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Uncertainty in the Neutron-Strength-Function Evaluation for a Small Number of Measured Resonances*

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When neutron reduced widths, Γ_n^0, are evaluated for N adjacent "single-population" resonances for an isotope, it is customary to express the fractional uncertainty in the s strength function, S_0, as $\pm(2.27/N)^{1/2}$ or $\pm(2/N)^{1/2}$. It is assumed that the Γ_n^0 follow a Porter-Thomas (P.T.) single-channel distribution with a common $\langle\Gamma_n^0\rangle$ for the interval, with no correlation between the different Γ_n^0. If the spacing distribution follows the Wigner formula for nearest neighbor spacings, but with no correlations, the $(2.27/N)^{1/2}$ fractional uncertainty applies for large N. For spacings following a statistical "orthogonal ensemble" (O.E.) behavior, the fractional uncertainty in $\langle D\rangle$ is $\approx N^{-1}$, so the fractional uncertainty in S_0 is $\approx(2/N)^{1/2}$ for large N. Experimentalists need easy to use rules for smaller N. We have used Monte Carlo methods with a P.T. form for Γ_n^0, and O.E. for spacings to establish the upper- and lower-bound values for S_0, divided by $\bar{S}_0 \equiv \bar{\Gamma}_n^0/\bar{D}$ (the ratio of the measured averages of Γ_n^0 and D). The method of confidence intervals was used. We also suggest a "best choice" ratio for true S_0 to measured $\bar{S}_0$, all for a range of small N. The results should also apply for "single populations" of p levels. The behaviors for Γ_n^0 and D were also studied separately.

I. INTRODUCTION

When a single population of s neutron resonances is studied over a region containing N resonances, one measures the reduced neutron width Γ_{nj}^0 for each resonance and has $N-1$ level spacings D_j between each nearest neighbor. We assume that single true mean $\langle\Gamma_n^0\rangle$ and true mean level spacing $\langle D\rangle$ apply for the population, that all s levels in the interval are detected, and that there is no contamination by p or spurious "levels." It is usually assumed that the individual Γ_{nj}^0 are distributed according to a single-channel Porter-Thomas (P.T.) formula

$$P_1(y, u)dy = (2\pi uy)^{-1/2}e^{-y/2u}dy\,, \qquad (1a)$$

where $y=\Gamma_n^0$ and u is the true mean $\langle\Gamma_n^0\rangle$. It is assumed that there are no correlations between the different y_j, although recent results[1] for ^{166}Er suggest that this may not be true. Equation (1a) has $\langle y\rangle = u$ and $\operatorname{var}y = 2u^2$.

If $y=\bar{\Gamma}_n^0$ is the average Γ_n^0 for a particular set of N levels, then the distribution for y is

$$P_N(y, u)dy = \frac{y^{N/2-1}(N/2)^{N/2}e^{-Ny/2u}dy}{u^{N/2}\Gamma(N/2)}\,, \qquad (1b)$$

where u is still $\langle\Gamma_n^0\rangle$. Equation (1b) has a χ^2 distribution form with $\langle y\rangle = u$ and $\operatorname{var}y = 2u^2/N$ [Standard deviation for $y=(2/N)^{1/2}u$]. Most authors assume that the distribution of nearest-neighbor level spacings about $\langle D\rangle$ follows a Wigner distribution

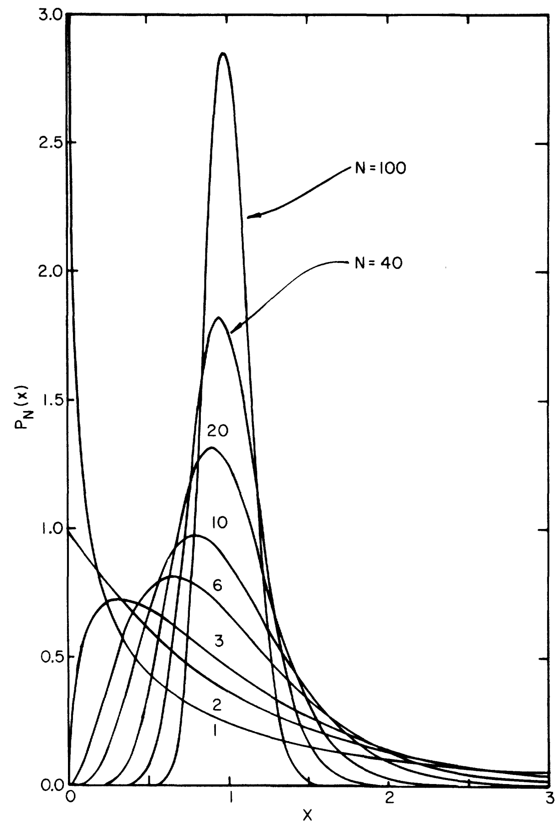

FIG. 1. The probability distribution $P_N(x)$ for the ratio of the sample average to true average for the reduced neutron width, Γ_n^0, according to P. T. single-channel theory for N adjacent single-population s levels.

with no correlation between them,

$$P(x)dx = \tfrac{1}{2}\pi x\, e^{-\pi x^2/4} dx\,, \tag{2}$$

where x is the nearest-neighbor level spacing in units of $\langle D\rangle$. This has $\langle x\rangle = 1$ and $\mathrm{var}\,x = (4-\pi)/\pi$. For $N-1$ spacings, the sample average $\bar{D}$ has $\langle \bar{D}\rangle = \langle D\rangle$ and $\mathrm{var}\,\bar{D} = (4-\pi)/\pi(N-1)$.

When N is very large, and one need not distinguish between N and $N-1$, $\bar{\Gamma}_n^0$ cannot differ much fractionally from $\langle \Gamma_n^0\rangle$, $\bar{D}$ cannot differ much fractionally from $\langle D\rangle$. In this case, a judgement of the relative likelihood that the true $\langle \Gamma_n^0\rangle$ has any given value in the neighborhood of $\bar{\Gamma}_n^0$ must have essentially the same "best choice" and "standard deviation" as has $\bar{\Gamma}_n^0$ about $\langle \Gamma_n^0\rangle$, with similar statements for $\langle D\rangle$ about measured $\bar{D}$. This leads to the usual expression

$$\frac{\langle \Gamma_n^0\rangle}{\langle D\rangle} \equiv S_0 = \frac{\bar{\Gamma}_n^0}{\bar{D}}\left[1 \pm (2.27/N)^{1/2}\right] \tag{3a}$$

after folding the fractional uncertainties of $\langle \Gamma_n^0\rangle$ and $\langle D\rangle$ in quadrature. For relatively small N, this can lead to a very poor treatment of the situation. It is our intent to discuss this small N region in this paper.

In addition, the previous analysis is wrong in the treatment of $\bar{D}$ relative to $\langle D\rangle$ in that there both long- and short-range order occurs in level spacings, which are described by the statistical-orthogonal-ensemble (O.E.) theory of Wigner, Dyson and Mehta, and others, and shown[1,2] to hold experimentally for our recent data for ^{166}Er and some other even-even nuclei. Reference 1 discusses this subject from the experimental and theoretical viewpoints, with references to the literature of

TABLE I. Values of $x_A(N)$ for the $\bar{\Gamma}_n^0/\langle \Gamma_n^0\rangle$ distribution, where A is the fractional probability of obtaining $\bar{\Gamma}_n^0/\langle \Gamma_n^0\rangle \geq x_A(N)$.

$N \backslash A$	0.5	0.841	0.159	0.9	0.1	0.95	0.05	0.99	0.01	0.995	0.005
1	0.4549	0.0401	1.987	0.0158	2.706	0.0039	3.841	0.000 16	6.635	0.000 04	7.879
2	0.6931	0.1728	1.841	0.1054	2.303	0.0513	2.996	0.010 05	4.605	0.005 01	5.298
3	0.7887	0.2780	1.729	0.1948	2.084	0.1173	2.605	0.038 28	3.782	0.023 91	4.279
4	0.8392	0.3541	1.650	0.2659	1.945	0.1777	2.372	0.074 28	3.319	0.051 75	3.715
5	0.8703	0.4113	1.591	0.3221	1.847	0.2291	2.214	0.1109	3.017	0.082 35	3.350
6	0.8914	0.4558	1.546	0.3674	1.774	0.2726	2.099	0.1453	2.802	0.1126	3.091
7	0.9065	0.4917	1.509	0.4047	1.717	0.3096	2.010	0.1770	2.639	0.1413	2.897
8	0.9180	0.5215	1.479	0.4362	1.670	0.3416	1.938	0.2058	2.511	0.1681	2.744
9	0.9270	0.5466	1.454	0.4631	1.632	0.3695	1.880	0.2320	2.407	0.1928	2.621
10	0.9342	0.5681	1.432	0.4865	1.599	0.3940	1.831	0.2558	2.321	0.2156	2.519
15	0.9559	0.6432	1.357	0.5698	1.487	0.4841	1.666	0.3486	2.039	0.3067	2.187
20	0.9669	0.6892	1.311	0.6221	1.421	0.5425	1.571	0.4130	1.878	0.3717	2.000
30	0.9779	0.7447	1.255	0.6866	1.342	0.6164	1.459	0.4984	1.696	0.4596	1.789
40	0.9834	0.7783	1.222	0.7263	1.295	0.6627	1.394	0.5541	1.592	0.5177	1.669
50	0.9867	0.8014	1.199	0.7538	1.263	0.6953	1.350	0.5941	1.523	0.5598	1.590

TABLE II. Ratios of $x_{1/2}(N)$ to $x_A(N)$ for $\bar{\Gamma}_n^0/\langle\Gamma_n^0\rangle$ distribution. The last column lists values of $C_N = 1/x_{1/2}(N)$ vs N. As discussed in the text, we suggest that the best choice of $\langle\Gamma_n^0\rangle$ is C_N times the experimental value of $\bar{\Gamma}_n^0$.

$N \backslash A$	0.841	0.159	0.9	0.1	0.95	0.05	0.99	0.01	C_N
1	11.35	0.2290	28.81	0.1682	115.7	0.1184	2896.0	0.0686	2.198
2	4.011	0.3766	6.579	0.3010	13.51	0.2314	68.97	0.1505	1.443
3	2.837	0.4563	4.049	0.3785	6.724	0.3028	20.60	0.2085	1.268
4	2.370	0.5087	3.156	0.4315	4.723	0.3538	11.30	0.2528	1.192
5	2.116	0.5470	2.702	0.4711	3.799	0.3931	7.850	0.2884	1.149
6	1.955	0.5766	2.426	0.5024	3.270	0.4247	6.133	0.3181	1.122
7	1.844	0.6006	2.240	0.5281	2.928	0.4511	5.122	0.3435	1.103
8	1.760	0.6205	2.105	0.5496	2.688	0.4736	4.460	0.3656	1.089
9	1.696	0.6375	2.002	0.5682	2.509	0.4931	3.996	0.3851	1.079
10	1.644	0.6522	1.920	0.5843	2.371	0.5103	3.652	0.4025	1.070
15	1.486	0.7044	1.678	0.6428	1.975	0.5737	2.742	0.4689	1.046
20	1.403	0.7375	1.554	0.6806	1.782	0.6156	2.341	0.5148	1.034
30	1.313	0.7790	1.424	0.7287	1.586	0.6702	1.962	0.5764	1.023
40	1.263	0.8049	1.354	0.7593	1.484	0.7055	1.775	0.6176	1.017
50	1.231	0.8232	1.309	0.7810	1.419	0.7308	1.661	0.6478	1.013

the field. For $N-1$ spacings, Dyson and Mehta[3] have shown that the fractional standard deviation of $\bar{D}$ about true $\langle D\rangle$ is $\approx 1/(N-1)$ for large N. For large N the uncertainty of S_0 is almost entirely due to that of $\langle\Gamma_n^0\rangle$, giving

$$\frac{\langle\Gamma_n^0\rangle}{\langle D\rangle} = S_0 \approx \frac{\bar{\Gamma}_n^0}{\bar{D}}[1 \pm (2/N)^{1/2}] \tag{3b}$$

for large N. This formula is still quite misleading for small N.

The two papers with which we are acquainted which treat this subject in detail are those of Muradyan and Adamchuk[4] and Slavinskas and Kennett.[5] Both of these papers use uncorrelated Wigner (U.W.) level spacings rather than O.E. theory. In Ref. 4 an improper implicit use is made of Bayes's theorem, which is not applicable, to evaluate certain expressions. The treatment in Ref. 5, aside from not using O.E. theory, seems to be basically correct, but did not present results in an easily usable form.

In this paper, we make use of our Monte Carlo spacing sets[1] which are in accord with O.E. generated by use of Dyson's Brownian motion model. This allows us to treat the distribution of $\bar{D}$ values (N levels) about $\langle D\rangle$ and make statements about $\langle D\rangle$ in terms of $\bar{D}$. The similar treatment for $\bar{\Gamma}_n^0$ about $\langle\Gamma_n^0\rangle$ was made partly from χ^2 tables and partly by computer analytically. We also make statements about how one should select $\langle\Gamma_n^0\rangle$, given $\bar{\Gamma}_n^0$. Finally, large numbers of computer-generated Monte Carlo sets were generated for

TABLE III. Values of ρ_A^+ and ρ_A^- for $\bar{\Gamma}_n^0/\langle\Gamma_n^0\rangle$ distribution, which are $(N/2)^{1/2}$ times the + or − fractional uncertainty of the chosen $\langle\Gamma_n^0\rangle$ for various confidence intervals.

N	$\rho_{0.841}^+$	$\rho_{0.159}^-$	$\rho_{0.9}^+$	$\rho_{0.1}^-$	$\rho_{0.95}^+$	$\rho_{0.05}^-$	$\rho_{0.99}^+$	$\rho_{0.01}^-$
1	7.317	0.5452	19.66	0.5882	81.10	0.6234	2047.	0.659
2	3.011	0.6234	5.579	0.6990	12.51	0.7686	67.97	0.849
3	2.249	0.6659	3.734	0.7612	7.011	0.8539	24.01	0.969
4	1.937	0.6948	3.049	0.8040	5.265	0.9139	14.56	1.057
5	1.765	0.7163	2.691	0.8362	4.425	0.9596	10.83	1.125
6	1.655	0.7333	2.471	0.8618	3.932	0.9964	8.890	1.181
7	1.578	0.7472	2.320	0.8829	3.607	1.027	7.711	1.228
8	1.521	0.7590	2.209	0.9007	3.375	1.053	6.921	1.269
9	1.476	0.7690	2.125	0.9161	3.201	1.075	6.355	1.304
10	1.441	0.7778	2.057	0.9295	3.065	1.095	5.929	1.336
15	1.332	0.8095	1.856	0.9783	2.670	1.168	4.771	1.454
20	1.274	0.8300	1.752	1.010	2.473	1.215	4.241	1.534
30	1.212	0.8560	1.643	1.051	2.271	1.277	3.725	1.640
40	1.178	0.8724	1.583	1.076	2.164	1.317	3.465	1.710
50	1.156	0.8841	1.545	1.095	2.096	1.346	3.304	1.761

the $\bar{\Gamma}_n^0/\bar{D}$ distribution about $\langle\Gamma_n^0\rangle/\langle D\rangle$ for a range of N values, using O.E. theory for the spacings.

II. BEST CHOICE OF $\langle\Gamma_n^0\rangle$ AND ITS UNCERTAINTIES

The problem can be stated generally as follows. We have x = sample average for a measured physical quantity which has some (unknown) true mean u, where the form of the probability distribution for x about u, $P(x, u)dx$, is known. If a given sample average x is obtained, what can we say about true mean u (best choice and uncertainty)? This is a problem in mathematical statistics rather than a forward probability calculation. While there is a tendency to apply maximum-likelihood theory, we believe that the approach of confidence intervals is more appropriate.

If $P(x, u)dx$ has a Gaussian distribution with standard deviation (S.D.) σ, the probability is 0.1587 of obtaining $x \leqslant u-\sigma$ or $x \geqslant u+\sigma$. This can be expressed in terms of the x_A values where the fractional probability of obtaining $x \geqslant x_A$ is A for a random sample. For the Gaussian case, we have $x_{0.1587}$ and $x_{0.8413}$. There is 0.6826 probability that x is between $u-\sigma$ and $u+\sigma$. The mean of x is also the median x. For a wider confidence interval, we might choose the pairs $A = 0.10$, 0.90, or 0.01, 0.99, etc.

For a non-Gaussian-distributed variable, we can similarly locate $x_{0.1587}$ and $x_{0.8413}$ and either $\langle x\rangle$ or the median value $x_{1/2}$. It is convenient to redefine x in units of u, so $\langle x\rangle = u = 1$ for the distribution unless the product or quotient of separate distributions is involved, as is the case for the strength function.

Figure 1 shows the distribution of $\bar{\Gamma}_n^0/\langle\Gamma_n^0\rangle$ for various choices of N. For very large N, there is a near-Gaussian shape centered about $x = 1$. For $N = 1, 2, 3$, etc., the peak is either at $x = 0$ or at $x < 1$, and the median x, $x_{1/2}$, is definitely at $x < 1$. Table I lists various x_A values vs N for this case. The experimentalists' problem is as follows. For a finite experimental sample of N levels for which $\bar{\Gamma}_n^0$ is obtained, how does one choose a "best choice" for $\langle\Gamma_n^0\rangle$, and + and − uncertainties in the "selected" $\langle\Gamma_n^0\rangle$? Using the confidence-interval approach, the upper and lower $\langle\Gamma_n^0\rangle$ choices are made by identifying x as measured $\bar{\Gamma}_n^0$ divided by unknown $\langle\Gamma_n^0\rangle$ and setting x equal to the lower and upper limits x_A values, respectively. For $N = 2$, we find $x_{0.8413} = 0.1728$ and $x_{0.1587} = 1.841$. The upper- and lower-bound $\langle\Gamma_n^0\rangle$ values for this confidence interval are $\bar{\Gamma}_n^0/0.1728 = 5.787\bar{\Gamma}_n^0$ and $\bar{\Gamma}_n^0/1.841 = 0.543\bar{\Gamma}_n^0$. We immediately notice how asymmetrical these limits are with respect to $\bar{\Gamma}_n^0$. This raises the question of a "best choice" for $\langle\Gamma_n^0\rangle$ if the upper and lower bounds for this confidence interval are so asymmetric. The choice of Ref. 5 was to use the maximum-likelihood method which corresponds here to choosing $\langle\Gamma_n^0\rangle = \bar{\Gamma}_n^0/\langle x\rangle = \bar{\Gamma}_n^0$. However, we note that for $N = 2$ there is 63.2% probability for $x < 1$ and only 36.8% probability for $x \geqslant 1$. For this reason, we favor the approach of

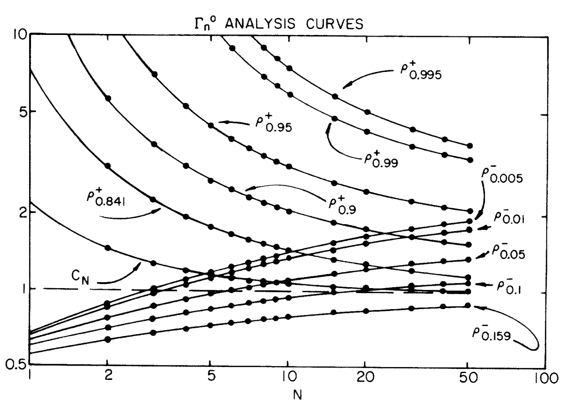

FIG. 2. Plot of $C_N \equiv 1/x_{1/2}(N)$ and ρ_A^+ and ρ_A^- vs N for the Γ_n^0 distribution. We suggest using $\langle\Gamma_n^0\rangle = C_N\bar{\Gamma}_n^0$ with unequal ± fractional uncertainties ρ_A^+ or ρ_A^- times $(2/N)^{1/2}$, respectively.

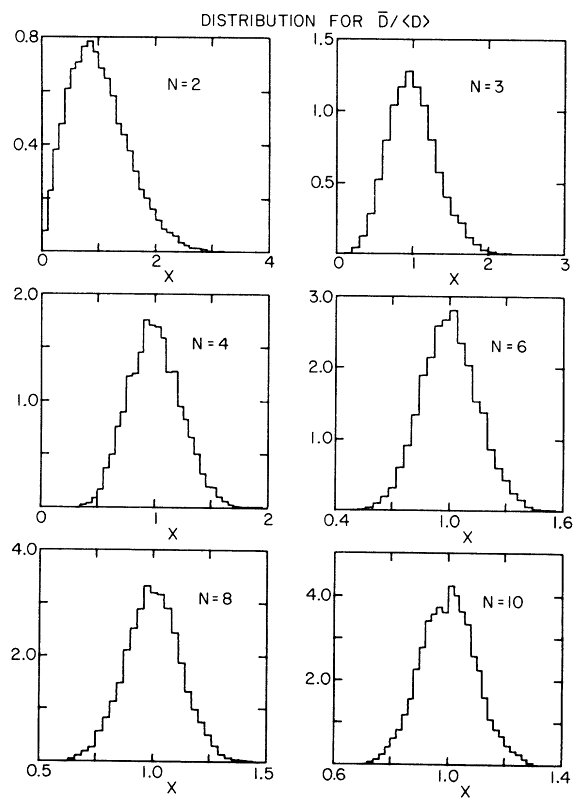

FIG. 3. Monte Carlo histograms for expected sample $\bar{D}$ to true $\langle D\rangle$ for the O.E. distribution for level spacings.

TABLE IV. Results for C_N and ρ_A^+ and ρ_A^- from Monte Carlo studies of the distribution of the sample mean level spacing $\bar{D}$ to the true mean $\langle D\rangle$ for the orthogonal-ensemble theory. $C_N \equiv 1/x_{1/2}(N)$ and we suggest that the best choice $\langle D\rangle = C_N\bar{D}$, with asymmetrical ± fractional uncertainties ρ_A^+ and ρ_A^- times $1/N$. The values are from the smooth curves of Fig. 4 which average out statistical fluctuations in the computed values.

N	C_N	$\rho_{0.841}^+$	$\rho_{0.159}^-$	$\rho_{0.9}^+$	$\rho_{0.1}^-$
2	1.07	1.99	0.770	3.04	0.909
3	1.02	1.32	0.774	1.88	0.929
4	1.01	1.16	0.790	1.62	0.965
5	1.00	1.11	0.809	1.52	0.997
6	1.00	1.08	0.827	1.47	1.02
7	1.00	1.07	0.843	1.44	1.04
8	1.00	1.06	0.856	1.41	1.06
9	1.00	1.05	0.868	1.39	1.07
10	1.00	1.05	0.878	1.38	1.08

setting the "best choice"

$$\langle\Gamma_n^0\rangle = \bar{\Gamma}_n^0/x_{1/2} = C_N\bar{\Gamma}_n^0 \text{ ("best choice")}, \tag{4a}$$

which has better symmetry.

The value of $x_{1/2}$ is very closely fitted by the empirical formula

$$x_{1/2} \approx 1 - \frac{2}{3N} + \frac{0.087}{N^2} + \frac{0.034}{N^3}. \tag{4b}$$

Values of $x_{1/2}$ are listed in Table I, and values of $C_N = 1/x_{1/2}$ and $x_{1/2}/x_A$ for various choices of A are listed in Table II. The various fractional + or − uncertainties Δ_A^+ or Δ_A^- about $C_N\bar{\Gamma}_N^0$ were also computed for matching confidence-interval pairs of A values. For $N=2$ we would write

$$\langle\Gamma_n^0\rangle = 1.443\bar{\Gamma}_n^0(1^{+3.011}_{-0.6234})$$

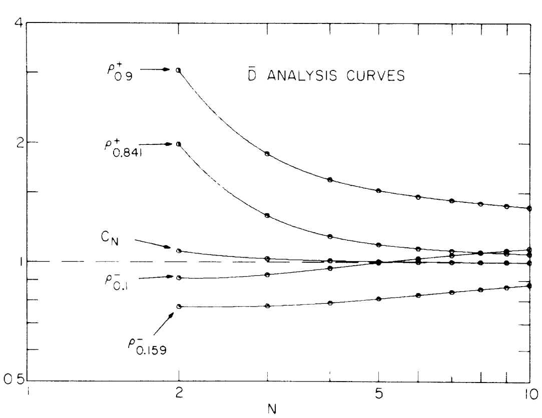

FIG. 4. Monte Carlo results for C_N and ρ_A^+ and ρ_A^- for the mean level spacing (N levels) according to O.E. theory. We suggest using $\langle D\rangle = C_N\bar{D}$, with unequal ± fractional uncertainties ρ_A^+ and ρ_A^- times $1/N$. The smooth curves and points are after averaging out statistical fluctuations in the computed values.

for $A=0.1587$ and 0.8413, the usual one-standard-deviation values for a Gaussian distribution. Note the very large asymmetry in the + and − fractional uncertainties for this small a sample.

Since for large N the $\Delta_{0.8413}^+$ and $\Delta_{0.1587}^+$ should approach $(2/N)^{1/2}$, it is useful to define new parameters ρ_A^+ and ρ_A^- as $(N/2)^{1/2}$ times the corresponding Δ_A^+ and Δ_A^-. These quantities are listed in Table III for various choices of N. The values of C_N and ρ_A^+ and ρ_A^- are plotted in Fig. 2.

III. MONTE CARLO RESULTS FOR THE BEST CHOICES AND UNCERTAINTIES OF $\langle D\rangle$ AND S_0

The similar distribution functions for $\bar{D}/\langle D\rangle$ generated by Monte Carlo methods for the O.E. case are shown in Fig. 3 for several values of N for which there are $N-1$ spacings. Note that, except for the case $N=2$ (one spacing), the distributions are fairly symmetrical and are centered about $x=1$. Table IV lists some "smoothed out" values for $C_N = 1/x_{1/2}$ and $\rho_A^\pm \equiv N\Delta_A^\pm$ for $2 \leqslant N \leqslant 10$. The quantities are plotted in Fig. 4 where smooth curves are shown to smooth out the statistical fluctuations due to the finite (10 800) number of

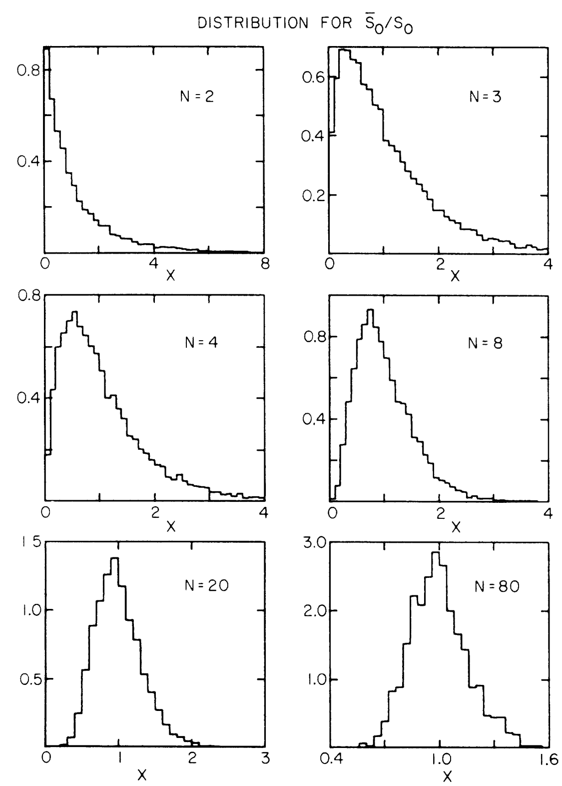

FIG. 5. Monte Carlo results for the distribution of x = ratio of $\bar{\Gamma}_n^0/\bar{D}$ to the true strength function using P.T. theory for Γ_n^0 and O.E. theory for the spacings.

Monte Carlo samples for each N. With about the same accuracy as our Monte Carlo results, the values of $\rho_A^{\pm}$ are given by the following empirical relations to $N\sim 20$:

$$\rho^+_{0.8413} \approx 1+\frac{0.662}{N}-\frac{2.73}{N^2}+\frac{10.7}{N^3}, \tag{5a}$$

$$\rho^-_{0.1587} \approx 1-\frac{1.54}{N}+\frac{3.44}{N^2}-\frac{2.56}{N^3}, \tag{5b}$$

$$\rho^+_{0.9} \approx 1.2+\frac{2.4}{N}-\frac{8.37}{N^2}+\frac{21.9}{N^3}, \tag{5c}$$

$$\rho^-_{0.1} \approx 1.2-\frac{1.34}{N}+\frac{1.71}{N^2}-\frac{0.378}{N^3}. \tag{5d}$$

Note that $N \geq 2$, so $N^3 \geq 8$. The values of C_N are all near unity. Within the accuracy of our evaluation, an empirical fit is given by

$$C_N \approx 1+0.52/N^3. \tag{5e}$$

Some histograms obtained for the distributions of $\overline{S}_0/S_0$ using 10 800 Monte Carlo samples for each $N \leq 10$, and 3600 or 1800 samples for each $N>10$ are shown in Fig. 5. The program separately generated $\overline{\Gamma}_n^0$ and $\overline{D}$ values each time, with the $\overline{D}$ selected according to O.E. theory. The values of $C_N \equiv 1/x_{1/2}$ and the various ρ_A^+ and ρ_A^-, defined as $(N/2)^{1/2}\Delta_A^{\pm}$, are listed in Table V and plotted in Fig. 6 after using a "smoothing" process as for the spacing distribution. These smooth curves for C_N, $\rho^+_{0.8413}$, and $\rho^-_{0.1587}$ are well fitted by

$$C_N \approx 1.0+\frac{0.858}{N}-\frac{1.18}{N^2}+\frac{1.28}{N^3}, \tag{6a}$$

$$\rho^+_{0.8413} \approx 1.0+\frac{7.73}{N+1.39}-\frac{39.7}{(N+1.39)^2}+\frac{139}{(N+1.39)^3}, \tag{6b}$$

$$\rho^-_{0.1587} \approx 1.0-\frac{10.7}{N+16.7}+\frac{249}{(N+16.7)^2}-\frac{3020}{(N+16.7)^3}. \tag{6c}$$

These figures, tables, and empirical formulas are intended for easy use by experimentalist in presenting the results of measurements. No attempt has been made to fold in the contributions due to experimental uncertainties in the measured quantities. A simple quadrature method might be suitable. More important is the problem of missed s levels, or the inclusion of some spurious or p "levels." Similarly, if several isotopes and/or spin states are involved there is the uncertainty as to whether or not all levels are properly identified by isotope and spin. For N not too small, the uncertainty in obtaining S_0 is even smaller if there are extra very weak missed s levels in regions where N-stronger s levels are detected. Both $\langle\Gamma_n^0\rangle$ and $\langle D\rangle$ are in error because of the wrong level count, but $\sum\Gamma_{nj}^0$ is not very much influenced by missing weak levels and the uncertainty in S_0 is reduced by the larger true N of the region using

$$\overline{S}_0=\left(\frac{N-1}{N}\right)\frac{\sum\Gamma_{nj}^0}{\Delta E},$$

where ΔE is the energy difference of the upper and lower measured levels and the sum is over the N measured levels even though the true num-

TABLE V. Results for C_N, ρ_A^+, and ρ_A^- from Monte Carlo studies of the distribution of the sample mean $\overline{\Gamma}_n^0/\overline{D}$ to the true strength function, S_0, using P.T. theory for the Γ_n^0, and O. E. theory for the spacings. Smoothed averages are used from the curves of Fig. 6.

N	C_N	$\rho^+_{0.841}$	$\rho^-_{0.159}$	$\rho^+_{0.9}$	$\rho^-_{0.1}$
2	1.295	3.39	0.676	6.29	0.771
3	1.200	2.34	0.705	4.09	0.820
4	1.158	1.96	0.725	3.21	0.848
5	1.137	1.77	0.740	2.79	0.871
6	1.117	1.66	0.752	2.53	0.897
7	1.106	1.59	0.765	2.34	0.914
8	1.093	1.54	0.774	2.21	0.934
9	1.088	1.50	0.782	2.11	0.948
10	1.076	1.47	0.789	2.04	0.960
15	1.053	1.36	0.813	1.86	1.01
20	1.034	1.29	0.836	1.74	1.04
30	1.017	1.20	0.855	1.64	1.08
40	1.014	1.16	0.873	1.58	1.11
50	1.013	1.13	0.886	1.53	1.12
60	1.012	1.12	0.896	1.50	1.14
70	1.011	1.11	0.905	1.47	1.16
80	1.010	1.10	0.911	1.44	1.18

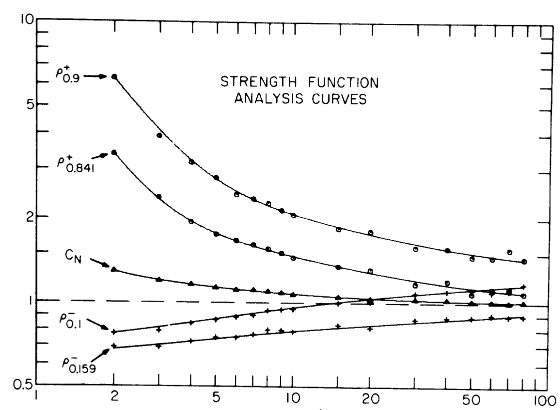

FIG. 6. Monte Carlo results for C_N and ρ_A^+ and ρ_A^- for determining a "best choice" and ± uncertainties for the "true" strength function S_0 in terms of the ratio of the observed averages $\overline{\Gamma}_n^0/\overline{D}$. We suggest using "true" $S_0=C_N(\overline{\Gamma}_n^0/\overline{D})$ with asymmetric (±) fractional uncertainties ρ_A^+ and ρ_A^- times $(2/N)^{1/2}$. The curves smooth out the indicated statistical fluctuations in the Monte Carlo points.

ber of levels is an unknown amount greater than N.

The present confidence-interval approach to the analysis is the one which we finally considered to be most appropriate for this problem. We initially explored various Bayes theorem approaches where one has the problem of choosing the appropriate *a priori* distribution form and where the expressions are plagued with infinities for small N. We also explored the maximum-likelihood approach and finally rejected it for reasons discussed in the text. As a final comment, it should be noted that our experimental studies[1,2] strongly support the O.E. theory only for level spacings for favorable cases in the $150 \leq A \leq 190$ mass region. It is not assured that the theory should apply to lighter nuclei or nuclei near closed shells where $\langle D\rangle$ is very large and where conditions may not yet be appropriate for such an extreme statistical treatment.

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Isospin-Forbidden and Allowed $^{12}C(d,\alpha)^{10}B$ Cross Sections*

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We report extensive new data on the isospin-forbidden reaction $^{12}C(d,\alpha_2)^{10}B(1.74)$ for 7.19 MeV $\leq E_d \leq$ 13.99 MeV. We also report extensive data for the isospin-allowed reactions $^{12}C(d,\alpha_{0,1,3})^{10}B$. All channels exhibit resonant behavior indicative of compound-nucleus formation. We find no evidence for appreciable direct or semidirect contribution to the α_2 cross sections. A partial-wave expansion of the α_2 data fixes J^π for a number of (isospin-mixed) ^{14}N compound-nuclear levels. Our data do not support either Noble's proposed ^6Li mechanism or Weller's modification to Noble's proposal.

I. INTRODUCTION

Since the deuteron, the α particle, and the ground state of ^{12}C all have zero isospin ($T=0$), and the second excited state of ^{10}B(1.74 MeV, $J^\pi=0^+$) has $T=1$, the reaction $^{12}C(d,\alpha_2)^{10}B(1.74)$ is isospin-forbidden. Thus, if the isospin quantum number is strictly conserved in nuclear reactions, the yield for this reaction is zero. Substantial cross sections for this and other isospin-forbidden reactions occur,[1-12] and are generally attributed to isospin mixing by Coulomb forces in the compound-nuclear states. However, Meyer-Schützmeister, von Ehrenstein, and Allas[2] and Jänecke *et al.*[3,12] suggest that their $^{12}C(d,\alpha_2)^{10}B$ data require a direct mechanism. Also Jänecke *et al.*[5,12] believe their $^{16}O(d,\alpha_1)^{14}N(2.31)$ data imply a direct or semidirect mechanism. Since direct