Equivalence of nonstatic two-pion-exchange nucleon-nucleon potentials

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Off-shell aspects of the one-pion-exchange potential and their relationship to different forms of the nonstatic (subleading-order) chiral two-pion-exchange nucleon-nucleon potential are discussed. Various types of off-shell behavior are categorized and numerous examples are given. Recently derived potentials based on chiral approaches are supplemented by a rather general form of the two-pion-exchange potential that is derived using old-fashioned methods and exhibits off-shell dependence. The latter potential is closely related to a general form of one-pion-exchange relativistic corrections and nonstatic two-pion-exchange three-nucleon forces developed long ago. [S0556-2813(99)07508-1]

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I. INTRODUCTION

Off-shell effects in nuclear potentials have long posed conceptual and technical problems in nuclear physics [1,2]. Although the off-shell problem often is caused by poorly posed input (and is therefore largely insoluble), even if the input is completely specified (e.g., a Lagrangian) there can still be ambiguity in final forms obtained for potentials. We will focus on the ambiguities, which are basically *choices* made by theorists. Thus, many different potentials are correct (to some order), but have different forms (in part). Nevertheless, these different forms lead to *equivalent* observables. We refer collectively to these ambiguities as ''off-shell'' effects. We wish to emphasize that different formalisms make these choices automatically. It is only in trying to relate different calculations that these ambiguities must be understood. That is our primary purpose in this work.

There are three basic types of ambiguity that have arisen in calculating nuclear potentials [3]. The first type is caused by an energy-dependent potential $V(E)$, where E is the eigenvalue of the appropriate equation of which $V(E)$ is part. Such forms occur naturally [4,5] when expanding energy denominators that occur in Schrödinger perturbation-theory treatments of mesons that are exchanged between nucleons: $V(E)\sim(1/E_m)J(1/[E_m+(H-E)])J$, where J is a meson-nucleon vertex (the two J's refer to different nucleons), E_m is the transiting-meson energy and $H-E$ is the nucleons' energy difference between intermediate and initial states. Expanding in powers of $(H-E)/E_m$, one finds the usual static potential (J^2/E_m^2) plus energy-dependent corrections. This is perfectly satisfactory (the Sturm-Liouville equation [6] is of this type), but energy-dependence is difficult to implement in anything other than the two-nucleon system. This energy-dependence was the origin of the Brueckner-Watson [7] vs TMO [8] controversy long ago (reviewed in Ref. [9]), and many results from chiral perturbation theory have this form [10]. The cost of this dependence is sufficiently high (particularly for $A\geqslant 3$) that we do not recommend keeping it, but rather one should eliminate it by any of various means developed over the years [3]. This has been done in all of the results of Sec. II. We will not further consider this type of off-shell dependence.

The second type of off-shell dependence arises from unitary ambiguities in the form of the potential, often from different choices of field variables [5,2] in the *same* Lagrangian. Thus if H is the nuclear Hamiltonian

$$(E-H)\Psi=0, \tag{1}$$

a transformation $\Psi'=e^{iU}\Psi\cong(1+iU)\Psi$ (for small U) produces

$$(E-H')\Psi'=0, \tag{2a}$$

where the transformed Hamiltonian

$$H'\cong H-i[H,U] \tag{2b}$$

is a satisfactory variant of H (for small U). Such transformations effect gauge transformations in electromagnetic problems and occur naturally in relativistic corrections to nuclear potentials [11,4,5].

One specific example of the latter is the quasipotential ambiguity [3], determined by an off-shell parameter ν. When a meson of mass m is exchanged between two nucleons (1 and 2) in an arbitrary reference frame, the relativistic propagator has the form

$$\frac{1}{q_0^2-(\vec{q}^2+m^2)}\cong-\frac{1}{\vec{q}^2+m^2}-\frac{q_0^2}{(\vec{q}^2+m^2)^2}, \tag{3}$$

where q_0^2 is nonvanishing in general and is given by ΔE_1^2 or ΔE_2^2 or $-\Delta E_1\Delta E_2$, where ΔE_i is the energy transferred by the meson to nucleon i. Each choice leads to a different off-shell potential, corresponding to the different ways of projecting out q_0, the time component of the four-momentum vector. A general result for q_0^2 is given by the linear combination: $[(\Delta E_1+\Delta E_2)^2(1-\nu)-2\Delta E_1\Delta E_2]$, which specifies the parameter ν [3] and demonstrates that the effect is a unitary ambiguity. Common choices of the parameter ν are $\nu=0$ (standard) [3,11], $\nu=1/2$ (no retardation) [4,12], and $\nu=1$ (soft). Most techniques lead to $\nu=0$, although a Foldy-Wouthuysen procedure [13] leads to $\nu=1$. The popular $\nu=1/2$ choice [5] simplifies the nuclear poten-

tial. Note that ΔE_i^2 is of order $(1/M^2)$, where M is the nucleon mass. This generates a relativistic correction to the potential.

Another unitary ambiguity arises when using pseudoscalar (PS) or pseudovector (PV) relativistic pion-nucleon interactions. The on-shell forms are identical, but they differ off-shell. These differences can be subsumed [4,5] to order $(1/M^2)$ by a collection of off-shell operators proportional to a parameter μ. Common choices are $\mu=-1$ (PS free spinors) [1,11,14], $\mu=0$ (minimal nonlocality) [15,16], $\mu=1$ (PV) [11], and $\mu=3$ (soft) [4]. Conventional chiral perturbation theory (CPT) [17,18] expansions (term-by-term chirality) correspond to $\mu=1$ and are preferred for that reason, although other criteria having to do with the final form of OPEP may mandate other values (viz., $\mu=0$, discussed below).

The third type of ambiguity is the form ambiguity. Potentials only make sense in the context of a scheme for iterating them to produce a final (to all orders) result for a binding energy or scattering amplitude. The nonrelativistic paradigm is to add the kinetic energy $T_{\rm NR}=\vec{p}^2/2M$ to the potential V to obtain the Hamiltonian $H_{\rm NR}$ that is used in Eq. (1) to obtain the desired observable. The naive (but obvious) relativistic generalization of this is to form the free energy [4]

$$T=\sum_{i=1}^{A}\sqrt{\vec{p}_i^{\,2}+M^2}\rightarrow 2\sqrt{\vec{p}^2+M^2}, \tag{4}$$

for a collection of identical-mass nucleons, where the second form applies only to the two-nucleon problem in its center-of-mass frame. Then $H_R=T+V$ is the appropriate relativistic form of the energy to be used in Eq. (1). A one-time formalism (linear in E) is always possible if one freezes out antinucleon degrees of freedom or by using a variety of projection schemes, which typically lead to different quasipotential equations [3].

An example of a quasipotential method is the popular Blankenbecler-Sugar [19] (BS) procedure, which is based on a two-nucleon Green's function of nonrelativistic form. Although this is a well-developed procedure that casts field-theory calculations into a Schrödinger-like appearance, the relationship between BS results and more conventional ones [using Eq. (4) for T] can be obtained by means of a trick [20]. The two-body relativistic Schrödinger equation (RSE, for lack of a better name) corresponding to the second form in Eq. (4) is

$$[2\sqrt{\vec{p}^2+M^2}+V(\vec{r})]\Psi_R=2\sqrt{k^2+M^2}\Psi_R, \tag{5}$$

where k is the momentum corresponding to the energy eigenvalue E and $\vec{r}$ is the nucleons' separation. Squaring both sides, subtracting $4M^2$ from both sides, and dividing by $4M$ lead to the remarkable and *equivalent* [20]

$$\left[\frac{\vec{p}^2}{M}+\overline{V}(\vec{r})\right]\Psi_{\rm BS}=\frac{k^2}{M}\Psi_{\rm BS}, \tag{6}$$

where

$$\overline{V}(\vec{r})=\left\{\frac{\sqrt{\vec{p}^2+M^2}}{2M},V(\vec{r})\right\}+\frac{V^2}{4M} \tag{7}$$

is the effective potential to be used in what appears to be a nonrelativistic Schrödinger equation. This equation requires relativistic kinematics (e.g., in relating k and E), however, and is not nonrelativistic. This procedure [using Eq. (6)] is at the heart of the Nijmegen partial-wave-analysis (PWA) program [21] for treating nucleon-nucleon scattering.

Any form change in the equation redefines the effective potential, which is hardly a surprise. Note the $V^2/4M$ term, which will be seen again since it typically sets the scale for corrections resulting from changes of form, and the factor $\sim(E/M)$ that multiplies V. Such factors occur everywhere when calculating potentials. This factor repositions relativistic corrections from T to $\overline{V}$. Clearly other form changes are possible. Although Eq. (6) is a decided advantage in treating the two-nucleon problem, it is not of any obvious use for the many-nucleon problem (there will be cross terms between the kinetic energies of different nucleons). A recent major advance has been the ability to handle $\sqrt{p^2+m^2}$ in configuration space [22], since modeling momentum-dependent operators in that space is the single most challenging numerical problem.

In Sec. II we review selected previous calculations and derive a new general form of the nonstatic (subleading order in CPT) two-pion-exchange nucleon-nucleon potential that manifests the unitary ambiguities discussed above. This form subsumes several previously calculated potentials as well as a number of cases not previously considered, and allows all of these results to be compared. In Sec. III results for specific off-shell choices are discussed, including the important case of the Nijmegen PWA program. Our summary is presented in Sec. IV.

II. CALCULATION

It should be obvious from the previous examples that we have been discussing variants of relativistic corrections. Using either the old rules of scale [5] (counting $T\sim 1/M$ or $1/\Lambda$, where Λ is the large-mass scale of QCD, and $V\sim T\sim 1/M$) or more general and sophisticated power-counting rules [17,23] leads to the same conclusion. Relativistic corrections to the nucleon kinetic energy ($\sim 1/M^3$ or $1/\Lambda^3$) can be juggled into the BS potential ($\sim V/M^2\sim 1/\Lambda^3$) or the BS correction term ($V^2/4M\sim 1/\Lambda^3$). The retardation corrections have the same intrinsic size ($V/M^2\sim 1/\Lambda^3$). This defines the limits of the expansions (in $1/M$) that will be made below.

With the exception of recent CPT calculations (or other chiral variants) much of the work on one- and two-pion-exchange potentials is quite old [24]. In particular, the work of TMO [8] and Sugawara and Okubo (SO) [11] stand out in their technical clarity. Both developed an energy-independent potential. The static (i.e., leading order in CPT) two-pion-exchange potential ($V_{2\pi}^0$) was developed by TMO and was recently reviewed in Ref. [9] in the context of

energy-dependent alternatives. The leading-order nonstatic (i.e., subleading order in CPT) corrections ($\Delta V_{2\pi}$) were developed in SO, even though they had no credible theory of the interactions of pions and nucleons, which came later. Nevertheless, with the exception of missing seagull terms ($\pi\pi$-N interactions) one of their models (PV coupling) produced correct and complete results. At the time of SO it was known that nucleon-antinucleon ''pair'' terms in PS coupling were unphysically large and had to be suppressed (they were simply thrown away). The reason for this suppression is now known to be chiral symmetry, which explains the success of their PV coupling model (derivative coupling with negligible pair terms). In addition SO supplemented their PV Lagrangian with two seagull interactions, one with a single derivative of Weinberg-Tomozawa type [25] and another (with no derivatives) of σ-term type [c_1 in Eq. (9) below]. Their results were correct for these interactions, as well. They did not consider seagulls with two derivatives, such as the c_3 and c_4 terms in Eq. (9).

The work of SO also emphasized (by example) that no unique form for $\Delta V_{2\pi}$ exists without choosing an appropriate off-shell form for ΔV_π, the relativistic corrections to the one-pion-exchange potential V_π^0 (OPEP). Moreover, this choice also affects the nonstatic two-pion-exchange three-nucleon forces (of nominal size V_π^{02}/M). The latter was developed in Ref. [5], except for seagull counter terms first incorporated completely in Ref. [10] (see also Refs. [1,2]). We emphasize that the former calculation treated the ''Born terms'' (in this case the leading-order π-N coupling and the Weinberg-Tomozawa $\pi\pi$-N coupling) to nonstatic (subleading) order, but important (dominant) terms in the chiral expansion were not treated until Ref. [10]. A review of one-pion-exchange contributions to the Hamiltonian and to the charge form factor (from electron scattering) was presented in the appendix to Ref. [5]. In addition, all necessary formulas were developed for determining $\Delta V_{2\pi}$, but the final integrals and spin-isospin commutators were not performed. This will be completed below.

The work of Refs. [4,5] is based on a relativistic Lagrangian for free nucleons plus a PV π-N interaction plus a Weinberg-Tomozawa [25] $\pi\pi$-N interaction. A Foldy-Wouthuysen reduction of this Lagrangian is made to the appropriate order in $1/M$ and this reduced set is used in time-dependent perturbation theory in the usual way. It yields potentials and pion-exchange-current operators. Because time-dependent perturbation theory leads to a variant of (time-independent) Schrödinger perturbation theory, the resulting potentials are energy-dependent. A mapping technique [or (alternatively) perturbation-theory manipulations] convert those forms to an energy-independent potential to be used in Eq. (1), with the kinetic energy given by Eq. (4).

For specificity we list next the required Lagrangian terms $\mathcal{L}^{(\Delta)}$, where $\Delta=0,1,...$, represents powers of $1/\Lambda$ (i.e., $\Lambda^{-\Delta}$) contained implicitly and explicitly in the coefficients of the products of pion and nucleon fields. The term $\mathcal{L}^{(3)}$ is only formally required to produce the necessary part of the free relativistic nucleon energy T listed in Eq. (4); it was not required to calculate the 2π-exchange force in Ref. [5]. The $\mathcal{L}^{(2)}$ terms [see Eq. (6f) of Ref. [4]] are required only to calculate the ΔV_π potential and are not needed for $\Delta V_{2\pi}$. Only $\mathcal{L}^{(0)}$ and $\mathcal{L}^{(1)}$ [see Eq. (5) of Ref. [5]] are required in order to produce $\Delta V_{2\pi}$. We have

$$\mathcal{L}^{(0)}=\frac{1}{2}[\dot{\boldsymbol{\pi}}^2-(\vec{\nabla}\boldsymbol{\pi})^2-m_\pi^2\boldsymbol{\pi}^2]+N^\dagger\left[i\partial_0-\frac{1}{4f_\pi^2}\boldsymbol{\tau}\cdot(\boldsymbol{\pi}\times\dot{\boldsymbol{\pi}})\right]N$$
$$+\frac{g_A}{2f_\pi}N^\dagger\vec{\sigma}\cdot\vec{\nabla}(\boldsymbol{\tau}\cdot\boldsymbol{\pi})N, \tag{8}$$

$$\mathcal{L}^{(1)}=\frac{1}{2M}\left[-N^\dagger\vec{p}^2N-\frac{1+g_A^2(\mu-1)}{4f_\pi^2}N^\dagger\{\vec{p},\cdot(\boldsymbol{\tau}\cdot\boldsymbol{\pi}\times\vec{\nabla}\boldsymbol{\pi})\}N\right.$$
$$+\frac{g_A(\mu+1)}{4f_\pi}N^\dagger\{\vec{\sigma}\cdot\vec{p},\boldsymbol{\tau}\cdot\dot{\boldsymbol{\pi}}\}N+\frac{g_A^2(\mu-1)}{4f_\pi^2}$$
$$\left.\times N^\dagger\boldsymbol{\pi}\cdot\vec{\nabla}^2\boldsymbol{\pi}N-\frac{g_A^2(\mu-1)}{8f_\pi^2}N^\dagger\{\vec{\sigma}\times\vec{p},\cdot\vec{\nabla}\boldsymbol{\pi}^2\}N\right]$$
$$+\frac{1}{f_\pi^2}N^\dagger\left[-2c_1m_\pi^2\boldsymbol{\pi}^2-c_3(\vec{\nabla}\boldsymbol{\pi})^2\right.$$
$$\left.-\frac{1}{2}\left(c_4+\frac{1}{4M}\right)\varepsilon_{ijk}\varepsilon_{abc}\sigma_k\tau_c\partial_i\pi_a\partial_j\pi_b\right]N, \tag{9}$$

$$\mathcal{L}^{(2)}=-\frac{g_A}{8f_\pi M^2}N^\dagger\{\vec{p}^2,\vec{\sigma}\cdot\vec{\nabla}(\boldsymbol{\tau}\cdot\boldsymbol{\pi})\}N$$
$$+\frac{g_A\mu}{16f_\pi M^2}N^\dagger\{\vec{\sigma}\cdot\vec{p},\{\vec{p},\cdot\vec{\nabla}(\boldsymbol{\tau}\cdot\boldsymbol{\pi})\}\}N, \tag{10}$$

$$\mathcal{L}^{(3)}=\frac{1}{8M^3}N^\dagger\vec{p}^4N. \tag{11}$$

Terms not required in what follows have not been listed, and this includes all short-range NN operators. Note that both terms in $\mathcal{L}^{(2)}$ depend on the average nucleon momentum $\vec{p}$ and are constrained by the requirements of relativity (as is $\mathcal{L}^{(3)}$ and four of the first five terms of $\mathcal{L}^{(1)}$). The (nucleon) momentum-dependent terms were developed by noting that the nucleon parts of $\mathcal{L}^{(0)}$ can be represented by a (covariant) free-nucleon term plus a PV π-N vertex supplemented by a Weinberg-Tomozawa $\pi\pi$-N interaction. Performing a Foldy-Wouthuysen reduction of this set [4,5] freezes out the antinucleon degrees of freedom and leads to all terms explicitly proportional to inverse powers of M. We will ignore all pion-momentum-dependent (form-factor) modifications of the PV vertex. Although the coefficient of $\vec{p}^2$ in Eq. (10) is completely specified, this is solely due to our knowledge that the pion is a pseudoscalar object. Scalar- and vector-meson exchanges produce similar terms of opposite sign to each other. Note the dependence on the off-shell parameter μ, which arises from a redefinition of nucleon fields. The result-

ing Lagrangian still satisfies chiral symmetry, but not on a term-by-term basis unless $\mu=1$.

Results for OPEP appropriate for Eq. (5) have been presented previously in many calculations and are summarized by Eq. (A12) of Ref. [5]. We restrict ourselves to the two-nucleon sector; the three-nucleon sector is discussed extensively in that reference. For continuity, we also revert to a convenient but old-fashioned notation for the effective π-N coupling constant f,

$$f=\frac{G}{2M}\approx\frac{g_A}{2f_\pi}, \tag{12}$$

where the second relation holds only if the Goldberger-Treiman [26] relation is exact. We find that to order $(1/\Lambda^3)$

$$V_\pi=V_\pi^0-\frac{1}{2M^2}\{\vec{p}^2,V_\pi^0\}+\left(\frac{1}{2}-\nu\right)i[T_{NR},U_G] +(\mu-2\nu+1)i[T_{NR},U_E], \tag{13}$$

where

$$V_\pi^0=f_0^2 m_\pi \boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\vec{\sigma}_1\cdot\vec{\nabla}\vec{\sigma}_2\cdot\vec{\nabla}h_0(x), \tag{14}$$

$$f_0^2=\frac{f^2m_\pi^2}{4\pi}, \tag{15}$$

$$\vec{x}=m_\pi(\vec{x}_1-\vec{x}_2), \tag{16a}$$

$$h_0=\frac{e^{-x}}{x}. \tag{16b}$$

Introduction of form factors into h_0 is straightforward, but for simplicity will not be done here. Our primary interest is the tail of this force, as analyzed by the Nijmegen PWA [27]. We also require the Gross (U_G) and equivalence (U_E) unitary transformations [5]

$$U_G=\frac{1}{4M}\{\vec{p}\cdot,\vec{x}V_\pi^0(\vec{x})/m_\pi\} \tag{17}$$

and

$$U_E=\frac{f_0^2}{8M}\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2(\{\vec{\sigma}_1\cdot\vec{p},\vec{\sigma}_2\cdot\vec{\nabla}h_0(x)\}+\{\vec{\sigma}_2\cdot\vec{p},\vec{\sigma}_1\cdot\vec{\nabla}h_0(x)\}), \tag{18}$$

where here and elsewhere $\vec{\nabla}$ signifies $\vec{\nabla}_x$.

Although the second term in V_π is simple in form [and represents the expansion of $(M/E_0)V_\pi^0(M/E_0)$, where $E_0=\sqrt{\vec{p}^2+m^2}$, in the c.m. frame], the other terms are much more complicated and involve the coupling of (nucleon) spin and momentum. This type of tensor coupling is the origin of the lower than usual P_D of the deuteron for the traditional Bonn potential models, and subsequently for a higher triton binding energy than other models [28]. Clearly, the last two terms in Eq. (13) can be neglected on-shell, or by *choosing* $\nu=\frac{1}{2}$ and $\mu=0$ they can be neglected off-shell, as well. This is a popular choice. Note that the two factors of (M/E_0) arise from our nucleon normalization. Rather than use the *covariant* normalization convention (current matrix elements are manifestly covariant and nucleon normalization factors are kept separate in *S*-matrix elements or are incorporated into covariant phase-space factors) we revert to the *invariant* convention (normalization factors are incorporated into fields and the total charge, for example, is a Lorentz invariant) that is consistent with a Foldy-Wouthuysen reduction and is conventional in nuclear problems.

We further note that Sugawara and Okubo encountered both U_G and U_E in the course of their derivation of the one- and two-pion-exchange potentials. Their conversion of intermediate results to $V_\pi^0-\{\vec{p}^2,V_\pi^0\}/2M$ is equivalent to choosing $\nu=\frac{1}{2}$ and $\mu=0$ in our result. That choice has been recently called the ''minimal nonlocality'' (MNL) choice [15], because it eliminates the complicated $\vec{\sigma}\cdot\vec{p}$ terms that otherwise arise. This type of complexity matters little for momentum-space computational approaches, but is a significant impediment to configuration-space calculations, which are now the dominant approach for high-accuracy computations of $A>3$ [29]. Minimal nonlocality is a significant simplification.

As stated earlier, all but the final integrals and spin-isospin manipulations for obtaining the nonstatic $\Delta V_{2\pi}$ were performed in Ref. [5] during the calculation of the nonstatic ''Born-term'' contributions to the 2π-exchange three-nucleon force. We require Eqs. (15) and (16) (direct-plus crossed-box diagrams), Eq. (18a) (''uncrossed'' or overlapping time-ordered diagrams), Eq. (20) (seagulls with no time derivatives), and Eq. (22a) (the Weinberg-Tomozawa seagull) of that reference. It was necessary to subtract the iterated contribution of OPEP (V_π) since this is automatically included in the solution of the RSE, Eq. (5). This subtraction led to an energy-dependent potential that was mapped into an energy-independent form (this can be achieved by many different equivalent techniques [30]). Unitary equivalences of 2π-range [such as in Eqs. (18a) and (22a) of Ref. [5]] were eliminated by converting them to 3π range (which terms we are consistently ignoring). By choosing to develop an energy-independent potential we have eliminated from consideration the first of the three ambiguities discussed in the introduction.

The resulting nonstatic 2π-exchange nucleon-nucleon force is given in its most general form appropriate for μ and ν off-shell ambiguities by

$$\Delta V_{2\pi}^{\mu,\nu}=\frac{f_0^4 m_\pi^2}{2M}\left(\vec{L}\cdot\vec{S}\left\{3[8g^2-2xgg'(\mu-2\nu-3)]-2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\left[8g^2\left(\frac{1}{g_A^2}-1\right)-2xgg'(\mu-2\nu+1)\right]\right\}\right.$$
$$+\left[3\{-xg(4g'+xg'')(\mu-3\nu)-(1-\nu)x^2g'(2g'+xg'')+16\overline{c_3}[3g^2+xg'(2g+xg')]\right.$$
$$+32\overline{c_1}x^2g^2\}-2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\left(-xg(4g'+xg'')(\mu-3\nu-2)+(1+\nu)x^2g'(2g'+xg'')+\frac{2}{g_A^2}[3g^2+xg'\right.$$
$$\left.\left.\times(2g+xg')]\right)\right]+\frac{1}{3}[3(1+\nu)+2(1-\nu)\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2]\{S_{12}[xgg'+x^2(g'^2+gg'')]-2\vec{\sigma}_1\cdot\vec{\sigma}_2[4xgg'+x^2$$
$$\left.\times(g'^2+gg'')]\}+\frac{8}{3}\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\left(4\overline{c_4}+\frac{1}{g_A^2}\right)[S_{12}(xgg')-\vec{\sigma}_1\cdot\vec{\sigma}_2(3g^2+2xgg')]\right), \tag{19}$$

where

$$g(x)=\frac{1}{x}\frac{dh_0(x)}{dx}=-e^{-x}\left(\frac{1}{x^2}+\frac{1}{x^3}\right) \tag{20}$$

and

$$\overline{c_i}=c_iM/g_A^2. \tag{21}$$

Each type of force in Eq. (19) (spin-orbit, central, tensor, and spin-spin) depends on the off-shell parameters. For comparison purposes [see Eq. (7)] we also find

$$\frac{V_\pi^2}{4M}=\frac{f_0^4 m_\pi^2}{12M}(3-2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)[2S_{12}(xgg')-2\vec{\sigma}_1\cdot\vec{\sigma}_2(3g^2$$
$$+2xgg')+3(3g^2+2xgg'+x^2g'^2)]. \tag{22}$$

Note that only two types of isospin operator are possible: direct type $(3-2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)$ and crossed type $(3+2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)$. Both uncrossed diagrams (with slanted time-ordered pion propagators) and reducible diagrams (with no pions at some intermediate time) are of direct type. This separation facilitates comparisons between different calculations of certain diagrams.

The crucial elements in this calculation are the chiral seagulls. These were first calculated in Ref. [10]. Written in the manner of Eq. (21), the $\overline{c_i}$ should be dimensionless numbers whose magnitudes are ~ 1 [23]. Because the low-mass Δ isobar plays such a large role in c_3 and c_4, these values are even larger and they play a dominant role in $\Delta V_{2\pi}$. The remaining terms depend only on f. They were first calculated in an energy-independent form by Sugawara and Okubo. Their calculation suppressed negative-energy states, which contribute to our π-N seagull amplitudes. They noted, however, that their PV calculation (corresponding to $\mu=1$ and $\nu=0$) had very weak pair terms, and indeed $\mathcal{L}^{(0)}$ and $\mathcal{L}^{(1)}$ have no seagulls if we eliminate the c_i, the Weinberg-Tomozawa terms (independent of g_A), and set $\mu=1$. For this reason their calculation of the relativistic direct- and crossed-box diagrams for PV coupling was complete.

We have verified their results for individual terms both before and after the unitary transformations they performed. Their final result corresponds to $\mu=0$, $\nu=1/2$ in Eq. (19) with the aforementioned seagull terms dropped. Thus, the SO PV calculation is the first correct calculation of the subtracted box and crossed-box graphs. Their PS calculation (corresponding to $\mu=-1$, $\nu=0$) is, however, missing some seagull terms, although we have verified all of their other calculated contributions. Although SO unnecessarily approximated $\sqrt{p^2+M^2}-M$ by $p^2/2M$ in the free-nucleon energy, it is clear from their calculation that $V_\pi+V_{2\pi}^0+\Delta V_{2\pi}^{\mathrm{MNL}}$ is to be substituted for V in Eq. (5).

III. SPECIFIC FORMS OF $\Delta V_{2\pi}$

Having developed a new and general form for $\Delta V_{2\pi}$ corresponding to the μ and ν off-shell ambiguities, we now relate this form to specific cases of potentials and procedures that are in common use today. For the special (and preferred) MNL case ($\mu=0$, $\nu=1/2$) one finds

$$\Delta V_{2\pi}^{\mathrm{MNL}}=\frac{f_0^4 m_\pi^2}{4M}\left\{16\vec{L}\cdot\vec{S}\left[3(g^2+xgg')-2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 g^2\left(\frac{1}{g_A^2}-1\right)\right]+\left[3\{3xg(4g'+xg'')-x^2g'(2g'+xg'')\right.\right.$$
$$+32\overline{c_3}[3g^2+xg'(2g+xg')]+64\overline{c_1}x^2g^2\}-2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\left(7xg(4g'+xg'')+3x^2g'(2g'+xg'')\right.$$
$$\left.\left.+\frac{4}{g_A^2}[3g^2+xg'(2g+xg')]\right)\right]+\frac{1}{3}(9+2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\{S_{12}[xgg'+x^2(g'^2+gg'')]-2\vec{\sigma}_1\cdot\vec{\sigma}_2[4xgg'$$
$$\left.+x^2(g'^2+gg'')]\}+\frac{16}{3}\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2(4\overline{c_4}+1/g_A^2)[S_{12}(xgg')-\vec{\sigma}_1\cdot\vec{\sigma}_2(3g^2+2xgg')]\right\}. \tag{23}$$

The form appropriate for the Nijmegen PWA [see Eq. (7)] is then given by

$$\Delta V_{2\pi}^{\rm Nij} = \Delta V_{2\pi}^{\rm MNL} + \frac{V_\pi^2}{4M}, \tag{24}$$

and should be used in Eq. (6) with an OPEP in the form

$$V_\pi^{\rm Nij} = \left\{ \frac{E_0}{2M}, \frac{M}{E_0} V_\pi^0(\vec{r}) \frac{M}{E_0} \right\}, \tag{25}$$

where $E_0 = \sqrt{\vec{p}^2 + M^2}$ is an operator.

The usual approach of the Nijmegen group, however, is to approximate the operator-valued factors of E_0 in Eq. (25) by their on-shell value: $E_{\rm os} = \sqrt{k^2 + M^2}$. This changes their OPEP to

$$\overline{V}_\pi^{\rm Nij} = \frac{M}{E_{\rm os}} V_\pi^0, \tag{26}$$

and the difference of $V_\pi^{\rm Nij}$ and $\overline{V}_\pi^{\rm Nij}$ modifies the required nonstatic 2π-exchange potential to (second-order perturbation theory is the easiest way to see this [31])

$$\Delta \overline{V}_{2\pi}^{\rm Nij} = \Delta V_{2\pi}^{\rm MNL} + \frac{3V_\pi^2}{4M}. \tag{27}$$

That is, changing from operator-valued kinematical factors to on-shell (c-number) ones merely adds $2V_\pi^2/4M$ to the ''form'' correction factor displayed in Eq. (24).

In a similar fashion we note that the new Urbana [16] relativistic potential model corresponds to Eq. (5) with

$$V^{\rm Urb} = \left(\frac{M}{E_0} V_\pi^0 \frac{M}{E_0} + V_{2\pi}^0 + \Delta V_{2\pi}^{\rm MNL} \right) + \cdots. \tag{28}$$

In addition, the Bonn potential [14] incorporated into an equation with relativistic kinematics requires a slightly different version of $\Delta V_{2\pi}^{\mu,\nu}$ corresponding to $\mu = -1$ and $\nu = \frac{1}{2}$ in Eq. (19):

$$\Delta V_{2\pi}^{\rm Bonn} = \Delta V_{2\pi}^{-1,1/2}. \tag{29}$$

Equations (23)–(29), as well as Eq. (19), are our principal results. Specifying the form of the equation to be solved and the form of OPEP eliminates the three types of ambiguity discussed in the Introduction.

Finally, a very interesting and unusual calculation (from the nuclear physics perspective) was recently performed [32]. The one-pion-exchange potential was calculated on-shell, and the (wave-function-normalization) factor $(M/E_0)^2$ was extracted as a kinematical (flux) factor. Thus, the OPEP was chosen to be the nonrelativistic V_π^0. Subtracting (M/E_0) times the iterated OPEP (using a nonrelativistic Green's function) they found a $\Delta V_{2\pi}$ in the form

$$\Delta V_{2\pi}({\rm Ref.\ [32]}) = \Delta V_{2\pi}^{\rm MNL} + \frac{3V_\pi^2}{4M}, \tag{30}$$

and did not relate this result to any particular dynamical equation. We have verified that Eq. (30) corresponds to the conditions of Ref. [32].

IV. SUMMARY

In summary, we have completed an old-fashioned calculation of the nonstatic two-pion-exchange nucleon-nucleon potential that emphasizes and highlights the off-shell nature of OPEP. Different versions were developed that correspond to different (μ,ν) off-shell parameters, and to using relativistic or nonrelativistic kinetic energies in the Schrödinger equation, or using operator-valued or on-shell kinematic factors. These new results would correspond to the Nijmegen [21], Urbana [16], and Bonn [14] approaches, and further manipulation leads to the Munich result [32]. These various approaches in perturbation theory correspond to different subtractions in the second-order iteration of OPEP and represent different off-shell versions of that potential.

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