Rescattering Effects and the Validity of the Sequential-Decay Model*

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A steady-state point of view is used to examine the conditions under which rescattering corrections to the sequential-decay model are important in considering direct reactions to unbound states. Expressions are derived which can give semiquantitative estimates of the size of these effects. A qualitative interpretation is provided for the factors which enter these expressions, and they are evaluated for two specific cases, one where the rescattering effects are expected to be large, and one where they are expected to be small.

I. INTRODUCTION

With the increasing use of experiments involving direct nuclear reactions to unbound final target states, there is interest in the analysis of data along the lines of the sequential-decay model (SDM).[1,2] This model is useful for two-stage reactions of the form

$$I+T\to R^*+3\to 1+2+3\,. \tag{I.1}$$

The first stage is treated as a direct reaction leading to the compound-nuclear state or resonance R^*, while the second stage is treated as the decay of R^*,

$$R^*\to 1+2\,. \tag{I.2}$$

It is the practice to incorporate the decay stage into the direct-reaction formalism by using the expression for the wave function of R^*, obtained by considering the resonant scattering,[2-5]

$$1+2\to R^*\to 1+2\,. \tag{I.3}$$

This procedure implies that in the full reaction of Eq. (I.1) the decay of R^* and the final distribution of particles 1 and 2 are treated as if particle 3 were absent. There are two types of corrections which might be required to improve upon this approach: (a) those which stem from interactions of particle 3 with R^* before decay (for small distances between 1 and 2); and (b) those which arise from the interaction of particle 3 with 1 and 2 after the decay (for large distances between 1 and 2). The corrections of the first type call for amendment of the assumption that the reaction, $I+T\to R^*+3$, is a direct reaction. This may require either the use of the multistage approach of Penny and Satchler,[6] or a more complete treatment of compound effects. These corrections may be important whether or not R^* undergoes a decay and will not be dealt with here. The corrections of the second type, often called "rescattering" corrections,[7,8] shall be our primary concern.

We shall discuss criteria for determining the situations under which rescattering effects may be ignored, i.e., conditions under which the use of the SDM may be valid with respect to these effects. We are especially concerned with determining these criteria without recourse to those widely used arguments which are based on a time-dependent view of the decay process.[1,2] The latter approach, while heuristically helpful, is difficult to use in making quantitative estimates. This would require assumptions about the localization of wave packets, and, in principle, the analysis of the time evolution of a three-body process. (Even for the much simpler two-body problem, the time dependence of the emerging flux is strongly dependent on the specific nature of the incoming wave packet.) Furthermore, since the practical consequence of the SDM is the simplification of T matrix elements, conditions for its valid use should not depend on special time effects. We, therefore, have sought the validity criteria from the T matrix itself.

Since the resonance width, or lifetime, of R is presumed to play an important role, we have tried to obtain the validity criteria in such a way as to highlight this dependence. As a substitute for the simple physical interpretation associated with the time-dependent arguments, we have tried to provide a qualitative interpretation relevant to the steady-state point of view.

In Sec. II, we shall briefly review the time-dependent arguments leading to criteria for the validity of the SDM. We then present alternative physical arguments appropriate to the steady-state approach. In Sec. III, we quantitatively establish the validity criteria for the steady-state approach by carrying out a model calculation involving the essential features of reaction (I.1). Finally, in Sec. IV, we apply the criteria obtained in Sec. III to specific situations and draw conclusions about the effects of rescattering.

II. SIMPLE PHYSICAL PICTURES FOR VALIDITY CRITERIA

Let us begin by considering the SDM from a time-dependent point of view. The labels we use for the particles correspond to those given in Eq. (I.1). The argument goes roughly as follows[1,2]: If particle 3 has a velocity v_3 such that during the lifetime of R^* (i.e., $\hbar/\Gamma$) it travels a distance large compared to the range (L) of immediate vicinity of R^*, then particle 3 is "gone" by the time of decay and does not interact with the decay products. The smallness of the parameter ρ,

$$\rho = \Gamma L/\hbar v_3 , \tag{II.1}$$

then gives the criterion for assuming the validity of the SDM. This picture has been modified by the introduction of the concept of proximity scattering,[7,8] which argues that it is possible for one of the decay products, e.g. particle 2, to overtake particle 3 if $v_2 > v_3$ and if v_2 is parallel to v_3. It is argued that since the decay products may leave in any direction, the rescattering is only appreciable if it occurs near particle 1, before the flux has been diminished by the $1/r^2$ consideration. These arguments are all reasonable, but can only be made quantitative if the decay were to occur at a well-defined time, and if all the particles are well localized. For consideration of approximations to the T matrix (a steady-state quantity) these considerations are not relevant.

We look next at the full T matrix for a process involving three bodies in the final state as indicated in Eq. (I.1),

$$\langle T\rangle = \langle \psi_f^{(-)} | V_i | \chi_i \rangle . \tag{II.2}$$

Here, χ_i describes particles I and T noninteracting, and $\psi_f^{(-)}$ describes the full wave function which asymptotically has incoming boundary conditions for particles 1, 2, and 3. The SDM gives

$$\langle T\rangle_{\mathrm{SDM}} = \langle \chi_3^{(-)} \phi_{1,2}^{(-)} | V_i | \chi_i \rangle , \tag{II.3}$$

where $\phi_{1,2}^{(-)}$ describes the scattering of particles 1 and 2 with appropriate boundary conditions.

$\langle T\rangle_{\mathrm{SDM}}$ will be large when the relative energy of particles 1 and 2 is near a resonance. Watson discussed this enhancement, and gave a simple picture to clarify this effect, by considering the process described in Eq. (I.1) run in reverse.[9] He argued that such a process would require three particles, originally separated, to come together and to interact in a single localized region. The probability of this occuring is enhanced if two of the three are resonating, thereby increasing the chance for all three of the particles to overlap in any one region.

To examine the importance of rescattering, we also find it convenient to consider the reaction, including rescattering, run in reverse,

$$1+2+3 \xrightarrow{a} (1+2)' + 3' \xrightarrow{b} R^* + 3' \xrightarrow{c} I + T . \tag{II.4}$$

Here, the process a represents "prescattering" (the reverse of rescattering). For processes b and c to be enhanced it is essential for the relative energy in the (1, 2) system, after a, to be within the width Γ of the resonance energy E_R. The chance that the prescattering process will lead to this situation will depend on the width Γ, or the effective "window" for resonance. For example, if Γ is very narrow and the (1, 2) system is already on resonance at energy E_R, then the scattering in process a is likely to remove the conditions for resonance, by producing a relative energy of the (1, 2) system which lies outside of the resonance peak. On the other hand, if the (1, 2) system is originally off resonance, then process a is essential, and again the entire process depends on the relative energy ending up within Γ of E_R.

If, for simplicity, we assume that particle 3 interacts with particle 2 but not particle 1, then process a must depend on the strength of the (2, 3) interaction. This may be taken into consideration by assuming process a is proportional to the (2, 3) scattering amplitude, $f(2, 3)$.

The T matrix for the prescattering process shown in Eq. (II.4) might well look something like

$$\langle T\rangle_{\mathrm{PS}} \sim \frac{\Gamma(1,2) f(2,3)}{\hbar v} \langle T\rangle_{\mathrm{SDM}}^{\max} , \tag{II.5}$$

where v is a characteristic velocity, required to give the appropriate dimensions. [We will show in Sec. III that an appropriate velocity is that of the (2, 3) system relative to 1.] Our qualitative arguments would indicate that the dimensionless quantity,

$$\rho = \frac{\Gamma(1,2) f(2,3)}{\hbar v} , \tag{II.6}$$

should give some indication of the importance of the prescattering. Considering now the sequential decay process in its natural order where particles 1, 2, and 3 are the final products, we would expect the smallness of a parameter, identical to ρ, to indicate when rescattering might be ignored.

In the succeeding section we will establish such a criterion in a more precise and quantitative fashion. The arguments given above are only to show the general form one might expect from steady-state arguments. In summary, one can expect the importance of rescattering to depend on: (a) the width, here related to a "window" for achieving resonance, rather than a lifetime; (b) a length, here the scattering amplitude, rather than some measure of the vicinity of R; and (c) a characteristic velocity which in the steady-state approach is not necessarily the velocity of particle 3.

III. DETAILED CALCULATION

In this section we establish more precisely the validity criteria by considering the full T matrix for process

$$I+T\to 1+2+3\,. \tag{III.1}$$

For simplicity, let us assume the interaction between particles 3 and 1 is zero, but that between 3 and 2 is finite. We will also assume that there is a narrow resonance in the (1, 2) system, for relative energy E_R and of width Γ. We choose to calculate the T matrix in a frame in which the asymptotic value of the momentum of particle 1 is zero. The choice of this particular frame is motivated by the desire to concentrate on the rescattering of particles 2 and 3 as they move relative to particle 1. If the mass of particle 1, m_1, is much larger than m_2+m_3, then this frame is nearly the same as the laboratory frame.

The full T matrix is

$$\langle T\rangle=\langle\psi^{(-)}(1,2,3)|V_i|\chi(I,T)\rangle, \tag{III.2}$$

where

$$(H_1+H_2+H_3+V_{12}+V_{32})\psi^{(-)}=E\psi^{(-)}, \tag{III.3}$$

and $\chi(I,T)$ describes the noninteracting initial particles, I and T. Let us consider the set of wave functions $\phi^{(-)}(1,2)$ which describes the (1, 2) system,

$$(H_1+H_2+V_{12})\phi^{(-)}(1,2)=E_{12}\phi^{(-)}(1,2)\,. \tag{III.4}$$

Here, $\phi^{(-)}(1,2)$ includes both relative and center-of-mass coordinates, and E_{12} includes both energy contributions. For particle 3, we have

$$H_3\chi^{(-)}(3)=E_3\chi^{(-)}(3)\,. \tag{III.5}$$

We shall ignore the internal degrees of freedom of each of the three particles, so that

$$\chi^{(-)}(3)=e^{i\vec{k}_3\cdot\vec{r}_3}, \tag{III.6a}$$

$$\phi^{(-)}(1,2)=\exp\left(i\vec{\lambda}\cdot\frac{m_1\vec{r}_1+m_2\vec{r}_2}{m_{12}}\right)\tilde{\phi}^{(-)}(\kappa(\vec{r}_1-\vec{r}_2)), \tag{III.6b}$$

where λ and κ are center-of-mass and relative momenta,

$$\vec{\lambda}=\vec{k}_1+\vec{k}_2\,, \tag{III.7a}$$

$$\vec{\kappa}=\frac{m_1\vec{k}_2-m_2\vec{k}_1}{m_{12}}. \tag{III.7b}$$

(We use the notation $m_{ijk\ldots}=m_i+m_j+m_k+\cdots$.) For the reference frame with $k_1=0$, we have in the asymptotic region (1 and 2 widely separated),

$$\phi^{(-)}(1,2)=\exp\left(i\vec{k}_2\cdot\frac{m_1\vec{r}_1+m_2\vec{r}_2}{m_{12}}\right)\left\{\exp\left[i\vec{k}_2\cdot\frac{m_1}{m_{12}}(\vec{r}_2-\vec{r}_1)\right]+\frac{\exp[-ik_2(m_1/m_{12})|\vec{r}_2-\vec{r}_1|]}{|\vec{r}_2-\vec{r}_1|}f^{(-)}\right\}, \tag{III.8}$$

where $f^{(-)}$ is the appropriate scattering amplitude.

Let us now express the full three-body wave function $\psi^{(-)}(1,2,3)$ as follows,

$$\psi^{(-)}(1,2,3)=\chi^{(-)}(3)\phi^{(-)}(1,2)+\frac{1}{E-H_{12}-H_3-i\epsilon}V_{23}\psi^{(-)}(1,2,3)\,. \tag{III.9}$$

The T matrix then becomes

$$\langle 1,2,3|T|I,T\rangle=\langle\chi^{(-)}(3)\phi^{(-)}(1,2)|V_i|\chi(I,T)\rangle + \sum_{1',2',3'}\frac{\langle\psi^{(-)}(1,2,3)|V_{23}|\phi^{(-)}(1',2')\chi^{(-)}(3')\rangle}{E-E_3'-E_{12}'+i\epsilon}\langle\chi^{(-)}(3')\phi^{(-)}(1',2')|V_i|\chi(I,T)\rangle\,. \tag{III.10}$$

The first term on the right-hand side is the result retained in the SDM (or Watson-Migdal[9,10] expression). We introduce the notation,

$$\langle T\rangle_{\text{SDM}}=\langle\chi^{(-)}(3)\phi^{(-)}(1,2)|V_i|\chi(I,T)\rangle\,. \tag{III.11}$$

We assume that this amplitude is enhanced near the resonance in the (1, 2) system, so that it has the form,

$$\langle T\rangle_{\text{SDM}}=\frac{N(\hat{k}_3)}{(\hbar^2/2m_2)(m_1/m_{12})(k_2^{\,2}-k_0^{\,2})+\frac{1}{2}i\Gamma}, \tag{III.12a}$$

$$N(\hat{k}_3)=\tfrac{1}{2}i\Gamma\langle T\rangle_{\text{SDM}}^{\max}, \tag{III.12b}$$

where $(m_1/m_{12})k_2$ is the relative momentum for particles 1 and 2; $(m_1/m_{12})k_0$ is the relative momentum for resonance; and $N(\hat{k}_3)$ is a numerator function, depending significantly only on the direction of particle 3.

The second term on the right-hand side of Eq. (III.10) represents the full correction due to the rescattering effects, $\langle T\rangle_{\rm RS}$,

$$\langle T\rangle_{\rm RS}=\int\frac{d^3k_3'd^3k_1'd^3k_2'\langle\psi^{(-)}(1,2,3)|V_{23}|\phi^{(-)}(1',2')e^{i\vec{k}_3'\cdot\vec{r}_3}\rangle}{E-\frac{\hbar^2}{2m_3}k_3'^2-\frac{\hbar^2(\vec{k}_2'+\vec{k}_1')^2}{2m_{12}}-\frac{\hbar^2}{(2m_1m_2/m_{12})}\left(\frac{m_1\vec{k}_2'-m_2\vec{k}_1'}{m_{12}}\right)^2+i\epsilon}\frac{1}{(2\pi)^9}\langle 1',2',3'|T|IT\rangle_{\rm SDM}. \qquad (\text{III.13})$$

The remainder of this section deals with estimating the size of this contribution.

To evaluate $\langle T\rangle_{\rm RS}$ we replace $\langle 1',2',3'|T|IT\rangle_{\rm SDM}$ by the expression in Eq. (III.12), and we replace $\psi^{(-)}(1,2,3)$ by $e^{i\vec{k}_3\cdot\vec{r}}\phi^{(-)}(1,2)$ (analogous to a second Born approximation). Before carrying out the integral in Eq. (III.13) we must evaluate the term

$$\langle V_{23}\rangle=\langle e^{i\vec{k}_3\cdot\vec{r}_3}\phi^{(-)}(1,2)|V_{23}|e^{i\vec{k}_3\cdot\vec{r}_3}\phi^{(-)}(1',2')\rangle. \qquad (\text{III.14})$$

This involves the interaction of 2 and 3 throughout all space. Here, we can simplify the result by assuming the bulk of the contribution to this matrix element comes when particle 2 is away from the vicinity of particle 1. We may then use the asymptotic form for $\phi^{(-)}(1,2)$,

$$\phi^{(-)}(1,2)=e^{i\vec{\lambda}_{12}\cdot\vec{R}_{12}}\left(e^{i\vec{\kappa}_{12}\cdot\vec{\rho}_{12}}+\frac{e^{-i\kappa_{12}\rho_{12}}}{\rho_{12}}f^{(-)}(\kappa_{12})\right), \qquad (\text{III.15})$$

where λ_{12} and κ_{12} are the center-of-mass and relative momenta, and R_{12} and ρ_{12} are the respective coordinates. This assumption is consistent with our concern for the effects of the interactions of 2 and 3, where 2 is significantly far from 1. We defined these as the "rescattering" corrections in the Introduction.

Putting Eq. (III.15) into the expression for the (2, 3) interaction, Eq. (III.14), we find there are four types of contributions: $\langle T\rangle_{\rm RS}^{(0,0)}$, with the plane waves on each side; $\langle T\rangle_{\rm RS}^{(1,1)}$, with scattered waves on each side; and $\langle T\rangle_{\rm RS}^{(0,1)}$ and $\langle T\rangle_{\rm RS}^{(1,0)}$, with plane waves on one side and scattered waves on the other.

Since the contribution from the scattered terms goes as $1/r$, we first evaluate $\langle T\rangle_{\rm RS}^{(0,0)}$, which involves only the plane-wave contributions to $\phi^{(-)}$ in Eq. (III.14). With some simplifying approximations outlined in the Appendix, we obtain

$$\langle T\rangle_{\rm RS}^{(0,0)}\approx\frac{m_{21}}{m_1}\frac{\Gamma}{\hbar^2}\frac{m_{23}f_B(R,0)}{|\vec{k}_2+\vec{k}_3|}\langle T\rangle_{\rm SDM}^{\max}\tfrac{1}{2}\ln\left(\frac{k_0'+C-R}{k_0'-C-R}\right), \qquad (\text{III.16})$$

where

$$k_0'\equiv\left(k_0{}^2-i\frac{\Gamma}{\hbar}\frac{m_2m_{12}}{m_1}\right)^{1/2},$$

and

$$R=|\vec{R}|=\left|\frac{m_3\vec{k}_2-m_2\vec{k}_3}{m_{23}}\right|, \qquad (\text{III.17a})$$

$$C=|\vec{C}|=\left|\frac{m_2}{m_{23}}(\vec{k}_2+\vec{k}_3)\right|. \qquad (\text{III.17b})$$

Here, $f_B(R,0)$ is the first Born approximation to the forward-angle scattering amplitude for the (2, 3) system, with R as the relative momentum. $\langle T\rangle_{\rm SDM}^{\max}$ is the maximum value (as a function of k_2) for the amplitude $\langle T\rangle_{\rm SDM}$ which is appropriate to the process [see Eq. (III.12b)]. The quantities R and C are parameters of a sphere which characterizes the elastic scattering of the (2, 3) system. For the scattering $\vec{k}_2+\vec{k}_3\rightarrow\vec{k}_2'+\vec{k}_3'$ the allowed values of $\vec{k}_2'$ are given by

$$|\vec{k}_2'-\vec{C}|=|\vec{R}|, \qquad (\text{III.17c})$$

as indicated graphically in Fig. 1.

The expression given in Eq. (III.16) has logarithmic singularities near $k_0=R+C$, for $C>R$, and near both $k_0=R+C$ and $k_0=R-C$, for $C<R$. From the Feynman-diagram treatment of the triangle singularity,[11] one would be led to expect only the singularity for $C>R$. The contribution to the integral leading to Eq. (III.16) comes primarily from $k_2'\approx k_0$. Thus for the condition $k_0=R+C$ and $R<C$, this contribution comes from $k_2'=R+C$. This is seen (Fig. 1) to be the case when $\vec{k}_2'$ is parallel to $\vec{k}_3'$, and $k_2'>k_3'$ (these are the well-known requirements for large rescattering).[7,11] In Eq. (III.16), however, two other singularities appear.

We next evaluate the contribution $\langle T\rangle_{\rm RS}^{(0,1)}$ which comes from including the scattered portion of the wave function on the right-hand side of Eq. (III.14). We use

$$f^{(-)}\left(\frac{m_1}{m_{12}}k_2'\right)=-\frac{\Gamma}{2}\frac{1}{k_2'm_1/m_{12}}\times\frac{1}{(\hbar^2/2m_2)(m_1/m_{12})(k_2'^2-k_0{}^2)-\frac{1}{2}i\Gamma} \qquad (\text{III.18})$$

to reflect the resonance in the (1, 2) system assumed in obtaining $\langle T\rangle_{\rm SDM}$. The following expression for $\langle T\rangle_{\rm RS}^{(0,1)}$ results from assumptions similar to those used in obtaining $\langle T\rangle_{\rm RS}^{(0,0)}$:

$$\langle T\rangle_{\rm RS}^{(0,1)}\approx-\frac{m_{21}}{m_1}\frac{\Gamma}{\hbar^2}\frac{f_B(R,0)}{|\vec{k}_2+\vec{k}_3|}\langle T\rangle_{\rm SDM}^{\max}\frac{m_{23}}{2}\ln(\dots), \qquad (\text{III.19})$$

where the argument of the logarithm is given by

$$(\ldots) = \frac{\left[\frac{m_{12}}{m_{123}}\frac{m_{23}}{m_2}\left(R^2+\frac{m_1 m_3}{m_{123}m_2}C^2\right) - \frac{m_1 m_3}{m_{123}m_2}k_0'^2\right]^{1/2} - k_0' - \frac{m_{12}}{m_{123}}\frac{m_{23}}{m_2}C}{\left[\frac{m_{12}}{m_{123}}\frac{m_{23}}{m_2}\left(R^2+\frac{m_1 m_3}{m_{123}m_2}C^2\right) - \frac{m_1 m_3}{m_{123}m_2}k_0'^2\right]^{1/2} - k_0' + \frac{m_{12}m_{23}}{m_{123}m_2}C}\,. \tag{III.20}$$

The contribution in Eq. (III.19) may be readily combined with $\langle T\rangle_{\mathrm{RS}}^{(0,0)}$ to yield

$$\langle T\rangle_{\mathrm{RS}}^{(0,0+1)} \cong \frac{m_{21}}{m_1}\frac{\Gamma m_{23}}{\hbar^2}\frac{f_B(R,0)}{|\vec{k}_2+\vec{k}_3|}\langle T\rangle_{\mathrm{SDM}}^{\max} F_1(k_0, k_2, k_3)\,, \tag{III.21}$$

where the factor F_1 is given by

$$F_1 = \tfrac{1}{2}\ln\left(\frac{\left[\frac{m_{12}}{m_{123}}\frac{m_{23}}{m_2}\left(R^2+\frac{m_1 m_3}{m_{123}m_2}C^2\right) - \frac{m_1 m_3}{m_{123}m_2}k_0'^2\right]^{1/2} + R + \frac{m_1 m_3}{m_{123}m_2}C}{\left[\frac{m_{12}}{m_{123}}\frac{m_{23}}{m_2}\left(R^2+\frac{m_1 m_3}{m_{123}m_2}C^2\right) - \frac{m_1 m_3}{m_{123}m_2}k_0'^2\right]^{1/2} + R - \frac{m_1 m_3}{m_{123}m_2}C}\right). \tag{III.22}$$

The effect of adding the scattered part of the resonance wave function is to remove the two singularities for $R > C$ and to yield a logarithmic expression identical with the Feynman-diagram expression of Ref. 11.

Finally, we look at the contributions coming from adding the scattering portions of $\phi^{(-)}(1, 2)$ to the left side of the matrix element given in Eq. (III.14). These contributions provide no new singularities. To estimate their size we consider the case in which $m_1 \gg m_{23}$, and obtain

$$\langle T\rangle_{\mathrm{RS}}^{(1,0+1)} = \langle T\rangle_{\mathrm{SDM}}\left(\frac{\Gamma}{\hbar^2}\frac{f_B(R,0)m_2}{k_0}F_2(k_3)\right), \tag{III.23}$$

where F_2 is given by

$$F_2 = \frac{m_{23}}{2m_2}\left[i\pi - \frac{k_0}{k_3}\ln\left(1+\frac{k_3}{k_0}\right) - \ln\left(1+\frac{k_0}{k_3}\right)\right]. \tag{III.24}$$

It should be noted that the term in Eq. (III.23) is proportional to the leading term, $\langle T\rangle_{\mathrm{SDM}}$, whereas $\langle T\rangle_{\mathrm{RS}}^{(0,0+1)}$ is proportional to the maximum value of $\langle T\rangle_{\mathrm{SDM}}$. If we combine all of the derived contributions, and replace $f_B(R, 0)$ by the full scattering amplitude, $f(R, 0)$, in an effort to improve on one of our approximations, we obtain the following general form for the total T matrix,

$$\langle T\rangle \cong \langle T\rangle_{\mathrm{SDM}}\left(1+\frac{\Gamma f(R,0)}{\hbar v_2}F_2\right) + \langle T\rangle_{\mathrm{SDM}}^{\max}\frac{\Gamma f(R,0)}{\hbar v_{23}}F_1\,, \tag{III.25}$$

where the factors F_1 and F_2 are given by Eqs. (III.22) and (III.24), and where v_2 and v_{23} are the velocities of particle 2 and the (2, 3) system center of mass. The factor F_1 contains the singularity associated with proximity scattering. Its maximum is determined by the width of the resonance in such a way that F_1 is proportional to $\ln\Gamma$ at the resonance. Since $\Gamma\ln\Gamma$ vanishes as Γ goes to zero,

$$\langle T\rangle \underset{\Gamma\to 0}{\longrightarrow} \langle T\rangle_{\mathrm{SDM}}\,, \tag{III.26}$$

even near the resonance condition.

We next examine those conditions under which the factor F_1, given in Eq. (III.22), is large, and those under which it is small. First note that for $k_3 \gg k_2 \approx k_0$, which is far from the singularity, the kinematic parameters become

$$R = C = (m_2/m_{23})k_3\,, \tag{III.27}$$

and we have for F_1,

$$F_1 = \tfrac{1}{2}\ln(m_{23}/m_2)\,. \tag{III.28}$$

This leads to the following expression for the T

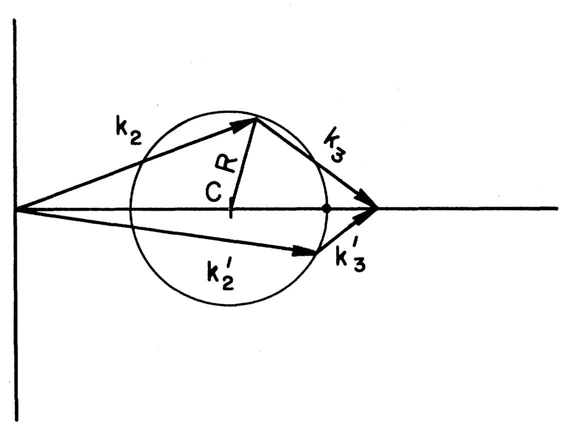

FIG. 1. Kinetic constraints for two-body elastic scattering.

matrix:

$$\langle T\rangle \sim \langle T\rangle_{\mathrm{SDM}}\left(1+\frac{\Gamma f((m_2/m_{23})k_3,0)}{\hbar v_2}\, i\pi \frac{m_{23}}{2m_2}\right) + \langle T\rangle_{\mathrm{SDM}}\frac{\Gamma}{\hbar v_3} f\left(\frac{m_2}{m_{23}}k_3, 0\right)\frac{m_{23}}{2m_2}\ln\left(\frac{m_{23}}{m_3}\right). \tag{III.29}$$

At the other extreme let us consider an upper bound for F_1. Near the singularity $k_0 = R + C$, F_1 becomes (for $m_1 \gg m_2 = m_3$)

$$F_1 \approx \tfrac{1}{2}\ln\left(2\hbar^2\frac{(k_3+k_0)k_3}{i m_2 \Gamma}\right). \tag{III.30}$$

Since $k_3 < k_0$, we obtain as an upper bound,

$$F_1 < \tfrac{1}{2}\ln\left[8\left(\frac{\hbar^2 k_0^2}{2m}\right)\Big/ i\Gamma\right] = \tfrac{1}{2}\ln\left(\frac{8E_R}{i\Gamma}\right). \tag{III.31}$$

Under the conditions that $E_R = 10\Gamma$, for example, we obtain

$$\tfrac{1}{2}\ln\left(\frac{8E_R}{i\Gamma}\right) \approx 2.2 - 0.8i\ . \tag{III.32}$$

Using the variables R and C, defined in Eq. (III.17), it is easy to display the conditions for approaching the logarithmic singularity. Conservation of total energy provides

$$R^2 + \frac{m_1 m_3}{m_{123} m_2} C^2 = A^2, \tag{III.33}$$

where A^2 depends only on the laboratory momentum of the incident particle, $P_{I(\mathrm{lab})}$, and the Q value for the entire process,

$$A^2 = \frac{2m_2 m_3}{m_{23}}\left(Q + \frac{1}{2m_I}\frac{M_T}{M_{IT}}(P_I^2)_{\mathrm{lab}}\right). \tag{III.34}$$

On a plot of $(m_1 m_3/m_{123} m_2)^{1/2}\, C$ versus R, energy conservation constrains the values of R and C to circles of radius A (see Fig. 2). It can be shown that for the reaction to occur at all,

$$A^2 > \frac{m_1 m_3}{m_{12} m_{23}} k_0^2\ . \tag{III.35}$$

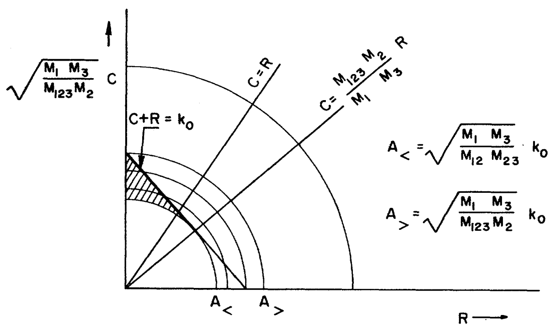

FIG. 2. Kinematic constraints relevant to the rescattering singularity (see text).

Furthermore, if the conditions for the logarithmic maximum in F_1 are to occur, then

$$A^2 < \frac{m_1 m_3}{m_{123} m_2} k_0^2\ . \tag{III.36}$$

In Fig. 2 are plotted the lines of constraint for different initial momenta, i.e., different values of A. The line $k_0 = R + C$ also appears on the graph. This line represents the conditions for approaching the singularity. The condition indicated in Eq. (III.35) insures that the smallest A gives an arc which is tangent to the line $k_0 = R + C$. The minimum A thus results in a peak at $R = (m_1 m_3/m_{123} m_2)C$. From Eq. (III.22) it may be seen that the singularities only occur for $R \leq (m_1 m_3/m_{123} m_3)C$. Therefore as A increases from its minimum value, the singularity moves to higher C and lower R, until $R = 0$ at $A^2 = (m_1 m_3/m_{123} m_2)k_0^2$. For A greater than this value, one can no longer approach the logarithmic singularity.

In addition to the conservation of energy, the definition of C,

$$C = \frac{m_2}{m_{23}}|\vec{k}_2 + \vec{k}_3|, \tag{III.37}$$

restricts its values to

$$\frac{m_2}{m_{23}}|\vec{k}_2 - \vec{k}_3| \leq C \leq \frac{m_2}{m_{23}}|\vec{k}_2 + \vec{k}_3|\ . \tag{III.38}$$

If either k_2 or k_3 is much larger than the other, then C is restricted to a very small range of values along the appropriate arc indicated in Fig. 2. If $k_3 \gg k_2$, then C must be near the intersection of the arc with the line $R = C$. If $k_2 \gg k_3$, C is limited to values near the line $R = (m_2/m_3)C$. For a large difference between k_2 and k_3, we move further from the line $k_0 = R + C$ with larger A (initial energy), and we also move to larger R. If it is the case that the (2, 3) scattering amplitude drops with increasing R, then raising the incident energy can reduce the rescattering effects in three ways: (a) by moving the kinematic conditions away from the logarithmic singularity; (b) by increasing R, thus reducing $f(R, 0)$; and (c) by increasing velocity v_{23} which is proportional to C.

With the T matrix we have derived, one can predict the differential cross section for observing both particle 2 and particle 3. We next develop an expression for the differential cross section obtained if only the angular distribution of particle 3 is observed. We begin with

$$d\sigma \sim |\langle T\rangle|^2 k_2^2 dk_2 d\Omega_2 k_3^2 dk_3 d\Omega_3 \delta(E_i - E_f), \tag{III.39}$$

and obtain

$$\frac{d\sigma}{d\Omega_3} \sim \int |\langle T\rangle|^2 k_3 k_2^2 dk_2 d\Omega_2 \frac{dk_3^2}{dE_f}\ . \tag{III.40}$$

For simplicity in estimating this cross section, we consider the case with $m_1 \gg m_2 = m_3$. The differential volume element may then be written in terms of the variables $\vec{C}$, $\vec{R}$, and A, as follows:

$$k_3 k_2^2 dk_2 d\Omega_3 \to 8\pi C^2 \sqrt{A^2 - C^2}\, dC\, d\omega\,, \qquad \text{(III.41)}$$

where

$$\omega = \vec{C} \cdot \vec{R}/CR\,. \qquad \text{(III.41}')$$

For $\langle T\rangle$, we use the expression given in Eq. (III.25), and then find those contributions to $(d\sigma/d\Omega_3)$ which are lowest order in Γ. The largest of these comes from taking $\langle T\rangle$ equal to $\langle T\rangle_{\text{SDM}}$, and gives

$$\left(\frac{d\sigma}{d\Omega_3}\right)^0 \sim \pi^2 |\langle T\rangle_{\text{SDM}}^{\max}|^2 k_{30} k_0^3 \frac{\Gamma}{\hbar^2 k_0^2/2m}\,, \qquad \text{(III.42)}$$

where k_{30} is the momentum of particle 3 which occurs when particle 2 has momentum k_0. There are other contributions to $(d\sigma/d\Omega_3)$ which involve $\langle T\rangle_{\text{SDM}}$ as a factor, and consequently arise primarily when $k_2 = k_0$ and $k_3 = k_{30}$. In the limit $k_{30} \gg k_0$ these yield

$$\left(\frac{d\sigma}{d\Omega_3}\right)^1 = \left(\frac{d\sigma}{d\Omega_3}\right)^0 \left(\text{Re}\,\frac{\Gamma}{E_R}\, i\pi k_0 f(\tfrac{1}{2}k_{30}, 0) + \text{Re}\,\frac{\Gamma}{E_R}\, i(\ln 2)\frac{k_0}{k_{30}} k_0 f(\tfrac{1}{2}k_{30}, 0)\right)\,, \qquad \text{(III.43)}$$

where we have taken

$$F_2 \to i\pi\,, \qquad \text{(III.44a)}$$

$$\frac{\int f(R,0) R\, dR}{\int R\, dR} \to f(\tfrac{1}{2}k_{30}, 0)\,, \qquad \text{(III.44b)}$$

$$\int dC\, F_1(C) \to \tfrac{1}{2}k_{30} \ln 2\,. \qquad \text{(III.44c)}$$

One final term must be considered among the portions of $(d\sigma/d\Omega_3)$ which are of lowest order in Γ. This term receives contributions from the entire range of k_2 and k_3, and arises from the square of the last term in Eq. (III.25). An estimate of the size of this term is given by

$$\left(\frac{d\sigma}{d\Omega_3}\right)^2 = \left(\frac{d\sigma}{d\Omega}\right)^0 \left\{\frac{\Gamma}{E_R}\,\frac{1}{8}\left[\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\right]^2 k_0 k_{30} |\bar{f}|^2\right\}\,, \qquad \text{(III.45)}$$

where

$$|\bar{f}|^2 = \frac{\int_0^A R\, |f(R,0)|^2 dC}{\int_0^A R\, dC}\,, \qquad \text{(III.46)}$$

and where we have used

$$A^2 \approx \tfrac{1}{2}k_{30}^2\,, \qquad \text{(III.47)}$$

and have evaluated F_1 at its maximum value, i.e., with $C = A$.

Putting all the contributions to $(d\sigma/d\Omega_3)$ together, we have

$$\left(\frac{d\sigma}{d\Omega_3}\right) = \left(\frac{d\sigma}{d\Omega_3}\right)^0 \left(1 + \frac{\Gamma}{E_R} G\right)\,, \qquad \text{(III.48)}$$

where, for large values of A,

$$G \to \text{Re}\left[k_0 f\left(\frac{k_{30}}{2}, 0\right)\left(i\pi + i\frac{k_0}{k_{30}} \ln 2\right)\right] + \left[\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\right]^2 \tfrac{1}{8} k_0 k_{30} |\bar{f}|^2\,. \qquad \text{(III.49)}$$

IV. CONCLUSIONS

We have found that rescattering introduces two corrections to the SDM scattering matrix. One of these is the addition of a contribution of the form

$$\langle T\rangle_{\text{SDM}}^{\max} \frac{\Gamma}{\hbar v_{23}} f(R,0) F_1\,, \qquad \text{(IV.1)}$$

where $f(R, 0)$ is the (2, 3) forward-angle scattering amplitude for relative momentum R, and where F_1 is given by Eq. (III.22). F_1 ranges in value,

$$F_1 \le \tfrac{1}{2} \ln\left(\frac{8E_R}{i\Gamma}\right)\,, \qquad \text{(IV.2)}$$

where E_R and Γ are the energy and width of the (1, 2) resonance. This is precisely the form of the expression anticipated in the introductory remarks of Sec. II. In addition, there is a second correction, a renormalization of the leading SDM term, given by

$$\langle T\rangle_{\text{SDM}} \frac{\Gamma}{\hbar v_2} f(R,0) F_2\,, \qquad \text{(IV.3)}$$

where F_2, given in Eq. (III.24), approaches $i\pi$ for large incident energies. At high energies the second correction may be larger than the first near the $\langle T\rangle_{\text{SDM}}$ resonance, but its effect is only to change the amplitude of the Watson-Migdal resonance peak of the SDM. Off resonance, the first correction term dominates.

For the differential cross section, $(d\sigma/d\Omega_3)$, we have found corrections to the SDM of the form

$$\left(\frac{d\sigma}{d\Omega_3}\right) = \left(\frac{d\sigma}{d\Omega_3}\right)^0 \left(1 + \frac{\Gamma}{E_R} G\right)\,, \qquad \text{(IV.4)}$$

where G is given in Eq. (III.49). The existence of such corrections must be a concern in the extraction of spectroscopic factors.

To estimate the size of the various rescattering terms, let us consider a case in which particle 2 is a proton, particle 3 is a neutron, $E_R = 1$ MeV, $\Gamma = 100$ keV, $R = 0$, and $C = k_0$. Then,

$$\frac{\Gamma}{\hbar v_{23}} f(R,0) F_1 \approx \frac{1}{4}\frac{\Gamma}{E_R} k_0 f(0,0) \ln\left(\frac{8E_R}{i\Gamma}\right)\,. \qquad \text{(IV.5)}$$

With $|f(0,0)| \sim 12$ fm, we obtain

$$\left|\frac{\Gamma}{\hbar v_{23}} f F_1\right| \sim 0.3 . \tag{IV.6}$$

Off resonance the cross section which is due to this rescattering term becomes

$$\sigma_{RS} \sim 0.09 \sigma_{max} . \tag{IV.7}$$

This effect may be observed in the data for $d + {}^{12}C \to {}^{12}C + p + n$ found in Ref. 8, where the actual physical parameters approach those assumed in the above example. There, it is seen that the cross section at the rescattering peak or shoulder (for E_p below the resonant value) is indeed on the order of 10% of the maximum cross section, at the resonance energy.

We next consider a stripping reaction leading to an unbound state ${}^{16}O(d,p){}^{17}O^*(5.08$ MeV). Let us take $\Gamma \sim 100$ keV, $\hbar^2 k_0^2/2m = E_R = 1$ MeV, $E_d = 12$ MeV, $Q \sim 2$ MeV. These parameters correspond roughly to those employed in Ref. 3. The above conditions lead to

$$E_{30} = \frac{\hbar^2 k_{30}^2}{2m} = 13 \text{ MeV} . \tag{IV.8}$$

With $R \approx \frac{1}{2} k_{30}$, we have $|f(R,0)| \sim 2.5$ fm which gives

$$\left|\frac{\Gamma}{\hbar v_{23}} f(R,0) F_1\right| \approx \left|\frac{\Gamma}{2E_{30}} (\ln 2) k_{30} f(R,0)\right| \approx 0.005 . \tag{IV.9}$$

We also have

$$\left|\frac{\Gamma}{\hbar v_2} f(R,0) F_2\right| \approx \left|\frac{1}{2}\frac{\Gamma}{E_R} k_0 f(R,0)\pi\right| \approx 0.1 . \tag{IV.10}$$

Finally, for $(d\sigma/d\Omega_3)$ we obtain

$$\left(\frac{d\sigma}{d\Omega_3}\right) = \left(\frac{d\sigma}{d\Omega_3}\right)_{SDM}\left(1 + \frac{\Gamma}{E_R} G\right) , \tag{IV.11}$$

where G is given by Eq. (III.49). Using the energies given above,

$$G = -2 + 0.07 |\bar{f}|^2 , \tag{IV.11$'$}$$

where $|\bar{f}|^2$ is defined by Eq. (III.46). Because $|\bar{f}|^2$ is less than 144 fm^2, i.e., $|f_{max}|^2$, G is less than 8. We thus obtain the following range for rescattering corrections:

$$0 \leq |(\Gamma/E_R) G| \leq 0.8 . \tag{IV.12}$$

The precise value depends on the value of $|\bar{f}|^2$. It is clear from this example that rescattering may have a significant effect on the determination of spectroscopic factors, which requires calculation of the absolute normalization of cross sections.

With the aid of the expressions discussed in the preceding sections, semiquantitative estimates may be made to determine: (a) to what degree rescattering effects may be ignored; (b) the approximate size of these effects if they must be included; and (c) the conditions under which the effects may be large. The expressions were obtained from a steady-state point of view, which seems to provide a good basis for arriving at quantitative criteria for testing the validity of the SDM.

APPENDIX

In obtaining the expressions given in Eqs. (III.16), (III.19), and (III.23), we have inserted Eq. (III.12) and Eq. (III.14) into the integral expression given in Eq. (III.13). As an approximation, we removed $\langle T\rangle_{SDM}^{max}$ from the integration taking

$$N(\hat{k}_3') \approx N(\hat{k}_3) . \tag{A1}$$

We used the following forms for $\langle V_{23}\rangle^{(i,j)}$ of Eq. (III.14) to obtain the corresponding $\langle T\rangle_{RS}^{(i,j)}$:

$$\langle V_{23}\rangle^{(0,0)} = (2\pi)^3 \langle V\rangle \delta(\vec{k}_2 + \vec{k}_3 - \vec{k}_2' - \vec{k}_3') , \tag{A2a}$$

$$\langle V_{23}\rangle^{(0,1)} = 4\pi \frac{\langle V\rangle (m_{12}/m_1)^2 f^{(-)}(\kappa')}{|\vec{k}_3 + \vec{k}_2 - \vec{k}_3'|^2 - |(m_{12}/m_1)\kappa'|^2 + i\epsilon} , \tag{A2b}$$

$$\langle V_{23}\rangle^{(1,0)} = 4\pi \frac{\langle V\rangle f^{(-)*}(k_2)}{|\vec{k}_3' + \vec{k}_2' - \vec{k}_3|^2 - k_2^2 - i\eta} , \tag{A2c}$$

$$\langle V_{23}\rangle^{(1,1)} = 4\pi \frac{\langle V\rangle f^{(-)*}(k_2) f^{(-)}(k_2')}{2i|\vec{k}_3 - \vec{k}_3'|} \times \ln\left(\frac{k_2 - k_2' - |\vec{k}_3 - \vec{k}_3'| + i\epsilon}{k_2 - k_2' + |\vec{k}_3 - \vec{k}_3'| + i\epsilon}\right) , \tag{A2d}$$

where

$$\langle V\rangle = -\frac{2\pi h^2 m_{23}}{m_2 m_3} f_B(R,0) \times (2\pi)^3 \delta(\vec{k}_2 + \vec{k}_3 - \vec{k}_1' - \vec{k}_2' - \vec{k}_3') , \tag{A2e}$$

$$\vec{k}' = \frac{m_1 \vec{k}_2' - m_2 \vec{k}_1'}{m_{12}} . \tag{A2f}$$

The expressions above for $\langle V_{23}\rangle^{(i,j)}$ are exact for a zero-range potential V_{23}, but are only approximate for a finite-range potential. [The forms given in Eqs. (A2c) and (A2d) assume $m_1 \gg m_{23}$ for simplification.] With the above expressions, the integrals in Eq. (III.13) can be evaluated using[12]

$$\int \frac{d^3k}{[(\vec{k} - \vec{a})^2 + m_1^2][(\vec{k} - \vec{b})^2 + m_2^2]} = \frac{\pi^2}{i|\vec{a} - \vec{b}|} \ln\left(\frac{m_1 + m_2 + i|\vec{a} - \vec{b}|}{m_1 + m_2 - i|\vec{a} - \vec{b}|}\right) \tag{A3}$$

and standard integrals with the assumption that Γ be vanishingly small.

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States in ^{15}O Between 8.8 and 9.0 MeV Excitation*

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The reaction $^{14}N(p,\gamma)^{15}O$ was used to examine the resonance structure in ^{15}O between 8.8 and 9.0 MeV. The resonance previously reported at $E_p = 1.742$ MeV was found to have two components corresponding to excitations in ^{15}O of 8.920 and 8.925 MeV, respectively. The decay properties of these two levels were studied with high-resolution Ge(Li) detectors, showing that they closely resemble the properties of probable mirror levels at 9.23 and 9.155 MeV in ^{15}N.

I. INTRODUCTION

The mass-15 nuclei ^{15}O and ^{15}N have been the subject of extensive study.[1] The properties of the $T=\frac{1}{2}$ states below 8.75 MeV are well understood, and the energies and decay properties of all except the two $J^{\pi}=\frac{1}{2}^{+}$ states are as expected for mirror nuclei. The situation is much less clear above this energy range, where experiments[2,3] have shown that the state in ^{15}N previously reported at 9.16 MeV is really a close-lying doublet, with the two members having energies of 9.152 and 9.155 MeV, and spins $J^{\pi}=\frac{3}{2}^{-}$ and $\frac{5}{2}^{+}$, respectively. More recently, Steerman and Young[4] have studied the reaction $^{13}C(^{3}He,p)^{15}N$ and determined the spin of the 9.22-MeV state to be $J^{\pi}=\frac{1}{2}^{-}$. Since Coulomb shifts between mirror nuclei much greater than 300 keV are uncommon, one should expect the three corresponding states in ^{15}O to fall between 8.8 and 9.1 MeV. However, only two states have been reported. Elastic proton experiments[5-8] show a large resonance at $E_p = 1.74$ MeV and a weaker one at $E_p = 1.81$ MeV, corresponding to 8.91 and 8.98 MeV excitation in ^{15}O. The analysis of these data has been inconclusive except for fixing the parity of the 8.91-MeV state as negative. Experiments using the $^{14}N(p,\gamma)$ reaction have confirmed the existence of these two resonances,[9] but the cross section in the capture channel has been too small to determine the spins and parities.

We have reexamined the resonance structure of ^{15}O between 8.8 and 9.1 MeV in the capture channel, to search for the existence of a third state. Since it appeared unlikely that previous experiments could have overlook an isolated state in this energy range, we have examined the two known resonance structures in detail to determine if one of these could possibly be a previously unresolved doublet.

II. EXPERIMENTAL PROCEDURE

The states in ^{15}O above 7.3 MeV are unbound. Information about these states is most readily obtained, therefore, from a detailed study of the observed resonance structure. Experimentally, one measures the excitation function either for one of the particle channels or the proton-capture reaction. In the energy region of interest (8.9 to 9.0 MeV) the analysis of such an excitation function is complicated for two reasons. First, it is well known[1] that there is a large nonresonant background, originating in part from two broad states at 9.48 and 9.72 MeV, respectively. Proper subtraction of this background requires that a complete γ-ray spectrum be taken at each proton