Pion-Nucleon Potentials for Use in Low-Energy Nuclear Physics

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One- and two-term nonlocal separable potentials are determined to fit all partial S-, P-, and D-wave experimental phase shifts up to the first energy resonance at about $E_\pi = 180$ MeV (lab). The theoretical and experimental phase shifts, scattering lengths, and volumes are in good agreement. The practical use of this potential in nuclear physics calculations is simplified by the choice of simple form factors and of identical ranges in all S, P, and D relative states.

1. INTRODUCTION. USEFULNESS OF A PION-NUCLEON POTENTIAL

In recent years, experimental and theoretical research in pion-nucleon (π-N) elastic and inelastic scattering has aroused considerable interest. Diverse complicated techniques are required to explain resonances and inelastic processes. A systematic utilization of the law of causality (dispersion theory), of the crossing symmetry, and of the S-matrix unitarity has brought about considerable progress in the understanding of π-N interaction phenomena. However, as soon as the π-N interaction is needed in a nuclear calculation, many simplifying hypotheses have to be made[1-4]: (i) A simplified form for the scattering amplitude is usually chosen[3,4]; (ii) in a multiple-scattering calculation as, for example, the pion-nucleus-potential calculation, the impulse approximation is used[3]; i.e., one assumes that the scattering amplitude of free pions on free nucleons and on bound nucleons in a nucleus are the same; this hypothesis is justified by the absence of poles at low energy and the short range of the π-N interaction; (iii) in numerous studies of the hole states in nuclei, a still more phenomenological approach to describe the pion-nucleus interaction is used. It is indeed well known that a pion is preferentially absorbed by two nucleons in a nucleus rather than by one because of energy and momentum conservation. Both experiment and theory show that if two nucleons practically at rest absorb a 140-MeV pion, they both leave the nucleus in opposite directions, each having energy of about 70 MeV. The final state in this reaction leads to a three-body problem which is hard to solve. Then, in the calculations, one neglects the interaction between the two ejected nucleons, or the interaction between these nucleons and the recoiling nucleus, or both interactions, using phenomenological pion-nucleus potentials (quasifree picture of pion absorption in nuclei).[4] These three types of approximations are often completed by the introduction of phenomenological parameters chosen without reference to the experiment.[3]

It seems to us that in many low-energy nuclear physics calculations, the number of approximations and phenomenology could be reduced if a realistic and simple analytic form for the π-N potential were available. In this paper we present a potential for the pion-nucleon interaction which is real (elastic), nonlocal, and separable.

2. DEFINITION OF THE POTENTIAL: ON-SHELL BEHAVIOR

A nucleon may be regarded as a cloud of pions, so that the problem of the scattering of a pion on a nucleon is a many-body problem; then the true π-N interaction is very probably of a nonlocal type. In most cases it is possible with a nonlocal force to go through the whole analytic calculation without any approximation. Moreover, as the nucleon-nucleon nonlocal potentials have been so helpful,[5] it seems natural to use them for the π-N interaction.

In the following, we assume that:
(i) Nonrelativistic quantum mechanics is suitable to describe the π-N interaction. This is probably true if the energy of the incident pion does not exceed the energy of the first resonance ($E_\pi = 180$ MeV in the laboratory system). When needed, it is always possible to include relativistic corrections, but pion-nuclei calculations suggest that they are of little importance.[3]
(ii) The π-N interaction may be described by a potential in the studied energy range.
(iii) π-N scattering is charge-independent[1] (isospin conservation).

For a two-body potential satisfying all the usual invariance and symmetry laws, the most general form may be written in relative coordinates:

$$W(\vec{r},\vec{r}\,') = \langle \vec{r}\,|W|\vec{r}\,'\rangle = \frac{\hbar^2}{2\mu}\sum_i \sum_{ljmt} B_{iljt}\langle r|v_{iljt}\rangle\langle v_{iljt}|r'\rangle \times \langle \hat{r}|l\tfrac{1}{2}jm\rangle\langle l\tfrac{1}{2}jm|\hat{r}'\rangle P_t . \quad (1)$$

P_t is the projection operator onto the isotopic state defined by

$$\vec{t} = \vec{t}_\pi + \vec{t}_N = \vec{I} + \frac{\vec{1}}{2}, \quad (t = \tfrac{1}{2}, \tfrac{3}{2}).$$

$s = \frac{1}{2}$ is the unique value for the spin of the π-N system ($s_\pi = 0$, $s_N = \frac{1}{2}$). The orbital angular momentum $\vec{l}$ and the spin $\vec{s}$ are coupled to give the total kinetic momentum $\vec{j} = \vec{l} + \vec{s}$:

$$|l\tfrac{1}{2};jm\rangle = \sum_{m_s(m_l)} |lm_l\rangle|\tfrac{1}{2}m_s\rangle\langle lm_l\tfrac{1}{2}m_s|jm\rangle .$$

The sum of the separable potentials ($i = 1, 2$) was chosen to allow for the possibility in certain waves of simultaneously having a long-range attraction ($B_1 < 0$) and a short-range repulsion ($B_2 > 0$). μ is the reduced mass of the system $\mu = m_\pi m_N/(m_\pi + m_N)$. The radial functions $v_{iljt}(r)$ are defined in the configuration space by

$$\langle r|v_{iljt}\rangle = r^{l-1}e^{-r\alpha_{iljt}}, \quad (2a)$$

or, alternatively, $v_{iljt}(k)$ in the impulse space are

$$\langle k|v_{iljt}\rangle = i^l\left(\frac{2}{\pi}\right)^{1/2}\int_0^\infty v_{iljt}(r)j_l(kr)r^2dr = i^l\left(\frac{2}{\pi}\right)^{1/2}\frac{l!(2k)^l}{(k^2+\alpha_{iljt}^2)} = i^l\left(\frac{2}{\pi}\right)^{1/2}p_{iljt}(k) . \quad (2b)$$

The α_{iljt} parameters (in fm^{-1}) are the inverse ranges of the potential, and the B_{iljt} parameters [in $\mathrm{fm}^{-(2l+3)}$] measure the depths of the potential in the ljt state. These parameters will be determined here to fit the experimental phase shifts.[6] This insures the correct behavior of the scattering amplitudes on the energy shell. Let us consider a scattering-wave solution of the Schrödinger equation:

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi(\vec{r}) + \int W(\vec{r},\vec{r}\,')\psi(\vec{r}\,')d^3r' = E\psi(\vec{r}) . \quad (3)$$

With the correct asymptotic behavior

$$\psi(\vec{r}) \xrightarrow[r\to\infty]{} e^{i\vec{k}\cdot\vec{r}} + t(\vec{k},\vec{k}\,')\frac{e^{ikr}}{r} .$$

Then, the scattering amplitude satisfies the Lippmann-Schwinger equation

$$t(\vec{k},\vec{k}\,') = W(\vec{k},\vec{k}\,') + \frac{2\mu}{\hbar^2}\int \frac{W(\vec{k},\vec{q})d^3q\, t(\vec{q},\vec{k})}{k_0^2 - q^2 + i\epsilon}, \quad (4)$$

where $k_0^2 = 2\mu E/\hbar^2$, but $\vec{k}$ and $\vec{q}$ may be arbitrary. In this case $t(\vec{q},\vec{k})$ is the scattering amplitude off the energy shell describing the inelastic scattering process. It is clear that we cannot deduce these scattering amplitudes from the experimental phase shifts, because the energy is always conserved in a collision of two free particles:

$$k^2 = k'^2 = 2\mu E/\hbar^2 ; \quad (5)$$

however, our realistic potential will have the correct behavior on the energy shell. Let us remember that the discontinuity on the cut $E_{\text{real}} \geq 0$ is given by the unitarity relation off the energy shell.[1,5]

3. PARTIAL-WAVE ANALYSIS - SOLUTION OF THE RADIAL SCHRÖDINGER EQUATION

A. Mathematical Solution

$E = \hbar^2k^2/2\mu$ stands for the energy of the relative particle [$E = \mu E_\pi(\text{lab})/m_\pi$]; then

$$(\nabla^2 + k^2)\psi(\vec{r}) = \frac{2\mu}{\hbar^2}\int W(\vec{r},\vec{r}\,')\psi(\vec{r}\,')d^3r' . \quad (6)$$

The partial-wave expansion of the wave function

$$\psi_{m,m_t}(\vec{r}) = \sum_{ljt}\frac{\mu_{ljt}(r)}{r}\langle\hat{r}|l\tfrac{1}{2}jm\rangle\chi_t^{m_t}, \quad (7)$$

where $\chi_t^{m_t}$ is the isospin function, and $s = \frac{1}{2}$, leads to the equation:

$$\sum_{ljt}\frac{1}{r}\left[\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + k^2\right]u_{ljt}(r)\langle\hat{r}|l\tfrac{1}{2}jm\rangle\chi_t^{m_t} = \sum_i\sum_{ljt}B_{iljt}\langle r|v_{iljt}\rangle\langle\hat{r}|l\tfrac{1}{2}jm\rangle\chi_t^{m_t}\lambda_{il} .$$

Finally, the radial equation for each ljt state is written (from now on jt subscripts will be omitted):

$$\left[\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + k^2\right]u_l(r) = \sum_i B_{il}rv_{il}(r)\int_0^\infty v_{il}(r')u_l(r')r'\,dr' = \sum_i B_{il}rv_{il}(r)\lambda_{il} = F_l(r) . \quad (8)$$

Equation (8) defines the integral λ_{il} and the function $F_l(r)$. The spherical Bessel and Neumann functions[7] are linearly independent solutions of the homogeneous equation [$F_l(r) = 0$] associated with Eq. (8). We choose the norm by setting the Wronskian equal to unity:

$$y_l(r) = \sqrt{k}\,rj_l(kr) \xrightarrow[r\to\infty]{} \frac{1}{\sqrt{k}}\sin\left(kr - l\frac{\pi}{2}\right),$$

$$z_l(r) = \sqrt{k}\,rn_l(kr) \xrightarrow[r\to\infty]{} \frac{1}{\sqrt{k}}\cos\left(kr - l\frac{\pi}{2}\right).$$

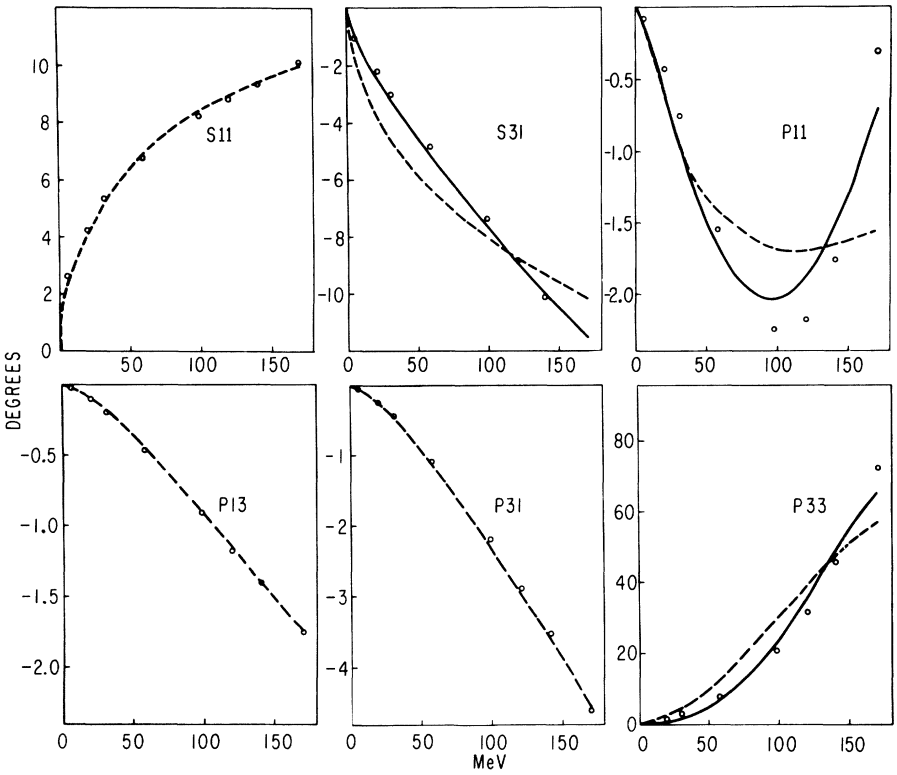

FIG. 1. Phase shifts versus energy of the pion in the laboratory. Circles are experimental values of the phase shifts given in Table IV of Ref. 3. The dashed lines are best fits with one-term potentials. For some waves a second term in the potential has been introduced to obtain a better fit (solid lines). The scattering lengths and volumes are listed in Table II, and the parameters of the potential in Table I.

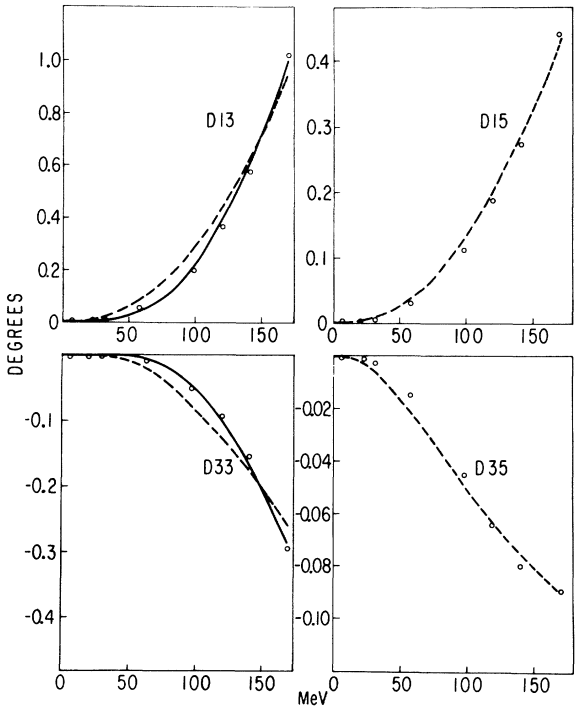

FIG. 2. See Fig. 1 caption.

The general solution of Eq. (8), which is regular at the origin $[u_l(0)=0]$ is

$$u_l(r) = x_l(k) r j_l(kr) + \int_0^\infty G_l(r, r') F_l(r') dr' , \qquad (9)$$

where $x_l(k)$ is an integration constant, and the Green's function $G_l(r, r')$ can be written in terms of the Heaviside function $\theta(r-r')$ [$\theta(x)=1$ if $x>0$, and $\theta(x)=0$ if $x<0$]:

$$G_l(r, r') = krr' n_l(kr) j_l(kr') \theta(r-r') + krr' j_l(kr) n_l(kr') \theta(r'-r) . \qquad (10)$$

The phase shift can be found from the asymptotic behavior of the solution given in Eqs. (9) and (10):

$$u_l(kr) \underset{r\to\infty}{\longrightarrow} x_l(k) r j_l(kr) + kr\, n_l(kr) \sum_i B_{il} \lambda_{il} p_{il}(k) , \qquad (11)$$

where $p_{il}(k)$ is defined in Eq. (2b). Hence, the phase shift is simply given by

$$\tan\delta_l(k) = -k \sum_i B_{il} \lambda_{il} p_{il}(k) . \qquad (12)$$

This result is independent of $x_l(k)$.

B. Numerical Solution for the Integral Equation

It is easy to see[5] that the solution of Eqs. (9) and (10)[5] can be written:

$$u_l(kr) = x_l(k) r j_l(kr) + kr \sum_i B_{il} \lambda_{il} H_{il}(r, k), \qquad (13)$$

with

$$H_{il}(r, k) = j_l(kr) J_{il}(r, k) + n_l(kr) N_{il}(r, k),$$

$$J_{il}(r, k) = \int_r^\infty n_l(kr') v_{il}(r') r'^2 dr',$$

$$N_{il}(r, k) = \int_0^r j_l(kr') v_{il}(r') r'^2 dr'.$$

We remark

$$N_{il}(\infty, k) = p_{il}(k).$$

Taking Eq. (13) into account, the λ_{il} integral becomes

$$\lambda_i = x_l(k) p_{il}(k) + \sum_j q_{ijl}(k) B_{jl} \lambda_{jl}, \qquad (14)$$

where

$$q_{ijl}(k) = k \int_0^\infty H_{jl}(r, k) v_{il}(r) r^2 dr$$

$$= q_{jil}(k). \qquad (15)$$

The linear inhomogeneous system [Eq. (14)] is easily solved (see Appendix), and one deduces the phase shift [Eq. (12)] and the scattering lengths

TABLE I. Values of the parameters. The ranges α_{il} are in fm^{-1}, the intensities B_{il} in fm$^{-(2l+3)}$ $(i=1, 2)$.

Wave	α_1	B_1	α_1	B_2
$S11$	...	...	3.5	−24.54
$S31$	...	...	3.5	45.99
	1.5	−2.37	3.5	127
$P11$	1.0	0.106	...	...
	1.5	3.19	3.5	−702
$P13$	...	...	3.5	307.9
$P31$	...	...	3.5	1088
$P33$	...	...	3.5	−1528
	1.5	3.79	3.5	−1755
$D13$	...	...	3.5	−1329
	1.5	0.097	3.5	−1725
$D15$	...	...	3.5	−667
$D33$	...	...	3.5	483
	1.5	−0.052	3.5	844
$D35$	1.5	0.0303	...	...

$a_{l=0}$ and volumes $a_{l=1}$ using, respectively,

$$k \cot\delta_0 \underset{k\to 0}{\longrightarrow} \frac{1}{a_0} \quad \text{and} \quad k^3 \cot\delta_1 \underset{k\to 0}{\longrightarrow} \frac{1}{a_1}. \qquad (16)$$

4. RESULTS AND DISCUSSION

A least-mean-squares calculation gives a perfect fit between the theoretical and experimental phase shifts reported by Roper and Wright[6] with only a one-term potential, for the waves $S11$, $P13$, $D15$, $P31$, and $D35$.[8] The inverse potential ranges α_{il} in this first group (except for the $D35$ wave) lie between 3. and 4. fm^{-1} for the best fit. We choose the unique value 3.5 fm^{-1} for the former waves and 1.5 fm^{-1} for the latter ($D35$), which also gives phase shifts in very close agreement with the experimental data. The $S11$ and $P13$ waves have reasonably weak B_{il} depths (−24.54 fm^{-3} and +307.9 fm^{-5}, respectively). On the other hand, the $P31$ wave corresponds to a strong repulsive depth (1088 fm^{-5}). The $D15$ and $D35$ waves are of little importance (see Table I).

For the other waves, it was necessary to include a second term in the potential ($i=1, 2$) in order to obtain close agreement with experimental results (see Figs. 1 and 2). In the case of the most important waves ($S31$ and $P33$) of this second group and for the $D13$ and $D33$ waves, this agreement is perfect. For the physically least important $P11$ wave the discrepancies between the theoretical and experimental phase shifts are everywhere lower than the experimental errors. We found that the χ^2 method gives only one solution α_{il}, B_{il} $(i=1, 2)$ for each of the second group of waves. Surprisingly, the best fits for these waves are characterized by $\alpha_1 \simeq 1.5$ fm^{-1} and also $\alpha_2 \simeq 3.5$ fm^{-1}. In agreement with experiment, the B_{il} depths mainly correspond to a strong repulsion in $S31$ states and a strong attraction in $P33$ and $D13$ states (see Table I). The scattering lengths $a_{l=0}$ and scattering volumes $a_{l=1}$ defined in Eq. (16) have been cal-

TABLE II. Scattering lengths for S waves, in $\lambda\!\!\!^{-}$ $(=\hbar/m_\pi c)$, and scattering volume for P waves in $\lambda\!\!\!^{-3}$.

Wave	Exp [a]	Exp [b]	Theory
$S11$	0.171 ± 0.005	from 0.157 to 0.158	0.162
$S31$	−0.088 ± 0.004	from −0.095 to −0.097	−0.176
$P11$	−0.101 ± 0.007	from −0.038 to −0.042	−0.124
$P13$	−0.029 ± 0.005	from −0.028 to −0.030	−0.017
$P31$	−0.038 ± 0.005	−0.034	−0.047
$P33$	0.215 ± 0.005	...	0.057

[a] See Hamilton and Woolcock (Ref. 1).
[b] See Table X (solutions A, B, C, D) of Ref. 6.

culated by using the parameters of the potential fitting the experimental phase shifts. The agreement between experimental and calculated values is good, except for the only wave presenting a resonance at low energy, $P33$ (see Table II).

In conclusion, the hypothesis that the low-energy elastic π-N scattering may be described nonrelativistically by a potential enables us to determine a realistic π-N potential which is valid for all experimental scattering data up to $E_\pi = 180$ MeV (lab). This potential may appear very useful in many applications in the nuclear physics field. The practical uses of this potential are considerably simplified by the two facts that it is nonlocal and separable and that the ranges of the potential are the same ($\alpha_1 = 1.5$ and $\alpha_2 = 3.5$ fm^{-1}) in all S, P, and D scattering states.

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APPENDIX

(1) In the case of a two-term potential ($i = 1, 2$), the expression for the phase-shift [Eq. (12)] is given by

$$k\cot\delta = \frac{-B_1B_2p_1^2q_{22} - B_1B_2p_2^2q_{11} + 2B_1B_2p_1p_2q_{12} + B_1p_1^2 + B_2p_2^2}{B_1B_2(q_{11}q_{22} - q_{12}^2) - B_1q_{11} - B_2q_{22} + 1},$$

where all ljt subscripts and the dependence on k are implicit.

(2) With the form factor defined in Eq. (2a), the p_{il} and q_{ijl} functions for the S, P, and D waves are successively:

$$p_{i,l=0} = \frac{1}{\alpha_i^2 + k^2}, \qquad q_{ij,l=0} = \frac{k^2 - \alpha_i\alpha_j}{(\alpha_i + \alpha_j)(\alpha_i^2 + k^2)(\alpha_j^2 + k^2)};$$

$$p_{i,l=1} = \frac{2k}{(\alpha_i^2 + k^2)^2}, \qquad q_{ij,l=1} = \frac{2}{(\alpha_j^2 + k^2)}\left[\frac{3\alpha_j^2 + \alpha_i^2 + 3\alpha_i\alpha_j + k^2}{(\alpha_i + \alpha_j)^3} - \frac{\alpha_i^3 + 3\alpha_ik^2}{(\alpha_i^2 + k^2)^2}\right];$$

$$p_{i,l=2} = \frac{8k^2}{(\alpha_i^2 + k^2)^3}, \qquad q_{ij,l=2} = \frac{-8(3\alpha_i^5 + 10\alpha_i^3k^2 + 15\alpha_ik^4)}{(\alpha_j^2 + k^2)^3(\alpha_i^2 + k^2)^3} + \frac{24}{(\alpha_j^2 + k^2)(\alpha_i + \alpha_j)^5}$$

$$+ \frac{24\alpha_j}{(\alpha_j^2 + k^2)^2(\alpha_i + \alpha_j)^4} + 8\frac{k^2 + 3(\alpha_i^2 + 3\alpha_i\alpha_j + 3\alpha_j^2)}{(\alpha_j^2 + k^2)^3(\alpha_i + \alpha_j)^3}.$$

(3) The derivation of scattering lengths and volumes [Eq. (16)] is straightforward from the above expression of $k\cot\delta$.

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[8]We use the usual notation for the pion-nucleon state l, $2t$, $2j$.