

Bremsstrahlung spectrum in α decay

M. I. Dyakonov*

Laboratoire de Physique Mathématique et Théorique, Université Montpellier II, F- 34095 Montpellier, France

(Received 18 May 1998; published 28 July 1999)

Using our previous approach to electromagnetic emission during tunneling, an explicit, essentially classical, formula describing the bremsstrahlung spectrum in α decay is derived. The role of tunneling motion in photon emission is discussed. The shape of the spectrum is a universal function of the ratio E_γ/E_0 , where E_γ is the photon energy and E_0 is a characteristic energy depending only on the nuclear charge and the energy of the α particle. [S0556-2813(99)07308-2]

PACS number(s): 23.60.+e, 03.65.Sq, 27.80.+w, 41.60.-m

During the α decay of a nucleus the α particle tunnels through the Coulomb barrier and is accelerated beyond the classical turning point to its final energy. Thus electromagnetic radiation should be emitted during the process. In spite of the fundamental nature of the γ emission during α decay, the corresponding bremsstrahlung spectrum was neither measured, nor considered theoretically until recent years. Normally soft γ emission accompanying the Coulomb interaction of heavy particles is well described by classical electrodynamics. The case of α decay is somewhat special, since part of the “trajectory” of the α particle lies within the underbarrier region and may be described classically only in some limited sense, as a motion in imaginary time. While the golden rule of quantum mechanics gives a straightforward recipe to calculate the emission probability, intuitively one would like to understand whether tunneling may be somehow incorporated in the general framework of classical electrodynamics and whether it makes any sense to say that photons are emitted during tunneling motion.

In Ref. [1] the electromagnetic radiation by a charge tunneling through a potential barrier was considered under the conditions that the motion is quasiclassical, i.e., the barrier is smooth compared to the particle wave length, and that the energy of the emitted photons is small compared to the particle energy. It was shown that in this case the emission spectrum is described by the well-known classical formula, involving the Fourier transform of the particle acceleration, with the only difference that the Fourier integral should be taken along a special contour in the complex plane of the time variable. The resulting formula for the bremsstrahlung spectrum per tunneling particle is *classical* (does not contain Planck constant \hbar), even though it describes radiation emitted in the tunneling process, which is unknown to classical mechanics. The result of Ref. [1] may be directly applied to α decay.

On the experimental side, D’Arrigo *et al.* [2] were the first to report bremsstrahlung in α decay. Their data for ^{226}Ra and ^{210}Po gave emission probabilities much larger than those predicted by the so-called Coulomb acceleration model, in which only acceleration outside the Coulomb barrier is taken into account, but close to those predicted by the sudden ac-

celeration model. These results are strange since even the Coulomb acceleration model grossly overestimates the emission probability. Recently Kasagi *et al.* [3] measured the γ emission associated with the α decay of ^{210}Po , obtaining probabilities which are almost two orders of magnitude smaller than those of Ref. [2], and which have a quite different dependence on the photon energy. Kasagi *et al.* applied the general formula of Ref. [1] and by numerical calculations obtained a good fit to their experimental data. However, numerical calculations for the specific case of ^{210}Po do not allow us to understand the general features of the spectrum and its dependence on the nuclear charge and α particle energy. Thus, there is a need for an explicit theoretical formula describing the bremsstrahlung spectrum.

Papenbrock and Bertsch [4] considered the problem and arrived at the conclusion that the main contribution to the photon emission stems from Coulomb acceleration, while the contribution of tunneling is negligible, contrary to the point of view expressed in Refs. [1] and [3]. Their theoretical results also agree well with the experimental data of Ref. [3].

The purpose of the present paper is to derive an explicit formula for the bremsstrahlung spectrum in α decay and to elucidate the role of tunneling. It will be shown that taking tunneling motion into account is very important to obtain the correct description of the bremsstrahlung.

The α particle moves in the Coulomb potential $U(r) = 2Ze^2/r$, which is truncated at the nuclear radius r_0 . The classical turning point is defined by $r_c = 2Ze^2/E_\alpha$, where $2e$ is the charge of the α particle, Ze is the charge of the daughter nucleus, $E_\alpha = m v^2/2$ is the α particle energy, m is the reduced mass, and v is the relative velocity at $r = \infty$. We assume r_0 to be much smaller than r_c . The initial state is described by an outgoing wave with the asymptotic form $\propto (1/r)\exp(ik_r r)$, where $k_i = (2mE_\alpha)^{1/2}/\hbar$. The final state at $r = \infty$ has the form $\propto (1/r)\cos(k_f r + \phi)$, where $k_f = [2m(E_\alpha - \hbar\omega)]^{1/2}/\hbar$, $\hbar\omega = E_\gamma$ is the energy of the emitted photon, and ϕ is a phase, related to the scattering phase. In the underbarrier region the wave function of the initial state increases exponentially when r changes from r_c inwards, while the wave function of the final state decreases from r_c inwards. (For a detailed discussion of the initial and final states entering the matrix element for photon emission see Ref. [4].)

The following conditions will be assumed:

*On leave from A.F. Ioffe Physico-Technical Institute, St. Petersburg, Russia. Electronic address: dyakonov@lpm.univ-montp2.fr

$$k_i r_c = 2Ze^2/\hbar v \gg 1, \quad (1)$$

$$\hbar \omega \ll E_\alpha. \quad (2)$$

Equation (1) says that the motion is quasiclassical which is generally very well justified, e.g., for $Z = 50$, $E_\alpha = 4$ MeV we have $k_i r_c = 100$. Equation (2) means that the energy losses are small, and we will see below that the main body of the bremsstrahlung spectrum lies in this energy range.

We can now use the quasiclassical approximation for the initial and final wave functions by replacing kr with $\int k(r)dr$, where $k_i(r) = [2m(E_\alpha - U(r))]^{1/2}$, $k_f(r) = [2m(E_\alpha - \hbar\omega - U(r))]^{1/2}$. The reflected part of the final state wave function with the asymptotic behavior $\propto \exp(-ik_f r)$ may be neglected [1], since its contribution to the matrix element is exponentially small, compared to the contribution of the outgoing wave $\propto \exp(ik_f r)$. To the first order in the small ratio $\hbar\omega/E_\alpha$ we have $k_i(r) - k_f(r) \approx \omega/v(r)$, where $v(r)$ is the classical velocity at point r . Proceeding further as in Ref. [1], one finally obtains the classical formula [5] describing the bremsstrahlung spectrum:

$$\frac{\partial E}{\partial \omega} = \frac{2}{3\pi} \frac{(Z_{\text{eff}}e)^2}{c^3} |a_\omega|^2, \quad (3)$$

where $\partial E/\partial \omega$ is the total energy emitted per decay per unit frequency range, $Z_{\text{eff}} = 2(A - 2Z)/(A + 4)$ is the effective charge, A is the mass number of the daughter nucleus,

$$a_\omega = \int_C a(t) \exp(-i\omega t) dt \quad (4)$$

is the ‘‘Fourier transform’’ of the acceleration $a(t)$, the integral is taken along the contour C which is defined by the manner in which the time variable $t = \int_{r_c}^r dr/v(r)$ changes when r runs from 0 to ∞ along real axis. In the underbarrier region t is imaginary, and it is easy to calculate the value of t for $r=0$: $t = -i\tau$, where $\tau = (\pi/2)r_c/v$ is the tunneling time. This value practically does not depend on the details of the potential curve at small distances and may be calculated by extending the Coulomb potential down to $r=0$. In the following we will use the relation $a_\omega = i\omega v_\omega$, where v_ω is defined through the classical velocity $v(t)$ analogous to Eq. (4).

The function v_ω generally depends on the details of the potential function in the region close to the nuclear radius. However, these details are irrelevant for sufficiently low frequencies, such that $\omega t_1 \ll 1$, where t_1 is the characteristic time the particle spends in the vicinity of r_0 . This time can be estimated as $t_1 \sim r_0/v_0$, with $m v_0^2 \sim Ze^2/r_0$, thus $t_1 \sim \tau(r_0/r_c)^{3/2}$, and the condition $\omega t_1 \ll 1$ may be rewritten as $\omega \tau \ll (r_c/r_0)^{3/2}$. Since, as shown below, the main part of the bremsstrahlung spectrum lies in the region $\omega \tau \sim 1$, and since $r_c/r_0 \gg 1$, it can be seen that the structure of the potential function in the vicinity of the nuclear radius may influence only the high-energy tail of the spectrum where the emission probability is already exponentially small. Thus, the main body of the spectrum may be calculated by extending the Coulomb potential down to $r=0$.

Introduce the dimensionless coordinate $\xi = r/r_c$, the dimensionless time $\eta = vt/r_c$, and the dimensionless frequency $\nu = (1/2)\omega r_c/v = \omega Ze^2/(E_\alpha v)$. Then $\partial E/\partial \omega$ may be expressed as

$$\frac{\partial E}{\partial \omega} = \frac{2(Z_{\text{eff}}e)^2 v^2}{3\pi c^3} |J(\nu)|^2, \quad (5)$$

$$J(\nu) = 2i\nu \int_{-i\pi/2}^{\infty} d\eta (d\xi/d\eta) \exp(-2i\nu\eta), \quad (6)$$

where the function $\xi(\eta)$ is the solution of the (dimensionless) Newton equation $d^2\xi/d\eta^2 = (2\xi^2)^{-1}$ with the conditions $\xi(0) = 1$, $(d\xi/d\eta)_{\eta=\infty} = 1$.

This may be compared with the results for two related classical problems. (a) The *Coulomb acceleration model* in which only acceleration outside the Coulomb barrier is taken into account. The lower limit for the integral in Eq. (6) should be $\eta = 0$, instead of $\eta = -i\pi/2$. (b) *Head-on collision of two particles with charges $2e$ and Ze* . In this case the lower limit in Eq. (6) should be $\eta = -\infty$, instead of $\eta = -i\pi/2$.

Introducing a new variable z , such that $\xi = (1 + \cosh z)/2$, $\eta = (z + \sinh z)/2$, we obtain

$$J(\nu) = i\nu \int_{-i\pi}^{\infty} dz \sinh z \exp[-i\nu(z + \sinh z)]. \quad (7)$$

The integration contour in Eq. (7) goes from $-i\pi$ to 0 along the imaginary axis and from 0 to ∞ along the real axis. It may be safely deformed to go from $-i\pi$ to $-i\pi + \infty$ (see Ref. [5]). Finally, introducing $x = z + i\pi$, we get

$$J(\nu) = -i\nu \exp(-\pi\nu) \int_0^{\infty} dx \sinh x \exp[i\nu(\sinh x - x)]. \quad (8)$$

By integration by parts $J(\nu)$ may be also expressed in a form which is more suitable for numerical calculations: $J(\nu) = \exp(-\pi\nu)[1 - i\nu I(\nu)]$, where

$$I(\nu) = \int_0^{\infty} dx (1 - e^{-x}) \exp[i\nu(\sinh x - x)]. \quad (9)$$

The integral in Eq. (9) converges more rapidly than the one in Eq. (8).

Equations (5) and (8) give an explicit formula for the bremsstrahlung spectrum in α decay, provided the conditions given by Eqs. (1) and (2) are fulfilled, and this is the main result of this work. For the head-on collision problem, mentioned above, $J(\nu)$ should be replaced by $J_{\text{coll}}(\nu)$, where $J_{\text{coll}}(\nu)$ is given by the same Eq. (8) but with the lower limit of integration replaced by $-\infty$ (see Ref. [5]). Thus $J_{\text{coll}}(\nu) = 2 \text{Re } J(\nu)$.

The Coulomb acceleration model, which neglects tunneling, predicts a spectrum that falls down at high frequencies as ω^{-2} . The reason is that in this model the acceleration has a singular behavior, increasing stepwise from a zero value at $t < 0$ to a finite value at $t = 0$. Taking tunneling into account

makes the function $a(t)$ smooth, and as a result, the spectrum falls down exponentially. One might say that there is a destructive interference between electromagnetic fields emitted during the tunneling motion at $r < r_c$ and during the classical acceleration stage at $r > r_c$. (Similarly, in the classical Coulomb collision problem, the interference between the incoming and outgoing parts of the trajectory also produces an exponentially decreasing spectrum at high frequencies [5].) Thus, as can be also seen from Eqs. (4), (6), taking account of the tunneling motion is very important to obtain the correct description of bremsstrahlung.

The opposite conclusion reached by Papenbrock and Bertsch [4] is due to a misinterpretation. When applied to the quasiclassical limit of the acceleration matrix element [Eq. (8) above], their reasoning goes as follows. Noticing that $\text{Im} J(\nu) = 0$ at $\nu = 0$ [6], they find that $\text{Im} J(\nu)$ is numerically small compared to $\text{Re} J(\nu)$ for arbitrary ν [for $\nu \gg 1$, one can see that the ratio $\text{Im} J(\nu)/\text{Re} J(\nu)$ is equal to $-1/\sqrt{3}$]. On these grounds the imaginary part of $J(\nu)$ is discarded, while the real part, as shown above, can be expressed as $(\frac{1}{2})J_{\text{coll}}(\nu)$. Since calculation of $J_{\text{coll}}(\nu)$ does not involve tunneling motion, the conclusion is made, that the contribution of tunneling is negligible. In fact, as one can verify by calculating the corresponding parts of the integral, it is the strong cancellation of the large contributions to $\text{Im} J(\nu)$ from the barrier region and the classically allowed region, that makes $\text{Im} J(\nu)$ relatively small. On the other hand, neglecting $\text{Im} J(\nu)$ is equivalent to replacing the true initial wave function [$\propto \exp(ik_r r)$ at $r = \infty$] by its real part which describes an α particle scattered by the nucleus [$\propto \cos(k_r r)$ at $r = \infty$], but normalized to $\frac{1}{2}$ incoming flux. Not surprisingly, the spectrum for α decay, obtained in Ref. [4] after neglecting $\text{Im} J(\nu)$, is equal to $\frac{1}{4}$ of the spectrum for the head-on Coulomb collision problem [7]. However, the difference between this result and the true spectrum is not very large (25% for $\nu \gg 1$). Clearly, the fact that the spectrum for the collision problem (with a factor $\frac{1}{4}$) is similar to the true spectrum in α decay, does not mean that tunneling is not important.

Papenbrock and Bertsch [4] have also performed a numerical calculation of the exact matrix element, without neglecting its real part and without using the quasiclassical approximation, for the specific case of ^{210}Po . The resulting spectrum is very close to the spectrum given by the explicit Eqs. (5), (8), (9) of the present work (see below). Thus the numerical results of Ref. [4] are correct and are in agreement with the above formulas, which explicitly take into account the tunneling part of the trajectory. However, the physical interpretation of these results in Ref. [4], leading to the conclusion that the main contribution to the photon emission stems from the Coulomb acceleration, is erroneous. In fact, it is the contribution of the tunneling region that accounts for the very large difference between the true spectrum and the predictions of the Coulomb acceleration model.

It is easy to find the asymptotic form of $J(\nu)$ both for small and for large values of ν . For $\nu \ll 1$, $J(\nu) = 1 - \pi\nu/2 - i\nu \ln(2\gamma/\nu)$, where $\gamma = 1.781$, so that $|J(\nu)|^2 \approx 1 - \pi\nu$, while for $\nu \gg 1$

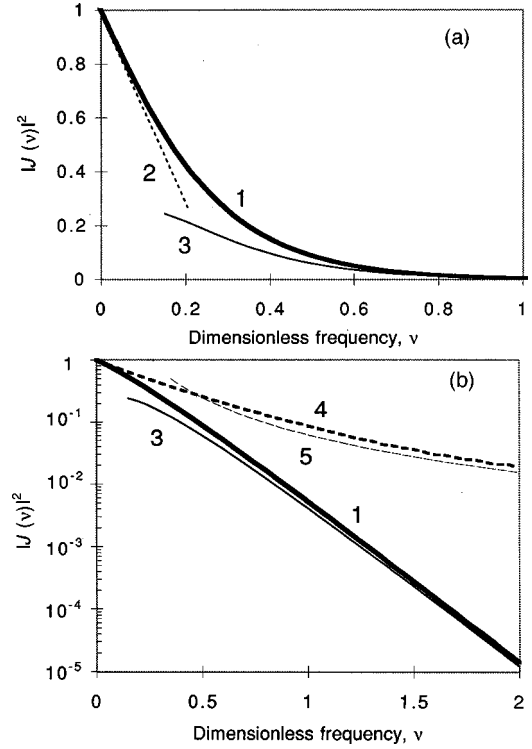


FIG. 1. Normalized bremsstrahlung spectrum. (a) Normal scale: (1) numerical calculation, (2) low frequency asymptote $|J(\nu)|^2 = 1 - \pi\nu$, (3) high frequency asymptote, Eq. (11). (b) Logarithmic scale: (1) numerical calculation, (3) high frequency asymptote, Eq. (11), (4) result for Coulomb acceleration model, (5) high frequency asymptote for Coulomb acceleration model $|J(\nu)|^2 = (4\nu)^{-2}$.

$$J(\nu) = -i \exp(i\pi/3) \Gamma(2/3) (4\nu/3)^{1/3} \exp(-\pi\nu), \quad (10)$$

$$\begin{aligned} |J(\nu)|^2 &= \Gamma^2(2/3) (4\nu/3)^{2/3} \exp(-2\pi\nu) \\ &= 2.22\nu^{2/3} \exp(-2\pi\nu). \end{aligned} \quad (11)$$

It follows from Eqs. (10), (11) that for $\nu \gg 1$ $|J_{\text{coll}}(\nu)|^2 = 3|J(\nu)|^2$. Figure 1 shows the normalized spectrum, given by the function $|J(\nu)|^2$, calculated numerically, together with the asymptotic curves for $\nu \ll 1$ and $\nu \gg 1$. It can be seen that Eq. (11) becomes a very good approximation for $\nu > 0.5$. In this region the $|J(\nu)|^2$ dependence is dominated by the exponential factor in Eq. (11), which may be expressed as $\exp(-E_\gamma/E_0)$, where $E_0 = (E_\alpha/2\pi)(\hbar\nu/Ze^2)$ is a characteristic energy, $2\pi\nu = E_\gamma/E_0$. For the experimental conditions of Ref. [3] ($Z = 82$, $E = 5.3$ MeV) one finds $E_0 \approx 75$ keV. Note that the shape of the spectrum is a universal function of E_γ/E_0 .

The experimentally measured quantity is the differential probability of photon emission per decay $\partial^2 N / \partial E_\gamma \partial \Omega = (4\pi\hbar E_\gamma)^{-1} \partial E / \partial \omega$. Thus

$$\frac{\partial^2 N}{\partial E_\gamma \partial \Omega} = \frac{Z_{\text{eff}}^2}{3\pi^2} \left(\frac{e^2}{\hbar c} \right) \frac{E_\alpha}{mc^2} \frac{1}{E_\gamma} |J(\nu)|^2. \quad (12)$$

This quantity is plotted in Fig. 2 as a function of E_γ for the case of α decay of ^{210}Po (heavy line), together with the ex-

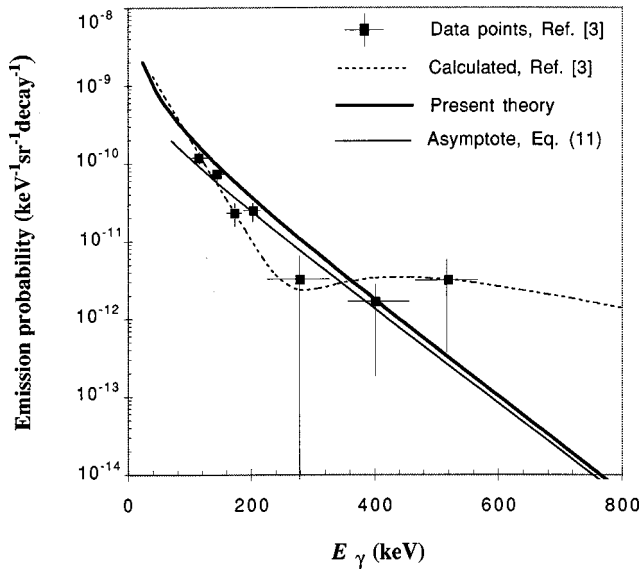


FIG. 2. γ ray emission probability in α decay of ^{210}Po . Results of the present theory (heavy solid line) are presented together with the experimental data points and results of numerical calculation from Ref. [3]. The thin solid line is the high frequency asymptote given by Eqs. (11) and (12).

perimental data taken from Ref. [3]. The result of the numerical calculation in Ref. [3] is presented by the dotted line. This calculation was done by using the general formula of Ref. [1], and thus should give the same result as the present work. However, one can see that the two results are qualitatively different: (a) the slopes at low γ ray energies differ by a factor ~ 2 and (b) while the present theory gives a spectrum that falls down exponentially at high E_γ , the calculation of Ref. [3] gives a broad maximum around $E_\gamma \sim 500$ keV. The difference in the slopes can not be easily understood. As to the high energy part of the spectrum, we should note that there is a limitation of the present theory related to our neglecting the details of the inner barrier region and the exten-

sion of the Coulomb potential down to $r=0$. As explained above, this is justified if $E_\gamma \ll E_0(r_c/r_0)^{3/2} \approx 860$ keV ($r_c/r_0 \approx 5$ for ^{210}Po). For energies comparable to or higher than this value the shape of the spectrum should depend on the parameters of the inner barrier region. Thus, the maximum on the curve calculated in Ref. [3] (where the cutoff of the Coulomb potential at $r=r_0$ was taken into account) might be related to this circumstance. More precise experimental data are needed to determine unambiguously the true slope at low energies, as well as the existence of any specific features at the high energy tail of the spectrum.

In summary, an explicit, essentially classical, formula, describing the bremsstrahlung spectrum in α decay, was derived, and it was shown that taking the tunneling motion into account is crucial for the correct description of the spectrum. The shape of the spectrum is a universal function of the ratio E_γ/E_0 , where E_γ is the photon energy and E_0 is a characteristic energy depending only on the nuclear charge and the energy of the α particle.

Note added in proof. Recently Takigawa *et al.* [8] performed numerical calculations of various contributions to the bremsstrahlung spectrum, which led them to the conclusion that the spectrum “results from subtle interferences of the contributions of the tunneling, mixed and classical regions as well as the wall of the potential well, each of which has a comparable magnitude.” In the present work both the mixed region, which does not exist in the quasiclassical approximation, and the inner barrier region were completely ignored on the grounds described above. In this context it should be noted that the final results of Ref. [8] for the total emission probability (curves denoted “total” in their Figs. 1–3) almost exactly coincide with the result of the present work, as given by Eqs. (5), (8) (heavy solid line in my Fig. 2).

I thank Michael Shur for his hospitality during my stay at Rensselaer Polytechnic Institute, where this work was completed. I appreciate useful discussions with Thomas Papenbrock.

- [1] M. I. Dyakonov and I. V. Gornyi, Phys. Rev. Lett. **76**, 3542 (1996).
- [2] A. D’Arrigo, N. V. Eremin, G. Fazio, G. Giardina, M. G. Glotova, T. V. Klochko, M. Sacchi, and A. Taccone, Phys. Lett. B **332**, 25 (1994).
- [3] J. Kasagi, H. Yamazaki, N. Kasajima, T. Ohtsuki, and H. Yuki, Phys. Rev. Lett. **79**, 371 (1997).
- [4] T. Papenbrock and G. F. Bertsch, Phys. Rev. Lett. **80**, 4141 (1998).
- [5] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon, Oxford, 1975), p. 181.
- [6] Indeed, tunneling does not contribute to the bremsstrahlung spectrum at very low frequencies ($\nu \ll 1$), because, as can be

seen from Eq. (4), at $\omega=0$ the contribution of the tunneling region to a_ω is equal to the difference of the initial relative velocity and the velocity at the barrier exit point r_c , both of which are equal to zero. This is no more true for frequencies ν on the order of, or much greater than 1. Note that the real part of the acceleration matrix element in Ref. [4] corresponds to the imaginary part of $J(\nu)$, and vice versa.

- [7] Ref. [4] does not contain an explicit formula for the spectrum in the quasiclassical approximation. It may be derived by taking the appropriate limit in Eq. (14) of Ref. [4] and by correcting the exponent in this equation by a factor of 2.
- [8] N. Takigawa, Y. Nozawa, K. Hagino, A. Ono, and D. M. Brink, Phys. Rev. C **59**, 593 (1999).