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Off-Shell Effects in the Three-Nucleon System*

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Results of calculations on the three-nucleon system are presented for two potentials: a spin-dependent square well potential acting in s waves, and a spin-dependent rank-one separable potential, which gives exactly the same s-wave phase shifts and deuteron binding energy as the square well. Significant differences between the T matrices arising from these potentials occur as soon as one goes off the energy shell by 40 or 50 MeV (c.m.). In the three-nucleon system the square well produces a triton binding energy of 9.11 MeV, a doublet scattering length of 0.414 fm, and quartet scattering length of 6.31 fm. The corresponding parameters for the separable potential are 9.92 MeV, −0.165 fm, and 6.33 fm. The results of calculations on the doublet and quartet s-wave elastic scattering amplitudes are presented for neutron lab energies up to 14.1 MeV. It is found that the doublet amplitudes are quite sensitive to off-shell effects, while the quartet amplitudes are not. A previously developed off-shell effective range theory for the half-off-shell T matrix is extended to the fully-off-shell case.

I. INTRODUCTION

It is by now well known in nuclear physics that elastic nucleon-nucleon scattering determines the on-shell T matrix, $T_l(k, k; k^2+i\epsilon)$, while what is needed to carry out many calculations is knowledge of the off-shell T matrix $T_l(p, q; s)$. In order to study the sensitivity of the results of calculations to the off-shell T matrix, several authors have turned their attention to the problem of constructing T matrices with the same on-shell values but different off-shell values.

It has been known for some time that knowledge of the S matrix does not uniquely determine the Hamiltonian of a system. In particular, Ekstein[1] demonstrated the existence of a large class of unitary transformations which produce Hamiltonians leading to the same S matrix. Necessary and sufficient conditions that must be fulfilled by unitary transformations in order to give phase-shift-equivalent potentials have been found by Mittelstaedt and Ristig.[2] Unitary transformations have been used to transform hard-core potentials into momentum-dependent or nonlocal potentials.[3]

Ristig and Kistler[4] have applied this transformation technique to Hartree-Fock calculations of nuclear matter, and found very different saturation curves for phase-shift-equivalent potentials. Other authors[5-7] have also used unitary transformations in order to study off-shell effects in nuclear matter. In particular, a very convenient unitary transformation has been introduced by Coester *et al.*[5] and applied to nuclear matter calculations, which assume local s-wave potentials with hard as well as Yukawa cores. In their work the difference between the unitary transformation and the identity is a short-range operator of rank two. This transformation has also been applied to nuclear-matter calculations with the Reid soft-core potential.[6] The general result of all these calculations is that off-shell effects are very important in nuclear matter.

The relation between the T matrices that arise from phase-shift-equivalent potentials produced by unitary transformations has been derived by Monahan, Shakin, and Thaler.[8] They have also constructed unitary transformations that leave the phase shifts unchanged and lead to a prespecified bound-state wave function, as well as transformations that leave both the phase shifts and the bound-state wave function unchanged.[9]

Another approach for studying off-shell effects, which eliminates explicit reference to the potential, has been introduced by Baranger *et al.*[10] They have shown that if the T matrix corresponds to a complete orthonormal set of two-particle wave functions, then the fully-off-shell T matrix can be obtained from the phase shifts and the symmetric part of the half-off-shell T matrix. This result assumes no bound states are present and does not take account of tensor forces. Extensions to bound states[11,12] and tensor forces[12] have been developed. Amado[13] has shown how these techniques can be used to construct a T matrix that is separated into a term that gives the exact phase shifts and binding energy, plus a term that vanishes on shell.

The approach of Picker, Redish, and Stephenson[14] also avoids dealing directly with a potential. They parametrize the wave function, and use it and the on-shell T matrix to calculate the half-off-shell

T matrix. In their approach the T matrix does not correspond to a complete orthonormal set of wave functions.

Several of the separable potentials that have been introduced[15] lead to an S matrix which is a rational function of the momentum. It has been known for some time[16] how to construct a local potential with a given rational S matrix. Using this procedure Kok, Erens, and van Wageningen[17] have carried out calculations on a model triton using the Yamaguchi potential[15] and the phase-shift-equivalent local potential. Their model triton consisted of three identical spinless bosons. They found differences of about 1 to 3 MeV in the triton binding energy, depending on which two-body parameters the potentials were fitted to. The separable potential always gave the larger binding energy. Srivastava *et al.*[18] have constructed the local potentials that are phase shift equivalent to the Tabakin and Mongan potentials and used them in nuclear-matter calculations. They only find significant differences in the case of Tabakin's one-term separable potential.[15]

Yet another approach to studying off-shell effects is based on the fact that the inverse scattering problem for rank-two separable potentials *does not* have a unique solution.[19,20] In particular, one of the two form factors in the potential can be chosen almost arbitrarily and the other one can be determined from the phase shifts. If the bound-state wave function is specified the arbitrariness is somewhat reduced. A unique potential is obtained if the form factor of the repulsive part of the potential is forced to be orthogonal to the given bound-state wave function. Using these rank-two separable potentials Fiedeldey[21] has calculated the binding energy of a model triton and found values ranging from 7.5 to 15.88 MeV.

Here we shall study off-shell effects in the three-nucleon system by comparing the results of calculations with two potentials: A spin-dependent square well potential acting in s waves and a spin-dependent one-term separable potential that gives the same $^3S_1$ and $^1S_0$ phase shifts as the square well. The method for constructing the rank-one separable potential that is phase shift equivalent to a purely attractive or repulsive local potential has been given previously,[20] and is based on choosing the Fredholm determinants corresponding to the two potentials to be the same. The square-well model used here is somewhat realistic in that it is known[22,23] to give good results for the low-energy properties of the three-nucleon system. The three-nucleon properties that we compare are the triton binding energies, the doublet and quartet scattering lengths, and the elastic s-wave scattering amplitudes up to a neutron lab energy of about 14 MeV.

TABLE I. Values for the off-shell lengths.

State	λ_0^{local} (fm)	$\lambda_0^{\text{separable}}$ (fm)
$^3S_1$	0.8884	0.9359
$^1S_0$	1.1299	1.1650

In Sec. II we compare the T matrices arising from the two potentials. Three things are compared: the parameters in off-shell effective range theory,[18,20,24] the half-off-shell extension functions,[25] and the fully-off-shell T matrices at negative energies. We find big differences as soon as one goes off the energy shell by 40 or 50 MeV (c.m.). In Sec. III we compare the three-nucleon quantities. The general result is that the doublet state is quite sensitive to off-shell effects, while the quartet state is not. Section IV is a summary and discussion of the results and gives some suggestions for future work. The generalization of off-shell effective-range theory from the half-off-shell case to the fully-off-shell case is given in the Appendix. Throughout we work in units in which $\hbar$ and the mass of the nucleon are one.

II. COMPARISON OF THE T MATRICES

The T matrix in partial wave l that arises from the one-term separable potential

$$V_l(p,q) = g_l(p)\lambda_l g_l(q) \tag{2.1}$$

is

$$T_l(p,q;s) = V_l(p,q)/D_l(k), \quad s = k^2 + i\epsilon, \tag{2.2}$$

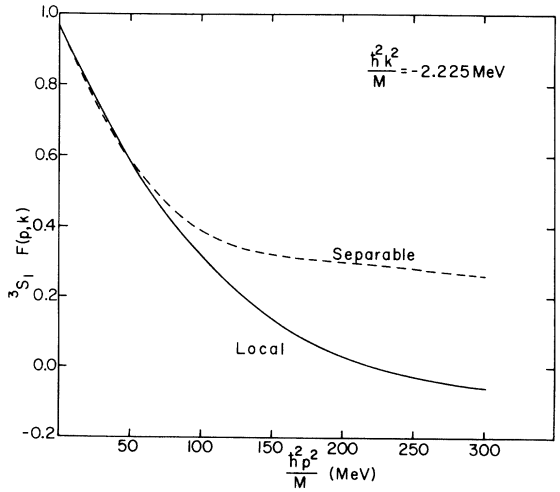

FIG. 1. Values of the $^3S_1$ half-off-shell extension function defined by (2.13).

where

$$D_l(k) = 1 + \int_0^\infty \frac{\lambda_l g_l^2(x) 4\pi x^2 dx}{x^2 - k^2 - i\epsilon} . \tag{2.3}$$

It is shown in Ref. 20, that in order for this T matrix to have the same on-shell ($p = q = k$) values as those of a local potential, we must have

$$\lambda_l g_l^2(k) = (2\pi^2 k)^{-1} \mathrm{Im} f_l(k) = -(2\pi^2 k)^{-1} |f_l(k)| \sin\delta_l(k) . \tag{2.4}$$

Here, $f_l(k)$ and $\delta_l(k)$ are the Jost function and phase shift for the local potential. It follows[20] from a dispersion relation for the Jost function that this choice of $g_l(k)$ implies that

$$D_l(k) = f_l(k) . \tag{2.5}$$

Thus the separable potential T matrix is easily obtained from the Jost function of the local potential. We see from (2.4) that $\sin\delta_l(k)$ must be of one sign for all real k, otherwise $g_l(k)$ will be real for some k and pure imaginary for others.

For a square well of depth V_0 and width b, the s-wave Jost function is

$$f_0(k) = e^{ikb}(K \cos Kb - ik \sin Kb)/K, \quad K = (V_0 + k^2)^{1/2} . \tag{2.6}$$

The parameters that appear here were taken in the triplet state to be $V_{0t} = 34.406$ MeV and $b_t = 2.0719$ fm, which correspond to the scattering parameters $a_t = 5.425$ fm and $r_{0t} = 1.749$ fm. In the singlet state the potential parameters were taken to be $V_{0s} = 14.017$ MeV and $b_s = 2.5895$ fm, which

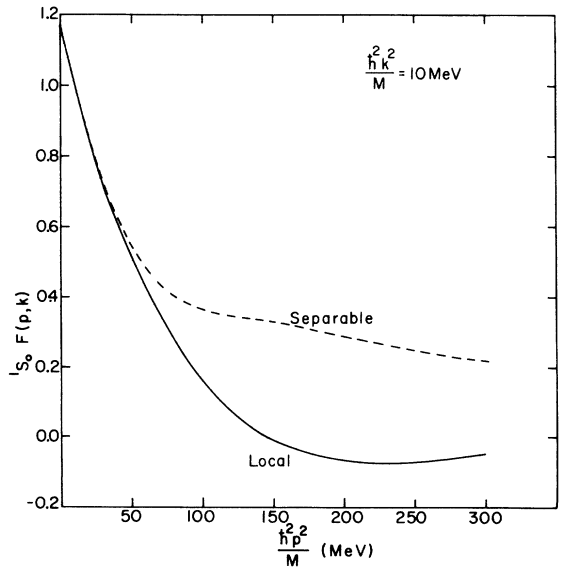

FIG. 2. Values of the $^1S_0$ half-off-shell extension function defined by (2.13).

were obtained by using $a_s = -23.714$ fm and $r_{0s} = 2.704$ fm. The T matrix that arises from the square-well potential has been given previously[26] and will not be repeated here.

In order to compare the two T matrices in the low-energy region, it is convenient to use off-shell effective-range theory.[18, 20, 24] According to this theory the low-energy T matrix is given by

$$T_l(p, q; s) = (p/k)^l T_l(k, k; s)(q/k)^l \times [1 + \tfrac{1}{2}\lambda_l^2(2k^2 - p^2 - q^2) + \text{higher-order terms in } p, q, \text{ and } k], \tag{2.7}$$

where for s waves the off-shell length, λ_0, is given by

$$\lambda_0^2 = 2\int_0^\infty r[1 - r/a - u(r)]dr . \tag{2.8}$$

Here $u(r)$ is the solution of the zero-energy s-wave Schrödinger equation with the boundary condition

$$u(r) \underset{r\to\infty}{\sim} 1 - r/a . \tag{2.9}$$

This low-energy approximation for the T matrix has been given previously for the half-off-shell case ($p = k$ or $q = k$),[20, 24] and is developed for the fully-off-shell case in the Appendix. The corresponding theory for the s-wave K matrix has been developed by Srivastava and Sprung.[18] The parameter I in their theory is related to the parameter λ_0 by

$$\lambda_0^2 = -2I/a . \tag{2.10}$$

For the square well, the off-shell length is given by

$$\lambda_0^{\text{local}} = b\left(1 - \frac{2b}{3a} - \frac{2}{V_0 b^2}\right)^{1/2} . \tag{2.11}$$

TABLE II. Values of $T_0(p, q; s)$ for the $^3S_1$ state with $p = 0$ fm^{-1} and $q = 0$ fm^{-1}.

s (fm^{-2})	Square well	Separable
0.00	0.27482	0.27482
−1.00	−0.19307	−0.18242
−2.00	−0.16164	−0.14783
−3.00	−0.15042	−0.13443
−4.00	−0.14452	−0.12694
−5.00	−0.14087	−0.12202
−6.00	−0.13838	−0.11849
−7.00	−0.13657	−0.11579
−8.00	−0.13519	−0.11366
−9.00	−0.13410	−0.11191

TABLE III. Values of $T_0(p, q; s)$ for the $^3S_1$ state with $p = 0$ fm^{-1} and $q = 2$ fm^{-1}.

s (fm^{-2})	Square well	Separable
0.00	0.02952	0.08853
−1.00	−0.01562	−0.05877
−2.00	−0.01182	−0.04762
−3.00	−0.01040	−0.04330
−4.00	−0.00965	−0.04089
−5.00	−0.00918	−0.03931
−6.00	−0.00886	−0.03817
−7.00	−0.00863	−0.03730
−8.00	−0.00846	−0.03661
−9.00	−0.00832	−0.03605

TABLE IV. Values of $T_0(p, q; s) \times 10^3$ for the $^1S_0$ state with $p = 1$ fm^{-1} and $q = 2$ fm^{-1}.

s (fm^{-2})	Square well	Separable
0.00	23.7225	−167.697
−1.00	−0.8814	−15.9304
−2.00	−0.9120	−14.4256
−3.00	−0.9042	−13.7691
−4.00	−0.8938	−13.3814
−5.00	−0.8848	−13.1184
−6.00	−0.8774	−12.9252
−7.00	−0.8714	−12.7756
−8.00	−0.8665	−12.6553
−9.00	−0.8624	−12.5559

For the separable potential obtained from (2.4) and (2.6), this parameter is given by

$$\lambda_0^{\text{separable}} = b\left(\frac{1}{2V_0 b^2} + \frac{b}{6a}\right)^{1/2}. \tag{2.12}$$

A comparison of these parameters, using the numbers given above is made in Table I. It is seen that the off-shell lengths in a given angular momentum state differ from each other by only a few percent. It should be kept in mind that in the low-energy region the second term in the square bracket on the right-hand side of (2.7) is typically no more than 10% of the first term, therefore a difference of a few percent in the off-shell lengths means a difference of a few tenths of a percent in the low-energy T matrices. We therefore conclude that the two T matrices are almost identical in the low-energy region.

In order to compare the T matrices farther off the energy shell, we have calculated the so-called half-off-shell extension function[25] as well as the fully-off-shell T matrices. The half-off-shell extension function[25] is given by

$$F_l(p, k) = T_l(p, k; s)/T_l(k, k; s). \tag{2.13}$$

The values of this function are shown in Fig. 1 for the $^3S_1$ state at the energy of the deuteron and in Fig. 2 for the $^1S_0$ state at a c.m. energy of 10 MeV. It is seen that the functions are very similar to each other as long as one does not go off the energy shell by more than 40 or 50 MeV. Beyond this point the functions differ significantly. We have found this pattern repeated for values of $\hbar^2k^2/M$ ranging from about −40 MeV up to about +40 MeV. One can show that $F_0(p, k)$ falls off for large p with k fixed like $p^{-2}\cos pb$ for the square-well potential and like p^{-1} for the separable potential. Thus the two potentials lead to very different off-shell behavior at high momenta. It is worth noting that in the upper half of the k plane and on the real axis[27]

$$f_l(k) \underset{|k|\to\infty}{\sim} 1 - (2ik)^{-1}\int_0^\infty dr\, V(r). \tag{2.14}$$

From this and (2.4) it follows that for the one-term separable potential that has the same phase shift as the local potential $V(r)$

$$g_l(k) \underset{|k|\to\infty}{\sim} (2\pi k)^{-1}\left[\pm\int_0^\infty dr\, V(r)\right]^{1/2}, \quad \text{Im}k \geq 0. \tag{2.15}$$

Using (2.1), (2.2), and (2.13) it is easy to see that the half-off-shell extension function for the separable potential will fall off like p^{-1}. This behavior for large momenta is quite different from that of the Yamaguchi potential[15] which leads to a p^{-2} behavior for large p.

We now turn our attention to the fully-off-shell T matrices. The two T matrices are compared in Tables II–IV. These tables illustrate what we have found to be true in general; i.e., when *both*

TABLE V. Three-nucleon parameters.

	Square-well potential	Separable potential	Experiment
Triton binding energy (MeV)	9.11	9.92	8.49
Doublet scattering length (fm)	0.414	−0.165	0.65 ± 0.04 [a]
Quartet scattering length (fm)	6.31	6.33	6.35 ± 0.02 [a]

[a] These values are taken from Ref. 30.

of the momenta p and q are small ($\lesssim 1$ fm^{-1}) the two T matrices are quite similar to each other; when one or both of the momenta become larger than about 1 fm there are large differences between the two T matrices.

In summary, we have found that when *all* of the momenta (p, q, and k) are less than, or approximately equal to 1 fm^{-1}, the two T matrices agree quite closely with each other. Reasonable agreement also occurs over a large range of k when *both* p and q are small. For almost all other momenta, significant differences between the two T matrices exist. We shall see in the next section, the effect these differences produce in a calculation of the properties of the three-nucleon system.

III. RESULTS FOR THE THREE-NUCLEON SYSTEM

For both of the potentials described in Sec. II, we have calculated the low-energy properties of the three-nucleon system using the Faddeev equations. As is well known, the separable potentials of the type we are considering reduce the Faddeev equations to coupled integral equations in one continuous variable. The properties of the three-nucleon system which arise from the square-well potentials given in Sec. II, have been calculated below the breakup threshold by Brayshaw and Buck[22] and have been calculated below and above this threshold by one of us[23] using the Osborn-Fuda[28] separable expansion of the T matrix. In order to check the previous results, we have repeated the calculations using the Weinberg series[29] for the T matrix. We found no significant difference between the results obtained with the two series, except in the case of the doublet scattering length. As was pointed out in Ref. 23 some difficulty was encountered in calculating the doublet scattering length with the Osborn-Fuda expansion, so the discrepancy was not unexpected. The results for the inelastic parameters which are given here, have not been given previously.

In Table V we give the calculated results for the triton binding energy, and for the doublet and quartet scattering lengths, as well as the experimental results. The experimental results for the scattering lengths are those of Dilg, Koester, and Nistler.[30] We see that the separable potential gives more binding in our model triton than the local potential. This is consistent with the results of Kok, Erens, and van Wageningen[17] who found that the Yamaguchi separable potential[15] gave more binding than the phase-shift-equivalent local potential. The difference between the doublet scattering lengths is quite dramatic in that they differ in sign. The fact that the larger scattering length goes with the smaller binding energy is consistent with the Phillips line,[31] which is an approximate linear relation between calculated values of the binding energy and doublet scattering length. We see from Table V that the calculated quartet scattering lengths are almost identical. We shall see that this insensitivity of the quartet results to the off-shell T matrix occurs also at nonzero neutron energies.

In Figs. 3 and 4 we give the calculated as well as the experimental values[32] for $k\cot[\mathrm{Re}(\delta)]$ for neutron lab energies ranging from 0 to 14.1 MeV. k is the relative wave number and is related to the neutron lab energy by $k^2 = (0.02141\ \mathrm{MeV}^{-1})E_{\mathrm{lab}}\ \mathrm{fm}^{-2}$. The results for the doublet inelastic parameter are given in Fig. 5; those for the quartet state are given in Table VI. The inelastic parameter is defined

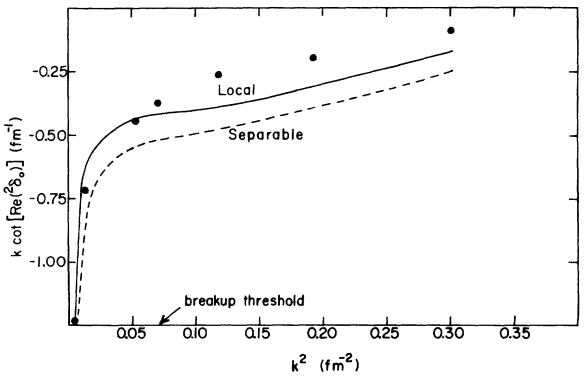

FIG. 3. Values of $k\cot[\mathrm{Re}(\delta)]$ for the doublet n-d scattering states versus k^2, where k is the relative wave number. Solid circles are from the phase-shift analysis of van Oers and Seagrave (Ref. 32).

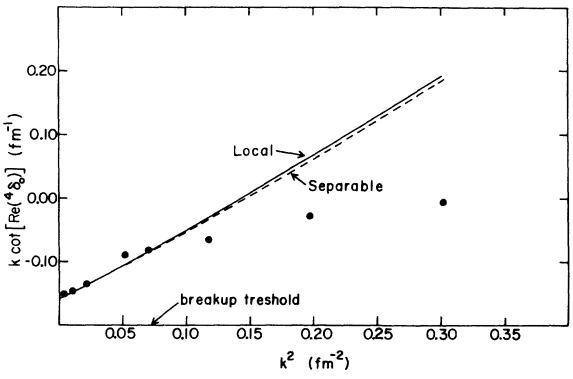

FIG. 4. Values of $k\cot[\mathrm{Re}(\delta)]$ for the quartet n-d scattering states versus k^2, where k is the relative wave number. Solid circles are from the phase-shift analysis of van Oers and Seagrave (Ref. 32).

as

$$y = |\exp(2i\delta)| . \tag{3.1}$$

Figures 3 and 5 show that the doublet scattering amplitude is quite sensitive to the off-shell T matrix. Figure 4 and Table VI indicate that the opposite is true for the quartet amplitude. The fact that the separable potential results for $k\cot[\mathrm{Re}(\delta)]$ lie below those for the local potential is consistent with the binding energy results, since one would expect the more attractive potential to give the larger phase shifts. We note that our results for the inelastic parameter are inconsistent with the phase-shift analysis of Van Oers *et al.*,[32] in which the doublet and quartet inelastic parameters are constrained to be the same. Our results suggest that one would do better to take the quartet inelastic parameter from a calculation, since it appears that it is not very model-dependent. The values of $k\cot[\mathrm{Re}(\delta)]$ in the doublet state calculated with the square well show good agreement with the phase-shift analysis below the breakup threshold but poor agreement above. In the quartet state, both potentials lead to good agreement below the breakup threshold and poor agreement above. The use of the constraint in the phase shift analysis[32] might account for the deterioration of the agreement between theory and experiment as one goes above the breakup threshold.

IV. SUMMARY AND DISCUSSION

For the phase-shift-equivalent potentials that we have considered, we have found that off-shell effects are important in calculating the triton binding energy, the doublet scattering length, and the s-wave doublet elastic scattering amplitudes.

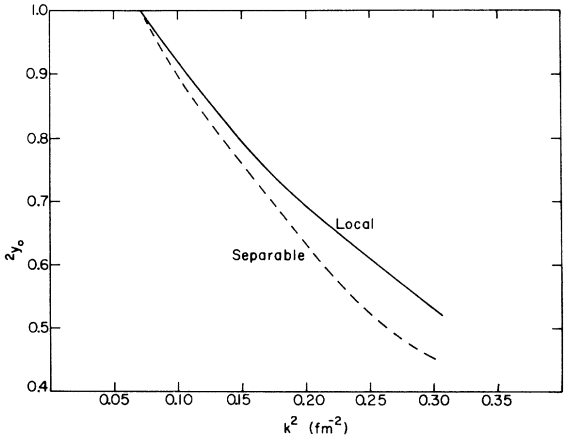

FIG. 5. Values of the inelastic parameter for the doublet n-d scattering states versus k^2, where k is the relative wave number.

The quartet s-wave elastic scattering amplitudes appear to be quite insensitive to the off-shell T matrix; at least for neutron lab energies up to about 14 MeV. This insensitivity in the quartet state is probably a result of the Pauli exclusion principle. The spin function that arises in this case is symmetric under the exchange of any pair of particles, therefore the spatial part of the three-particle wave function must be antisymmetric under exchange of the two neutrons. This means that the relative orbital angular momentum of the two neutrons must be odd. In our calculations we have used potentials which act in only relative s states, thus in the quartet state there is no potential acting between the two neutrons. This suggests that before drawing firm conclusions about the unimportance of off-shell effects in quartet n-d scattering, one should carry out calculations with forces in higher partial waves. It is our feeling that this would not make any difference in our results for the quartet state. The square-well calculations of Brayshaw and Buck[22] include d-wave forces and their results are very close to ours.

The explanation just given for the unimportance of off-shell effects in the quartet state can probably also account for the quartet s-wave inelastic parameter being so close to one in the low-energy region. Since breakup of the deuteron can be thought of as a transition to one of its continuum states, the highly elastic nature of low-energy quartet scattering suggests that the deuteron is not very distorted in quartet scattering.

It is clear that calculations of the breakup amplitudes should also be carried out with phase-shift-equivalent potentials. We have seen that the inelastic parameter for doublet s-wave scattering is quite model-dependent even at the low energies we have considered, therefore one expects the doublet breakup amplitudes to be sensitive to the off-shell T matrix. Techniques for carrying out breakup calculations with separable potentials have been developed,[33] and it should be possible to apply these techniques to the separable expansions considered.

TABLE VI. Inelastic parameters $^4y_0$ for the quartet state.

k^2 (fm^{-2})	Square-well potential	Separable potential
0.10	1.	1.
0.15	1.	1.
0.20	1.	1.
0.25	0.99	0.99
0.30	0.97	0.98

In studying off-shell effects one should also consider potentials which besides producing the same on-shell T matrix also produce the same deuteron wave function, since this wave function can be obtained in principle from knowledge of the electromagnetic form factor. Using the results of Refs. 19 and 20 it should be fairly straightforward to construct a rank-two separable potential that produces the same bound state wave function and phase shifts as the square well. Our guess is that such a separable potential would give almost the same results for the low-energy properties of the three-nucleon system as the square well, since the two T matrices would have exactly the same residue at the deuteron pole. We are planning to carry out such a calculation.

Another problem that needs to be studied in conjunction with off-shell effects, is the dependence of nuclear properties on the high-energy nucleon-nucleon phase shifts. Some work has been done along these lines. In particular, Monahan, Shakin, and Thaler[34] have developed a systematic procedure for producing two-nucleon interactions which have exactly the same phase shifts below a certain cutoff energy but different phase shifts above this cutoff. Also, Fiedeldey[35] has done a study of the influence of the high-energy phase shifts on the triton binding energy. He finds that the triton binding energy is not very sensitive to large differences in the high-energy tail as long as the phase shifts are obtained from a reasonable potential. However, for arbitrary variations of the phase shifts there are large effects.

Clearly, much more work needs to be done in studying off-shell and high-energy effects in the three-nucleon system.

APPENDIX

Here we extend the derivation of the off-shell effective-range theory from the half-off-shell case to the fully-off-shell case. We begin by introducing the free waves whose space representation is

$$\langle \vec{r} | plm \rangle = (2\pi^2)^{-1/2} j_l(pr) Y_{lm}(\hat{r}) . \tag{A1}$$

With these basis functions the on-shell T matrix has the normalization

$$\langle klm | T(k^2 + i\epsilon) | klm \rangle = -(2\pi^2 k)^{-1} e^{i\delta_l(k)} \sin\delta_l(k) . \tag{A2}$$

The fully-off-shell T matrix can be written in the form

$$T_l(p, q; s) \equiv \langle plm | T(s) | qlm \rangle = \int \langle plm | \vec{r} \rangle d\vec{r} \langle \vec{r} | T(s) | \vec{r}' \rangle d\vec{r}' \langle \vec{r}' | qlm \rangle . \tag{A3}$$

From this and the power series representation of the spherical Bessel functions, it follows that $p^{-l}\langle plm | T(s) | qlm \rangle q^{-l}$ is a function of p^2 and q^2. If we expand this function in a two-dimensional Taylor series about the point $p^2 = k^2$, $q^2 = k^2$, we arrive at

$$T_l(p, q; s) = (p/k)^l T_l(k, k; s)(q/k)^l \times \left[1 + \frac{\partial}{\partial p^2}(k/p)^l F_l(p, k)\big|_{p=k}(p^2 - k^2) + \frac{\partial}{\partial q^2}(k/q)^l F_l(q, k)\big|_{q=k}(q^2 - k^2) + \cdots \right] , \tag{A4}$$

where we have used (2.13). From (2.9), (2.10), (2.12), and (2.14) of Ref. 20 it follows that

$$\frac{\partial}{\partial p^2}(k/p)^l F_l(p, k)\big|_{p=k} \xrightarrow[k\to 0]{} -\tfrac{1}{2}\lambda_l^2 , \tag{A5}$$

where

$$\lambda_l^2 = -\frac{1}{(2l+3)!!}\int_0^\infty r^{l+3} V(r) u_l(0, r) dr . \tag{A6}$$

Here $u_l(k, r)$ is the solution of the radial Schrödinger equation for the lth partial wave with energy k^2 which outside the range of the force becomes

$$u_l(k, r) = k^{l+1} r \left[\frac{e^{-i\delta_l(k)} j_l(kr)}{\sin\delta_l(k)} + h_l^{(+)}(kr) \right] \xrightarrow[k\to 0]{} \frac{(2l+1)!!}{(2l+1)} r^{-l} - \frac{1}{(2l+1)!!}\frac{r^{l+1}}{c_l} . \tag{A7}$$

The parameter c_l is defined by

$$c_l^{-1} = -\lim_{k\to 0} k^{2l+1} \cot\delta_l(k) \tag{A8}$$

and is the usual scattering length for $l = 0$. We can eliminate the potential from (A6) by using the identity

$$V(r) u_l(0, r) = \left[\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} \right] \times \left[u_l(0, r) - \frac{(2l+1)!!}{(2l+1)} r^{-l} + \frac{1}{(2l+1)!!}\frac{r^{l+1}}{c_l} \right] , \tag{A9}$$

which follows from the zero-energy Schrödinger equation. Putting (A9) into (A6) and integrating by parts we arrive at

$$\lambda_l^2 = \frac{2}{(2l+1)!!}\int_0^\infty dr\, r^{l+1} \times \left[\frac{(2l+1)!!}{(2l+1)} r^{-l} - \frac{1}{(2l+1)!!}\frac{r^{l+1}}{c_l} - u_l(0, r) \right] . \tag{A10}$$

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