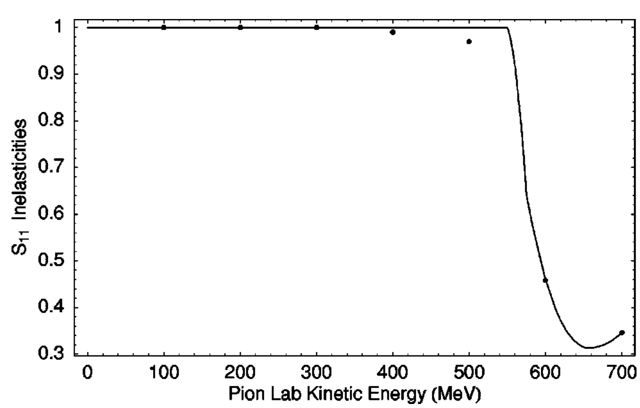

FIG. 2. Fit of the S_{11} inelasticities to the SAID-SP95 analysis.

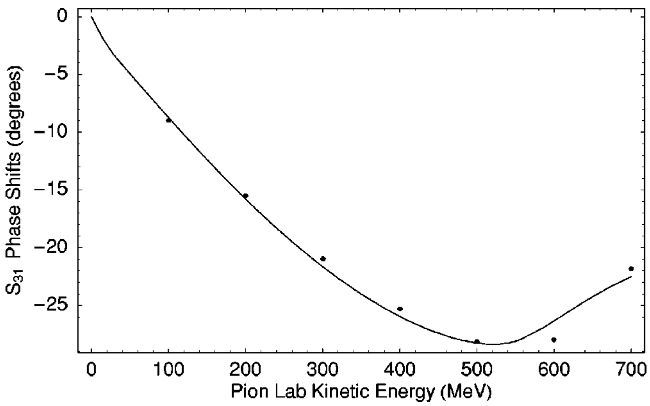

FIG. 3. Fit of the S_{31} phase shifts to the SAID-SP95 analysis.

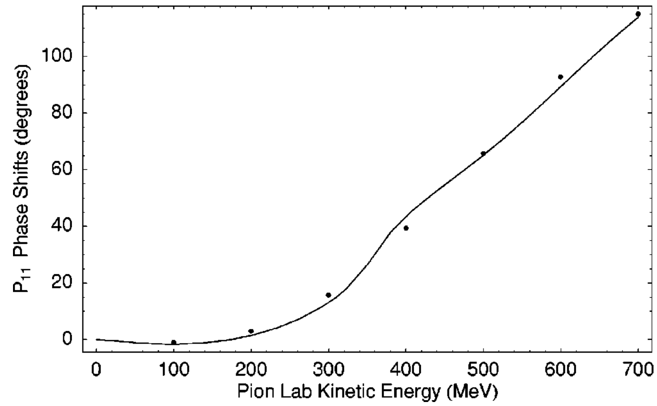

FIG. 5. Fit of the P_{11} phase shifts to the SAID-SP95 analysis.

includes $\varkappa_\rho$, which determines the relative strength of the tensor and vector ρNN coupling. The exponent n that appears in the form factors, Eqs. (3.38a) and (3.40), is $n=10$ in all cases. This gave us excellent numerical stability in the numerical solution of the partial wave integral equations obtained from Eq. (4.7).

The parameters that play a role in the P_{11} partial wave are constrained not only by the fit to the P_{11} elastic amplitude, but also by the requirement that this amplitude should have a pole at $W=m_N$ with a residue determined by the dressed or physical coupling constant $g_{\pi NN}$. It follows from Eqs. (3.29), (4.8), and (4.13) that at the nucleon pole

$$T^{1/2}_{1,1/2}(W)\rightarrow -\frac{12\pi m_\pi^2 g^2_{\pi\pi N}}{2m_N(W-m_N)}. \tag{5.1}$$

We have ensured that the residue that appears here is such that $g^2_{\pi\pi N}/4\pi=14.3$, which is consistent with the value given in Table III for the N-exchange potentials.

VI. DISCUSSION

It is clear that the model presented here gives a reasonable description of the pion-nucleon elastic scattering amplitude up to a pion laboratory kinetic energy of 700.0 MeV. The model accounts for the rapid variation of the amplitudes due to the presence of the various resonances, as well as the opening of the inelastic channels.

At this point it is appropriate to compare our exchange model with the models created by other workers. To set the stage for the comparison, we restate the ingredients of our model in terms of the various contributions to the effective μB-$\mu' B'$ potentials that appear in the coupled Lippmann-Schwinger equations (4.7), and which are defined by Eq. (4.8). The first term on the right hand side of Eq. (4.8) is one of the interactions $U_{\pi N,\pi N}$, $U_{\pi N,\pi\Delta}$, or $U_{\pi\Delta,\pi N}$, where $U_{\pi N,\pi N}$ is due to N, Δ, σ, and ρ exchange, while $U_{\pi N,\pi\Delta}$, and $U_{\pi\Delta,\pi N}$ are due to N exchange. These potentials contribute in all partial waves. The second term on the right hand side of Eq. (4.8) consists of energy dependent potentials, the so-called pole contributions, which arise from the processes $\mu B \Leftrightarrow B'' \Leftrightarrow \mu' B'$. In our model $B''=\{N,\ \Delta=P_{33}(1232),\ R=P_{11}(1440),\ D=D_{13}(1520),\ S=S_{11}(1535),\ S'=S_{31}(1620)\}$. Perusal of Table II shows how these baryons mediate coupling to the inelastic channels.

In general the models constructed by other workers contain interactions which play a role similar to our $U_{\pi N,\pi N}$, and are due to N, Δ, σ, and ρ exchange. They also have interactions which correspond to the pole terms in Eq. (4.8), where the intermediate baryons are an N or a Δ. This is true for the models reported in Refs. [6–8] and [10–15], except that the Sato-Lee model [8] has no σ exchange; and the Gross-Surya model [13,14] has no Δ exchange as the result of an approximation that is made. Also, the Schütz *et al.* models [10–12] have a more sophisticated treatment of σ

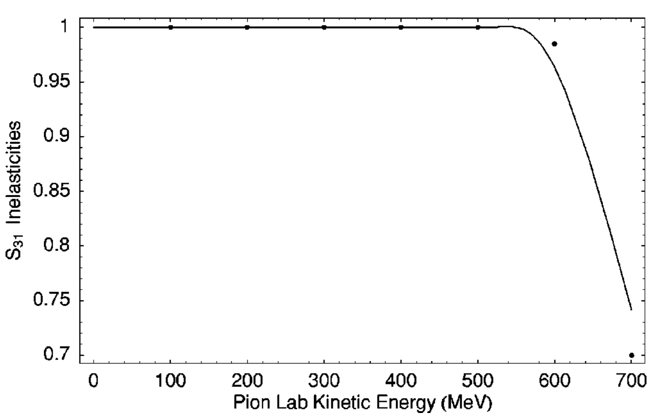

FIG. 4. Fit of the S_{31} inelasticities to the SAID-SP95 analysis.

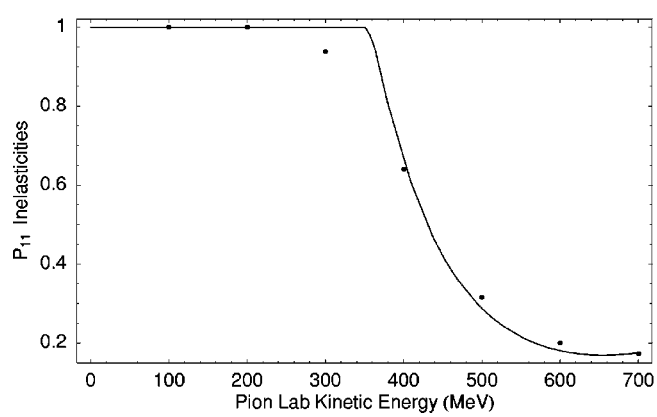

FIG. 6. Fit of the P_{11} inelasticities to the SAID-SP95 analysis.

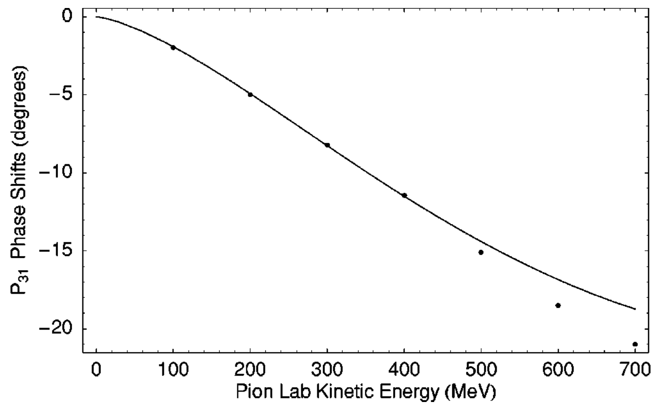

FIG. 7. Fit of the P_{31} phase shifts to the SAID-SP95 analysis.

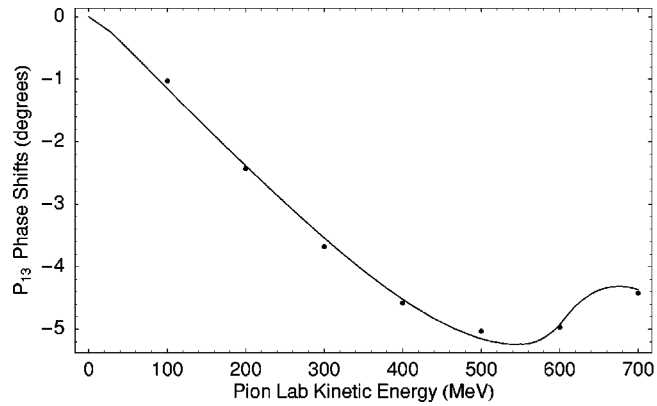

FIG. 9. Fit of the P_{13} phase shifts to the SAID-SP95 analysis.

and ρ exchange in that the scalar-isoscalar (σ) and vector-isovector (ρ) terms are viewed as arising from a correlated pair of two pions in the $J=0$ (σ) and $J=1$ (ρ) t channels. Some models also include the effects of baryons other than the N and Δ. In particular, the model of Ref. [12] also includes a pole contribution due to the $S_{11}(1535)$ resonance. The Gross-Surya model [13,14] includes R and D poles, as well as R exchange. The Pascalutsa-Tjon model [15] takes into account the R pole and R exchange.

The models reported in Refs. [6–8,15] do not allow for inelastic channels. The original Gross-Surya model [13] includes inelasticity through the processes $R \Leftrightarrow \pi\Delta'$ and $D \Leftrightarrow \pi\Delta'$, where Δ' is an effective baryon with the spin and isospin of the actual Δ, while in their second model [14] the inelasticity arises through the processes $R \Leftrightarrow \sigma^* N$ and $D \Leftrightarrow \sigma^* N$ where σ^* is an isoscalar-scalar particle with a mass equal to that of two pions. The Schütz *et al.* model reported in Ref. [12] allows for $\pi\Delta$, ηN, and σN inelastic channels with various mechanisms for coupling to these channels. Their πN-ηN interaction is due to N and a_0 exchange, and the direct process $\pi N \Leftrightarrow S_{11}(1535) \Leftrightarrow \eta N$; while their ηN-ηN interaction is due to N and f_0 exchange, and $\eta N \Leftrightarrow S_{11}(1535) \Leftrightarrow \eta N$. Their πN-$\pi\Delta$ and $\pi\Delta$-$\pi\Delta$ interactions are due to N, Δ, and ρ exchange. Their πN-σN interaction arises from N exchange, while their σN-σN interaction arises from N and σ exchange. The emphasis in their work [12] is on trying to decide whether the $S_{11}(1535)$ and $P_{11}(1440)$ resonances are genuine three-quark states, or if they are generated dynamically by the interactions in the πN model. This is an issue which other workers, including one of us, have also considered [40,46], but in the present work we have been more concerned with obtaining a good accounting of the data with as simple a model as possible.

The Gross-Surya model [14], the Schütz *et al.* model [12], and the present model account for the πN, elastic partial wave amplitudes up to pion laboratory kinetic energies of 600, 744, and 700 MeV, respectively. In their common energy range, the Gross-Surya fits [14] and ours are of similar quality. The S_{11}, P_{11}, and P_{33} elastic amplitudes obtained in the Schütz *et al.* model [12] and the present model are similar. Their P_{31} phase shifts rise above the experimental phases at the high energy end, while ours fall below. Our P_{13} phase shifts are in noticeably better agreement with the experimental phases than theirs. Since Schütz *et al.* [12] do not include $S_{31}(1620)$ and $D_{13}(1520)$ poles in their model, they do not obtain a good accounting of the S_{31} and D_{13} phase shifts and inelasticities in the neighborhood of these resonances.

Trying to establish the nature of the πN resonances is a difficult and rather old problem. As pointed out in Sec. I , the original Chew-Low model [5] was able to produce the $P_{33}(1232)$ resonance without introducing a single-baryon state corresponding to the Δ. Nowadays, most people agree that the Δ is a three-quark state, and would not question including it in our model on a footing equivalent to the

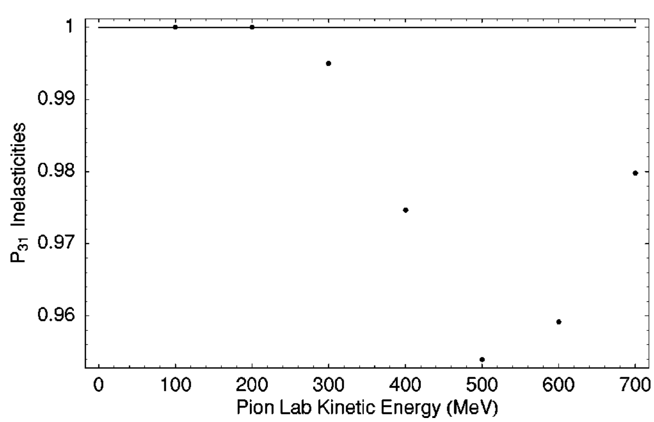

FIG. 8. Fit of the P_{31} inelasticities to the SAID-SP95 analysis.

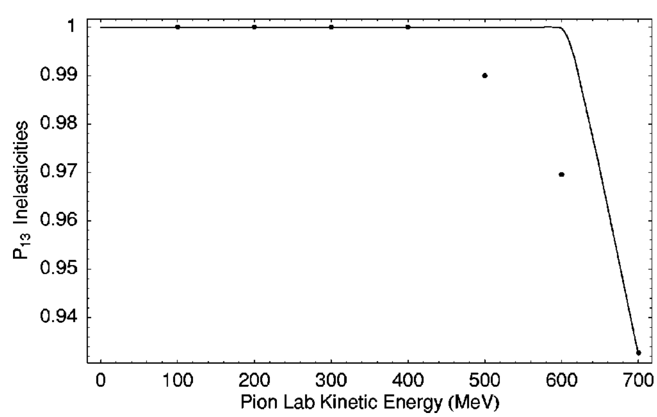

FIG. 10. Fit of the P_{13} inelasticities to the SAID-SP95 analysis.

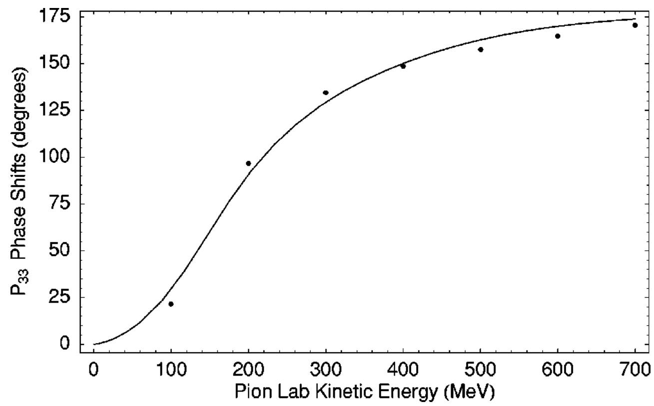

FIG. 11. Fit of the P_{33} phase shifts to the SAID-SP95 analysis.

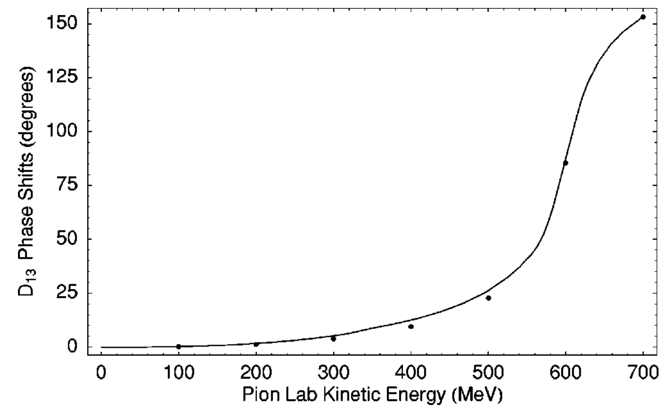

FIG. 13. Fit of the D_{13} phase shifts to the SAID-SP95 analysis.

nucleon. At higher energies the importance of the inelastic channels makes it even more difficult to pin down the nature of the resonances, as it is clear that in certain cases the coupling to the inelastic channels is sufficient to produce a resonance [40,46,12]. Unless an unambiguous procedure is developed for deriving effective few-particle equations from QCD that are valid over a wide energy range, the nature of the πN resonances will probably remain ambiguous. An analysis of the inelastic cross sections, as well as calculations of pion photoproduction and electroproduction, should make it possible to reduce the ambiguities.

In its present form, our model will make it possible to calculate η production up to pion laboratory energies of 700 MeV, and pion photoproduction up to photon laboratory energies of 850 MeV. As pointed out in Sec. I, one of us has extended the mass operator–Okubo method used in the present work to allow for a photon-nucleon channel [42]. This extension is both Poincaré and gauge invariant, so a satisfactory formalism exists for applying our present model to photoproduction calculations. An effort is underway to extend the formalism so as to make possible electroproduction calculations. This involves replacing real photons with virtual photons.

Extending our model to higher energies will obviously entail including more resonances and inelastic channels. In an isobar model analysis of $\pi N \rightarrow \pi\pi N$ for total c.m. energies in the range 1.32–1.93 GeV, Manley *et al.* [47] found it necessary to treat the inelasticity as arising from a coherent superposition of the two-body channels; $\pi\Delta$, ρN, σN, and πR. In a more recent multichannel resonance parametrization of πN scattering, Manley and Saleski [48] extended this set to include ηN, $K\Lambda$, ωN, and $\rho\Delta$ channels. Even though at higher energies an exchange model of the type developed here becomes quite complex, it is clear that with present day computing capabilities it is tractable.

Our present model can be used as an ingredient in models of the $NN\pi$ and $N\pi\pi$ systems. In three-particle systems such as these, besides Poincaré invariance, it is also necessary to consider *cluster separability*. This is the requirement that when subsystems are separated by large spacelike intervals, the subsystems should become dynamically independent. In particular, the Hamiltonian, as well as the other Poincaré generators, should reduce to the sum of the subsystem generators. For a nonrelativistic system it is rather trivial to satisfy cluster separability, but for relativistic models based on the Bakamjian-Thomas construction [36] this requirement can be problematic. It is possible to construct unitary transformations which when applied to the generators obtained with the Bakamjian-Thomas construction, lead to generators that satisfy cluster separability [37,38,49,50]. These unitary operators are called Sokolov transformations or packing operators. Unfortunately the construction of these operators is rather complicated. For a true three-particle system, i.e., one in which no particle creation or annihilation

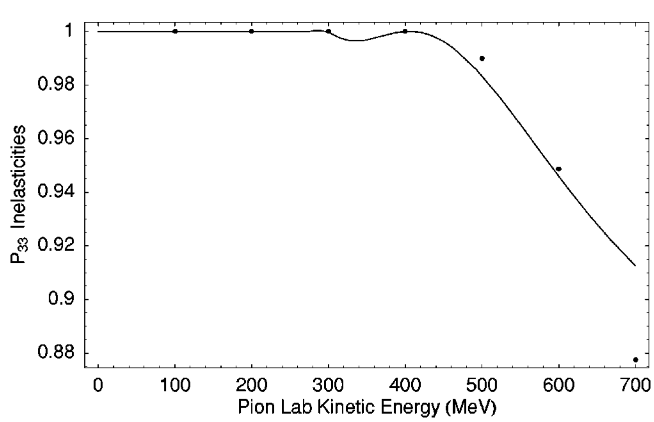

FIG. 12. Fit of the P_{33} inelasticities to the SAID-SP95 analysis.

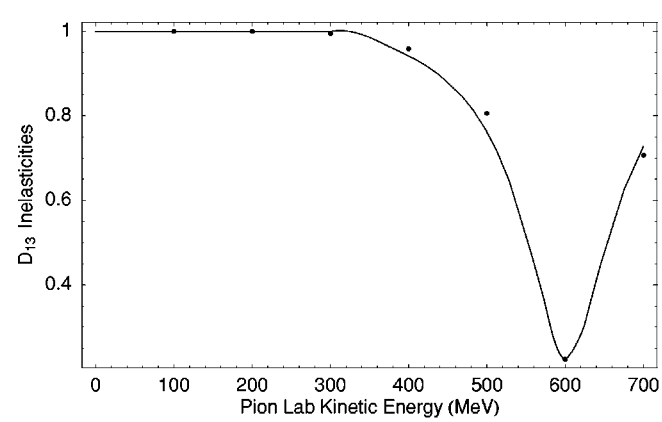

FIG. 14. Fit of the D_{13} inelasticities to the SAID-SP95 analysis.

takes place, it has been shown that the Sokolov transformation has no effect on the bound state spectrum or the S matrix [37], thus calculations of three-particle observables can be carried out without ever actually constructing the packing operators. Fadeev-like integral equations for three-particle systems described by Bakamjian-Thomas models have been derived by several workers [37,51,52]. The kernels of these equations are related to the two-particle t matrices, however, the connection is not as direct as it is in the nonrelativistic Fadeev equations.

It is possible to construct Bakamjian-Thomas, three-particle models in which particle creation and annihilation takes place, and cluster separability is satisfied. These are essentially relativistic Lee models [53], whose superselection rules lead to the satisfaction of the cluster separability requirement [37,38]. The Betz-Coester model of the $NN\pi$ system [54] is a model of this type. This model allows for the inclusion of a $\pi N \Leftrightarrow \Delta$ vertex, so it can accommodate the type of model for the πN system that has been presented here. The calculations of Betz and Lee [55] demonstrate that a Betz-Coester type of model provides a practical framework for analyzing the $NN\pi$ system. The $V\theta$-$N\theta\theta$ sector of the Lee model [53] has many of the features of the $N\pi\pi$ system [56], so it should also be possible to construct a Bakamjian-Thomas model of the $N\pi\pi$ system that satisfies cluster separability. We are presently exploring the application of our πN model to the $N\pi\pi$ system.

ACKNOWLEDGMENT

This work was supported in part by National Science Foundation Grant No. PHY-9605215.

APPENDIX

In this appendix, we give the interaction Lagrangian densities $\mathcal{L}_I(x)$, as well as the various vertex functions, $U_{\mu B,B'}(\mathbf{q})$, and potentials, $U_{\mu B,\mu' B'}(\mathbf{q},\mathbf{q}')$. The fields for the various baryons are notated according to $P_{11}(938)$-N, $P_{33}(1232)$-Δ, $P_{11}(1440)$-R, $D_{13}(1520)$-D, $S_{11}(1535)$-S, and $S_{31}(1620)$-S'. The meson fields are notated in the obvious way; i.e., π, η, σ, and ρ.

The interaction Lagrangians are

$$\mathcal{L}_{\pi NN}(x) = -ig_{\pi NN}\overline{N}(x)[\Gamma_{NN}(i\partial)\boldsymbol{\tau}\cdot\boldsymbol{\pi}(x)]N(x), \quad \text{(A1)}$$

$$\mathcal{L}_{\pi NR}(x) = -ig_{\pi NR}\overline{R}(x)[\Gamma_{NR}(i\partial)\boldsymbol{\tau}\cdot\boldsymbol{\pi}(x)]N(x) + (\dagger), \quad \text{(A2)}$$

$$\Gamma_{NB}(q) = \left(\lambda_{NB} + \frac{1-\lambda_{NB}}{m_N + m_B}\not{q}\right)\gamma_5, \quad \text{(A3)}$$

$$\mathcal{L}_{\pi B\Delta}(x) = -(g_{\pi B\Delta}/m_\pi)\overline{\Delta}^\mu(x)[\partial_\mu \mathbf{T}^\dagger_{B\Delta}\cdot\boldsymbol{\pi}(x)]B(x) + (\dagger),$$
$$B = N, R \quad \text{(A4)}$$

$$\mathcal{L}_{\pi ND}(x) = (g_{\pi ND}/m_\pi)\overline{D}^\mu(x)[\partial_\mu\boldsymbol{\tau}\cdot\boldsymbol{\pi}(x)]\gamma_5 N(x) + (\dagger), \quad \text{(A5)}$$

$$\mathcal{L}_{\pi\Delta D}(x) = (g_{\pi\Delta D}/m_\pi)\overline{D}^\mu(x)[\not{\partial}\mathbf{T}_{D\Delta}\cdot\boldsymbol{\pi}(x)]\Delta_\mu(x) + (\dagger), \quad \text{(A6)}$$

$$\mathcal{L}_{\pi NS}(x) = g_{\pi NS}\overline{S}(x)[\Gamma_{\pi NS}(i\partial)\boldsymbol{\tau}\cdot\boldsymbol{\pi}(x)]N(x) + (\dagger), \quad \text{(A7)}$$

$$\mathcal{L}_{\eta NS}(x) = g_{\eta NS}\overline{S}(x)[\Gamma_{\eta NS}(i\partial)\eta(x)]N(x) + (\dagger), \quad \text{(A8)}$$

$$\Gamma_{\mu NB}(q) = \lambda_{\mu NB} + \frac{1-\lambda_{\mu NB}}{m_B - m_N}\not{q}, \quad \text{(A9)}$$

$$\mathcal{L}_{\pi NS'}(x) = g_{\pi NS'}\overline{S'}(x)[\Gamma_{\pi NS'}(i\partial)\mathbf{T}^\dagger_{NS'}\cdot\boldsymbol{\pi}(x)]N(x) + (\dagger), \quad \text{(A10)}$$

$$\mathcal{L}_{\pi\Delta S'}(x) = i(g_{\pi\Delta S'}/m_\pi)\overline{\Delta}^\mu(x)[\partial_\mu\mathbf{T}^\dagger_{S'\Delta}\cdot\boldsymbol{\pi}(x)]\gamma_5 S'(x) + (\dagger), \quad \text{(A11)}$$

$$\mathcal{L}_{\sigma NN}(x) = g_{\sigma NN}\overline{N}(x)\sigma(x)N(x), \quad \text{(A12)}$$

$$\mathcal{L}_{\sigma\pi\pi}(x) = \left[g_{\sigma\pi\pi}\frac{m_\pi}{2}\boldsymbol{\pi}(x)\cdot\boldsymbol{\pi}(x) + \tilde{g}_{\sigma\pi\pi}\frac{1}{2m_\pi}[\partial^\mu\boldsymbol{\pi}(x)]\cdot[\partial_\mu\boldsymbol{\pi}(x)]\right]\sigma(x), \quad \text{(A13)}$$

$$\mathcal{L}_{\rho NN}(x) = g_{\rho NN}\overline{N}(x)(1/2)\boldsymbol{\tau}\cdot[\gamma^\mu\boldsymbol{\rho}_\mu(x) + (\kappa_\rho/2m_N)\sigma^{\mu\nu}\partial_\mu\boldsymbol{\rho}_\nu(x)]N(x), \quad \text{(A14)}$$

$$\mathcal{L}_{\rho\pi\pi}(x) = g_{\rho\pi\pi}\boldsymbol{\rho}^\mu(x)\cdot[\boldsymbol{\pi}(x)\times\partial_\mu\boldsymbol{\pi}(x)]. \quad \text{(A15)}$$

In writing the results for the vertex functions, $U_{\mu B,B'}(\mathbf{q})$, and potentials, $U_{\mu B,\mu' B'}(\mathbf{q},\mathbf{q}')$ we use the shorthand notation $\varepsilon_B = \varepsilon_B(\mathbf{q})$, $\omega'_\mu = \omega_\mu(\mathbf{q}')$, $\varepsilon''_B = \varepsilon_B(\mathbf{q}+\mathbf{q}')$, etc.

The $\pi N,\Delta$ vertex function is derived from Eq. (3.32), and is found to be

$$\langle t(ih)_N|U_{\pi N,\Delta}(\mathbf{q})|(i'h')_\Delta\rangle = -i(g_{\pi N\Delta}/m_\pi)(\varepsilon_N + m_N)^{1/2}(2m_\Delta)^{1/2} \times(\boldsymbol{\varepsilon}^*_t\cdot\mathbf{T}_{N\Delta})_{ii'}(\mathbf{q}\cdot\mathbf{S}_{N\Delta})_{hh'}, \quad \text{(A16)}$$

where $\boldsymbol{\varepsilon}_t$ is defined by Eq. (3.26), and $\mathbf{T}_{N\Delta}$ and $\mathbf{S}_{N\Delta}$ are defined by Eq. (3.33).

The $\pi N,\pi\Delta$ transition potential due to nucleon exchange is derived from Eq. (3.35), and is given by

$$\begin{aligned} U^N_{\pi N,\pi\Delta}(\mathbf{q},\mathbf{q}') = {}& g_{\pi NN}(g_{\pi N\Delta}/m_\pi)(-\sqrt{8/3}P^{\pi N,\pi\Delta}_{1/2} \\ &+ \sqrt{5/3}P^{\pi N,\pi\Delta}_{3/2})(\varepsilon_N + m_N)^{1/2}(\varepsilon'_\Delta + m_\Delta)^{1/2} \\ &\times[F(\mathbf{q},\mathbf{q}';m_N,m_\Delta)\boldsymbol{\sigma}\cdot\mathbf{x}_N \\ &+ F(\mathbf{q},\mathbf{q}';-m_N,-m_\Delta)\boldsymbol{\sigma}\cdot\mathbf{x}'_\Delta] \\ &\times[\mathbf{q}\cdot\mathbf{S}_{N\Delta} + (1/m_\Delta)(\omega_\pi + \mathbf{q}\cdot\mathbf{x}'_\Delta)\mathbf{q}'\cdot\mathbf{S}_{N\Delta}], \end{aligned}$$

$$F(\mathbf{q},\mathbf{q}';m_N,m_\Delta)=\frac{1}{2\varepsilon_N''}\left[\frac{\omega_\pi'+\varepsilon_\Delta'-m_\Delta}{\varepsilon_N-\omega_\pi'-\varepsilon_N''}+1-\frac{1-\lambda_{NN}}{2m_N}\right.$$
$$\left.\times(\varepsilon_N-\varepsilon_\Delta'+\varepsilon_N''+m_\Delta+2m_N)\right], \tag{A17}$$

$$\mathbf{x}_N=\mathbf{q}/(\varepsilon_N+m_N),\quad \mathbf{x}_\Delta'=\mathbf{q}'/(\varepsilon_\Delta'+m_\Delta). \tag{A18}$$

Here $P_I^{\pi N,\pi\Delta}$ is an operator that connects states of the πN system of total isospin I to states of the $\pi\Delta$ system of total isospin I, and its matrix elements between normalized states is simply 1.

The $\pi N,\pi N$ potential due to Δ exchange is derived from Eqs. (3.36) and (3.37), and is found to be

$$U^{\Delta}_{\pi N,\pi N}(\mathbf{q},\mathbf{q}')=(g_{\pi N\Delta}/m_\pi)^2\left(\frac{4}{3}P^{\pi N}_{1/2}+\frac{1}{3}P^{\pi N}_{3/2}\right)$$
$$\times(\varepsilon_N+m_N)^{1/2}(\varepsilon_N'+m_N)^{1/2}$$
$$\times[G(\mathbf{q},\mathbf{q}';m_N,m_\Delta)+(\boldsymbol{\sigma}\cdot\mathbf{x}_N)$$
$$\times(\boldsymbol{\sigma}\cdot\mathbf{x}_N')G(\mathbf{q},\mathbf{q}';-m_N,-m_\Delta)]$$
$$\times\frac{1}{4\varepsilon_\Delta''}\left(\frac{1}{\varepsilon_N-\varepsilon_\Delta''-\omega_\pi'}+\frac{1}{\varepsilon_N'-\varepsilon_\Delta''-\omega_\pi}\right),$$

$$G(\mathbf{q},\mathbf{q}';m_N,m_\Delta)=[(\varepsilon_\Delta''+m_\Delta)-(\varepsilon_N-m_N)-(\varepsilon_N'-m_N)]$$
$$\times[(1-\alpha-\alpha')\mathbf{q}\cdot\mathbf{q}'+(2/3)$$
$$\times(\alpha\alpha'|\mathbf{q}+\mathbf{q}'|^2-\alpha\mathbf{q}^2-\alpha'\mathbf{q}'^2)]$$
$$+(1/3)(1-\alpha-\alpha')(\varepsilon_N-m_N)$$
$$\times(\varepsilon_N'-m_N)[(\varepsilon_\Delta''-m_\Delta)$$
$$-(\varepsilon_N+m_N)-(\varepsilon_N'+m_N)], \tag{A19}$$

$$\alpha=\alpha(\mathbf{q},\mathbf{q}')=\frac{1}{m_\Delta}\left[\omega_\pi+m_\Delta+\frac{\mathbf{q}\cdot(\mathbf{q}+\mathbf{q}')}{\varepsilon_\Delta''+m_\Delta}\right],$$

$$\alpha'=\alpha(\mathbf{q}',\mathbf{q}).$$

The $\pi N,B$ and $\pi\Delta,B$ vertex functions, where B is the nucleon or the Roper resonance, are calculated from matrix elements such as Eqs. (3.25) and (3.36), and are found to be

$$\langle t(ih)_N|U_{\pi N,B}(\mathbf{q})|(i'h')_B\rangle$$
$$=ig_{\pi NB}(\boldsymbol{\varepsilon}_t^*\cdot\boldsymbol{\tau})_{ii'}(\varepsilon_N+m_N)^{1/2}(2m_B)^{1/2}$$
$$\times\left[\lambda_{NB}+(1-\lambda_{NB})\frac{W_{\pi N}+m_N}{m_B+m_N}\right](\boldsymbol{\sigma}\cdot\mathbf{x}_N)_{hh'}, \tag{A20}$$

$$\langle t(ih)_\Delta|U_{\pi\Delta,B}(\mathbf{q})|(i'h')_B\rangle$$
$$=-i(g_{\pi B\Delta}/m_\pi)(\boldsymbol{\varepsilon}_t^*\cdot\mathbf{T}^\dagger_{B\Delta})_{ii'}(\varepsilon_\Delta+m_\Delta)^{1/2}(2m_B)^{1/2}$$
$$\times(W_{\pi\Delta}/m_\Delta)(\mathbf{q}\cdot\mathbf{S}^\dagger_{B\Delta})_{hh'}, \tag{A21}$$

$$B=N,R.$$

The $\pi N,D$ and $\pi\Delta,D$ vertex functions, where D is the $D_{13}(1520)$ resonance, are calculated from the matrix elements

$$\langle\mathbf{k}t(\mathbf{p}ih)_N|H_{\pi ND}|(\mathbf{p}'i'h')_D\rangle$$
$$=(2\pi)^3\delta^3(\mathbf{Q}-\mathbf{p}')i(g_{\pi ND}/m_\pi)$$
$$\times(\boldsymbol{\varepsilon}_t^*\cdot\boldsymbol{\tau})_{ii'}\overline{u_N}(p,h)\gamma_5k_\mu u_D^\mu(p',h'), \tag{A22}$$

$$\langle\mathbf{k}t(\mathbf{p}ih)_\Delta|H_{\pi\Delta D}|(\mathbf{p}'i'h')_D\rangle$$
$$=(2\pi)^3\delta^3(\mathbf{Q}-\mathbf{p}')(-i)(g_{\pi\Delta D}/m_\pi)$$
$$\times(\boldsymbol{\varepsilon}_t^*\cdot\mathbf{T}^\dagger_{D\Delta})_{ii'}\overline{u^\Delta_\mu}(p,h)\not{k}u_D^\mu(p',h'), \tag{A23}$$

and are found to be

$$\langle t(ih)_N|U_{\pi N,D}(\mathbf{q})|(i'h')_D\rangle$$
$$=-i(g_{\pi ND}/m_\pi)(\boldsymbol{\varepsilon}_t^*\cdot\boldsymbol{\tau})_{ii'}(\varepsilon_N+m_N)^{1/2}$$
$$\times(2m_D)^{1/2}(\boldsymbol{\sigma}\cdot\mathbf{x}_N\,\mathbf{S}_{ND}\,.\mathbf{q})_{hh'}, \tag{A24}$$

$$\langle t(ih)_\Delta|U_{\pi\Delta,D}(\mathbf{q})|(i'h')_D\rangle$$
$$=i(g_{\pi\Delta D}/m_\pi)(\boldsymbol{\varepsilon}_t^*\cdot\mathbf{T}^\dagger_{D\Delta})_{ii'}(\varepsilon_\Delta+m_\Delta)^{1/2}(2m_D)^{1/2}$$
$$\times(W_{\pi\Delta}-m_\Delta)\left[1+\frac{(\mathbf{q}\cdot\mathbf{S}^\dagger_{N\Delta})(\mathbf{q}\cdot\mathbf{S}_{ND})}{m_\Delta(\varepsilon_\Delta+m_\Delta)}\right]_{hh'}. \tag{A25}$$

The $\pi N,S$ and $\eta N,S$ vertex functions, where S is the $S_{11}(1535)$ resonance, are calculated from the matrix elements

$$\langle\mathbf{k}t(\mathbf{p}ih)_N|H_{\pi NS}|(\mathbf{p}'i'h')_S\rangle$$
$$=(2\pi)^3\delta^3(\mathbf{Q}-\mathbf{p}')(-g_{\pi NS})$$
$$\times(\boldsymbol{\varepsilon}_t^*\cdot\boldsymbol{\tau})_{ii'}\overline{u_N}(p,h)\Gamma_{\pi NS}(k)u_S(p',h'), \tag{A26}$$

$$\langle\mathbf{k}(\mathbf{p}ih)_N|H_{\eta NS}|(\mathbf{p}'i'h')_S\rangle$$
$$=(2\pi)^3\delta^3(\mathbf{Q}-\mathbf{p}')(-g_{\eta NS})\delta_{ii'}\overline{u_N}(p,h)$$
$$\times\Gamma_{\eta NS}(k)u_S(p',h'), \tag{A27}$$

and are given by

$$\langle t(ih)_N|U_{\pi N,S}(\mathbf{q})|(i'h')_S\rangle = -g_{\pi NS}(\boldsymbol{\varepsilon}_t^*\cdot\boldsymbol{\tau})_{ii'}(\varepsilon_N+m_N)^{1/2}(2m_S)^{1/2} \times\left[\lambda_{\pi NS}+(1-\lambda_{\pi NS})\frac{W_{\pi N}-m_N}{m_S-m_N}\right]\delta_{hh'}, \tag{A28}$$

$$\langle (ih)_N|U_{\eta N,S}(\mathbf{q})|(i'h')_S\rangle = -g_{\eta NS}\delta_{ii'}(\varepsilon_N+m_N)^{1/2}(2m_S)^{1/2} \times\left[\lambda_{\eta NS}+(1-\lambda_{\eta NS})\frac{W_{\eta N}-m_N}{m_S-m_N}\right]\delta_{hh'}. \tag{A29}$$

The $\pi N,S'$ and $\pi\Delta,S'$ vertex functions, where S' is the $S_{31}(1620)$ resonance, are calculated from the matrix elements

$$\langle \mathbf{k}t(\mathbf{p}ih)_N|H_{\pi NS'}|(\mathbf{p}'i'h')_{S'}\rangle = (2\pi)^3\delta^3(\mathbf{Q}-\mathbf{p}')(-g_{\pi NS'}) \times(\boldsymbol{\varepsilon}_t^*\cdot\mathbf{T}_{NS'})_{ii'}\overline{u}_N(p,h)\Gamma_{\pi NS'}(k)u_{S'}(p',h'), \tag{A30}$$

$$\langle \mathbf{k}t(\mathbf{p}ih)_\Delta|H_{\pi\Delta S'}|(\mathbf{p}'i'h')_{S'}\rangle = (2\pi)^3\delta^3(\mathbf{Q}-\mathbf{p}')(g_{\pi\Delta S'}/m_\pi) \times(\boldsymbol{\varepsilon}_t^*\cdot\mathbf{T}_{\Delta S'})_{ii'}\overline{u}_\Delta^\mu(p,h)k_\mu\gamma_5 u_{S'}(p',h'), \tag{A31}$$

and are given by

$$\langle t(ih)_N|U_{\pi N,S'}(\mathbf{q})|(i'h')_{S'}\rangle = -g_{\pi NS'}(\boldsymbol{\varepsilon}_t^*\cdot\mathbf{T}_{NS'})_{ii'}(\varepsilon_N+m_N)^{1/2}(2m_{S'})^{1/2} \times\left[\lambda_{\pi NS'}+(1-\lambda_{\pi NS'})\frac{W_{\pi N}-m_N}{m_{S'}-m_N}\right]\delta_{hh'}, \tag{A32}$$

$$\langle t(ih)_\Delta|U_{\pi\Delta,S'}(\mathbf{q})|(i'h')_{S'}\rangle = -(g_{\pi\Delta S'}/m_\pi)(\boldsymbol{\varepsilon}_t^*\cdot\mathbf{T}_{\Delta S'})_{ii'}(\varepsilon_\Delta+m_\Delta)^{1/2}(2m_{S'})^{1/2} \times(W_{\pi\Delta}/m_\Delta)[(\mathbf{q}\cdot\mathbf{S}_{S'\Delta}^\dagger)(\boldsymbol{\sigma}\cdot\mathbf{x}_\Delta)]_{hh'}. \tag{A33}$$

The $\pi N,\pi N$ potential due to σ exchange is derived from Eq. (3.38), and is found to be

$$U^\sigma_{\pi N,\pi N}(\mathbf{q},\mathbf{q}') = (g_{\sigma NN}/2m_\pi)[g_{\sigma\pi\pi}m_\pi^2+\tilde{g}_{\sigma\pi\pi} \times(\omega_\pi\omega'_\pi-\mathbf{q}\cdot\mathbf{q}')](\varepsilon_N+m_N)^{1/2} \times(\varepsilon'_N+m_N)^{1/2}[1-(\boldsymbol{\sigma}\cdot\mathbf{x}_N)(\boldsymbol{\sigma}\cdot\mathbf{x}'_N)] \times\left[\frac{1}{(\omega_\pi-\omega'_\pi)^2-(\mathbf{q}-\mathbf{q}')^2-m_\sigma^2} +\frac{1}{(\varepsilon_N-\varepsilon'_N)^2-(\mathbf{q}-\mathbf{q}')^2-m_\sigma^2}\right]. \tag{A34}$$

The $\pi N,\pi N$ potential due to ρ exchange is derived from the Feynman-like amplitude

$$\langle t(ih)_N|U^\rho_{\pi N,\pi N}(\mathbf{q},\mathbf{q}')|t'(i'h')_N\rangle = g_{\rho\pi\pi}g_{\rho NN}\left[\frac{1}{2}\boldsymbol{\varepsilon}_{t'}\cdot\boldsymbol{\tau},\frac{1}{2}\boldsymbol{\varepsilon}_t^*\cdot\boldsymbol{\tau}\right]_{ii'}\frac{1}{2}\overline{u}_N(p,h) \times\left[\frac{\Gamma_\mu(p-p')\Delta^{\mu\nu}(p-p')}{(p-p')^2-m_\rho^2}+\frac{\Gamma_\mu(k'-k)\Delta^{\mu\nu}(k'-k)}{(k'-k)^2-m_\rho^2}\right] \times(k_\nu+k'_\nu)u_N(p',h'), \tag{A35}$$

$$\Gamma_\mu(q)=\gamma_\mu+\frac{\kappa_\rho}{2m_N}i\sigma_{\mu\nu}q^\nu,\quad \Delta^{\mu\nu}(q)=-g^{\mu\nu}+\frac{q^\mu q^\nu}{m_\rho^2},$$

$$\mathbf{k}+\mathbf{p}=\mathbf{k}'+\mathbf{p}'=\mathbf{0},$$

and is found to be

$$U^\rho_{\pi N,\pi N}(\mathbf{q},\mathbf{q}')=g_{\rho\pi\pi}g_{\rho NN}\left(P_{1/2}^{\pi N,\pi N}-\frac{1}{2}P_{3/2}^{\pi N,\pi N}\right) \times(\varepsilon_N+m_N)^{1/2}(\varepsilon'_N+m_N)^{1/2} \times\frac{1}{2}[R(\mathbf{q},\mathbf{q}';m_N)+(\boldsymbol{\sigma}\cdot\mathbf{x}_N)(\boldsymbol{\sigma}\cdot\mathbf{x}'_N) \times R(\mathbf{q},\mathbf{q}';-m_N)],$$

$$R(\mathbf{q},\mathbf{q}';m_N) = \frac{S(\mathbf{q},\mathbf{q}';m_N)+(\kappa_\rho/2m_N)(W_{\pi N}-W'_{\pi N})(\varepsilon_N-\varepsilon'_N)}{(\omega_\pi-\omega'_\pi)^2-(\mathbf{q}-\mathbf{q}')^2-m_\rho^2} +\frac{S(\mathbf{q},\mathbf{q}';m_N)}{(\varepsilon_N-\varepsilon'_N)^2-(\mathbf{q}-\mathbf{q}')^2-m_\rho^2}, \tag{A36}$$

$$S(\mathbf{q},\mathbf{q}';m_N)=W_{\pi N}+W'_{\pi N}-2m_N-\frac{\kappa_\rho}{2m_N}[(W_{\pi N}+W'_{\pi N}) \times(\varepsilon_N+\varepsilon'_N-2m_N)-2(\varepsilon_N\varepsilon'_N-\mathbf{q}\cdot\mathbf{q}'-m_N^2)].$$

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