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Muon Absorption by Nuclei with $N > Z$*

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Muon absorption is investigated for nuclei with an excess of neutrons in order to answer the question: Are the studies carried out in previous works on $N=Z$ nuclei valid for a larger class of nuclei with $N>Z$ and possibly heavier? If so, to what extent? The present study indicates that the conventional shell-model techniques yield similar results for a more complex class of nuclei. Several selected nuclei are taken as examples; the ^{88}Sr nucleus is thoroughly studied.

INTRODUCTION

Since the isotriplet theory of Feynman and Gell-Mann,[1] the concept of isospin plays a key role in relating weak and electromagnetic processes. In particular, during the last few years, the muon absorption process by light ($N=Z$) nuclei has been investigated under the light of its isospin connection with photon (real and virtual) absorption. Since an important feature of nuclear-photoabsorption cross sections in light nuclei is the giant dipole resonance (gdr) which exhausts most of the transition strength, the first detailed investigation involved this gdr. The above-mentioned analogy was used to extract matrix elements from well-known experimental data on the gdr and transpose them into muon-capture-rate expressions.[2] Since then, several calculations have been made in general concentrating on the collective dipole state obtained by muon capture with specific models, other multipoles being essentially considered as corrections.[3]

Most of these activities are concentrated on nuclei without extra neutrons. The few works concerning muon absorption (or neutrino-induced nuclear reactions) on $N=Z$ nuclei are more concerned with total muon capture rates without detailed interest for dipole (or other multipole) states. As far as muon capture is concerned, general isospin sum rules published in the recent years[4,5] do not go beyond giving clear indications about the relations between the electromagnetic and the polar vector part of muon-absorption reduced matrix elements. The purpose of this paper is to investigate whether results obtained for $N=Z$ nuclei are still valid for a more extended class of $N>Z$ and possibly heavier nuclei.[6]

Thus, in Sec. I, the expression $|M_V|^2$ (to be defined at the end of this introduction along with several other relevant expressions) is investigated; commutation relations display its relation to an electromagnetic counterpart; its expansion into multipole components yields the relative importance of multipole strengths as a function of the numbers of protons and neutrons of the muon absorbing nucleus; and finally the dipole case is more particularly investigated. Section II is devoted to the validity of equalities between $|M_{V,A,P}|^2$ and derived quantities in the j-j shell model through the use, first, of spin-isospin commutation relations and then of a detailed shell-model analysis. Also, the effect of spin-orbit energy splitting is considered at the end of Sec. II on the basis of previous work on light $N=Z$ nuclei. Various aspects of muon absorption by ^{88}Sr are given in Sec. III in view of studying the breaking effects of switching on a nucleon-nucleon interaction. The Tamm-Dancoff method is used in the calculations, and the results are systematically compared to previous works on $N=Z$ nuclei. A conclusion follows to sum up this work and give its limitations and perspectives for the future.

Standard assumptions concerning weak-interaction muon capture process by a nucleon, assuming the absence of exchange currents, lead to the well-known expression for the total muon capture rate:

$$\Lambda_\mu = \frac{\nu_\mu{}^2}{2\pi\hbar^2 c}|\Phi_\mu|^2[G_V{}^2|M_V|^2 + 3G_A{}^2|M_A|^2 + (G_P{}^2 - 2G_PG_A)|M_P|^2] + \Lambda'_\mu \quad (1)$$

the conventions and expressions being the same as in FW, except for slightly different values of two constants[3] which appear in the detailed expressions of the $G_{V,A,P}$, namely,

$$G = \frac{1.023 \pm 0.002}{M_p{}^2}10^{-5}, \quad F_A(0) = -1.23 \pm 0.1 .$$

$|\Phi_\mu|^2$ is the average muon-squared wave function. Λ'_μ comes from relativistic terms and amounts to about 10% of Λ_μ.[7] G_V, G_A, G_P are effective constants, linear combinations of the fundamental weak-interaction form factors:

$$|M_{V,A,P}|^2 = \overline{\sum_a}\sum_b\left(\frac{\nu_{ab}}{\nu_\mu}\right)^2\int\frac{d\hat\nu}{4\pi}\left|\left\langle b\left|\sum_{i=1}^A \tau_i^- \mathcal{O}_i^{V,A,P}e^{-i\vec\nu_{ab}\cdot\vec x_i}\right|a\right\rangle\right|^2, \quad (2)$$

with

$$\mathcal{O}_i^V = 1_i, \quad \mathcal{O}_i^A = \frac{\vec\sigma_i}{\sqrt3}, \quad \mathcal{O}_i^P = \vec\sigma_i\cdot\hat\nu .$$

$|a\rangle$ and $|b\rangle$ are initial and final states of the nuclear system. $\nu_{ab}/\nu_\mu = (hc\nu_{ab})/m_\mu c^2$ is the energy of the outgoing neutrino for the nuclear transition over the rest mass of the muon. Several times in the following sections, the expression $|M_{V,A,P}|_L{}^2$ appears; it is the form obtained for $|M_{V,A,P}|^2$ on replacing $e^{-i\vec\nu_{ab}\cdot\vec x_i}$ by the Lth multipole expansion term $\omega_L(i)$.

The following way of writing the expression $|M_{V,A,P}|^2$ yields quantities of interest:

$$\begin{aligned}|M_V|^2 &= \int\frac{d\hat\nu}{4\pi}\overline{\sum_a}\sum_b\left(\frac{\nu_{ab}}{\nu_\mu}\right)^2\left|\left\langle b\left|\int e^{-i\vec\nu\cdot\vec x}\left[\sum_i\tau_i^-\delta(\vec x - \vec x_i)\right]d^3\vec x\right|a\right\rangle\right|^2 \\ &= \int\frac{d\hat\nu}{4\pi}\overline{\sum_a}\sum_b\left(\frac{\nu_{ab}}{\nu_\mu}\right)^2\left|\left\langle b\left|\int e^{-i\vec\nu\cdot\vec x}[\mathcal{V}^-(\vec x)]\,d^3\vec x\right|a\right\rangle\right|^2 \\ &= \int\frac{d\hat\nu}{4\pi}\overline{\sum_a}\sum_b\left(\frac{\nu_{ab}}{\nu_\mu}\right)^2|\langle b|V^-(\vec\nu)|a\rangle|^2 \end{aligned} \quad (3)$$

with

$$V^-(\vec\nu) = \int e^{-i\vec\nu\cdot\vec x}\mathcal{V}^-(\vec x)d^3\vec x = \sum\tau_i^- e^{-i\vec\nu\cdot\vec x_i}, \quad (3')$$

$$\mathcal{V}^-(\vec x) = \sum_i \tau_i^-\delta(\vec x - \vec x_i), \quad (3'')$$

and

$$T^- = V^-(\vec\nu = 0) = \sum_i\tau_i^- . \quad (3''')$$

Similar quantities can be deduced from $|M_A|^2$,

$$\begin{aligned}|M_A{}^\lambda|^2 &= \int\frac{d\hat\nu}{4\pi}\overline{\sum_a}\sum_b\left(\frac{\nu_{ab}}{\nu_\mu}\right)^2\left|\left\langle b\left|\int e^{-i\vec\nu\cdot\vec x}\left[\sum_i\tau_i^-\sigma_i^\lambda\delta(\vec x - \vec x_i)\right]d^3\vec x\right|a\right\rangle\right|^2 \\ &= \int\frac{d\hat\nu}{4\pi}\overline{\sum_a}\sum_b\left(\frac{\nu_{ab}}{\nu_\mu}\right)^2\left|\left\langle b\left|\int e^{-i\vec\nu\cdot\vec x}[\mathcal{A}_\lambda^-(\vec x)]\,d^3\vec x\right|a\right\rangle\right|^2 \\ &= \int\frac{d\hat\nu}{4\pi}\overline{\sum_a}\sum_b\left(\frac{\nu_{ab}}{\nu_\mu}\right)^2|\langle b|A_\lambda^-(\vec\nu)|a\rangle|^2 \end{aligned} \quad (4)$$

with

$$A_\lambda^-(\vec\nu) = \int e^{-i\vec\nu\cdot\vec x}\mathcal{A}_\lambda^-(\vec x)\,d^3\vec x = \sum_i\tau_i^-\sigma_i^\lambda e^{-i\vec\nu\cdot\vec x_i}, \quad (4')$$

$$\mathcal{A}_\lambda^-(\vec x) = \sum_i\tau_i^-\sigma_i^\lambda\delta(\vec x - \vec x_i), \quad (4'')$$

and

$$Y^- = A_\lambda^-(\vec{\nu}=0) = \sum_{i=1}^{A} \sigma_i^\lambda \tau_i^- . \tag{4'''}$$

$\mathcal{V}^-(\vec{x})$ and $\mathcal{A}_\lambda^-(\vec{x})$ defined above belong to the set of quantities $\mathcal{V}^\alpha(\vec{x})$ and $\mathcal{A}_\lambda^\beta(\vec{x})$ (α and β being isospin indices) which form a $SU_2 \otimes SU_2$ current algebra.[8] Such an algebra yields simple commutation relations between the Fourier transforms $V^-(\vec{\nu})$, $A_\lambda^-(\vec{\nu})$. In particular,

$$\delta(\vec{x}-\vec{x}')\mathcal{V}^-(\vec{x}) = +\tfrac{1}{2}[\mathcal{V}^-(\vec{x}), \mathcal{V}^3(\vec{x}')] \Rightarrow V^-(\vec{\nu}) = +\tfrac{1}{2}[T^-, V^3(\vec{\nu})], \tag{5}$$

$$\delta(\vec{x}-\vec{x}')\mathcal{A}_\lambda^-(\vec{x}) = \tfrac{1}{2}[\mathcal{A}_\lambda^-(\vec{x}), \mathcal{V}^3(\vec{x}')] \Rightarrow A_\lambda^-(\vec{\nu}) = \tfrac{1}{2}[Y^-, V^3(\vec{\nu})], \tag{5'}$$

$$\delta(\vec{x}-\vec{x}')\mathcal{A}_\lambda^-(\vec{x}) = -\tfrac{1}{2}[\mathcal{A}_\lambda^3(\vec{x}), \mathcal{V}^-(\vec{x}')] \Rightarrow A_\lambda^-(\vec{\nu}) = -\tfrac{1}{2}[Y^3, V^-(\vec{\nu})]. \tag{5''}$$

The Eqs. (5), (5′), (5″) are still valid if $e^{-i\vec{\nu}\cdot\vec{x}}$ is replaced by a component ω_L of its multipole expansion; they are then also useful for investigation of the $|M_{V,A,P}|_L{}^2$.

I. ANALYSIS OF $|M_V|^2$

A. Relation to Electromagnetic Processes

Equation (5) sandwiched between the initial and final nuclear states $|a\rangle$ and $|b\rangle$ yields:

$$\langle b|V^-(\vec{\nu})|a\rangle = \tfrac{1}{2}\sum_{b'}\langle b|T^-|b'\rangle\langle b'|V^3(\vec{\nu})|a\rangle - \tfrac{1}{2}\sum_{b''}\langle b|V^3(\vec{\nu})|b''\rangle\langle b''|T^-|a\rangle . \tag{6}$$

$|b'\rangle$ and $|b''\rangle$ are a complete set of intermediate states, solutions (as well as $|a\rangle$ and $|b\rangle$) of the nuclear Hamiltonian $H = H_0 + V_c$, where V_c is the Coulomb interaction and $[H, T^-] \simeq [V_c, T^-]$. Assuming that $|a\rangle$ represents a stable nucleus (the only interesting case in view of muon absorption experiments), its isospin coordinates are $T^3 = -(N-Z)/2 = -T$, which corresponds to the lowest energy member of the isomultiplet. Therefore, the second term on the right-hand side gives zero since $T^-|a\rangle = 0$. Also, the state $|b\rangle$ having $\{T+1, -(T-1)\}$, the set of states $|b'\rangle$ is $\{T+1, -T\}$. Thus,

$$\begin{aligned}|M_V|^2 &= \overline{\sum_a}\sum_b \left(\frac{\nu_{ab}}{\nu_\mu}\right)^2 \int \frac{d\hat{\nu}}{4\pi} \left|\left\langle b\left|\sum_i \tau_i^- e^{-i\vec{\nu}_{ab}\cdot\vec{x}_i}\right|a\right\rangle\right|^2 \\ &= \overline{\sum_a}\sum_{b'} \left(\frac{\nu_{ab}}{\nu_\mu}\right)^2 \int \frac{d\hat{\nu}}{4\pi} \left|\left\langle b'\left|\sum_i \tau_i^3 e^{-i\vec{\nu}_{ab}\cdot\vec{x}_i}\right|a\right\rangle\right|^2 \left(\frac{T+1}{2}\right).\end{aligned} \tag{7}$$

This is the Eq. (3.5) of Ref. 2 for the case $T = T^3 = 0$.

B. Multipole Expansion

The exponential may be expanded in multipole terms yielding:

$$|M_V|_L{}^2 = \overline{\sum_a}\sum_b \left(\frac{\nu_{ab}}{\nu_\mu}\right)^2 \int \frac{d\hat{\nu}}{4\pi} \left|\left\langle b\left|\sum_i \tau_i^- \omega_L(i)\right|a\right\rangle\right|^2 \tag{7'}$$

$$= \overline{\sum_a}\sum_b \left(\frac{\nu_{ab}}{\nu_\mu}\right)^2 \int \frac{d\hat{\nu}}{4\pi} \left|\left\langle b'\left|\sum_i \tau_i^3 \omega_L(i)\right|a\right\rangle\right|^2 \left(\frac{T+1}{2}\right). \tag{7''}$$

Table I illustrates the relative importance of the various multipole terms $|M_V|_L{}^2$ estimated in the framework of a pure shell model. Two factors may guide the reading of this table. The retardation effect increases with the nuclear radius, and the excess neutrons have a specific quenching effect on proton-hole–neutron-particle transitions. Thus, monopole ($0^+ \to 0^+$) transitions, uniquely due to retardation effects, are very small for light nuclei and get more important in heavier nuclei; dipole ($0^+ \to 1^-$) transitions, predominant in light nu-

TABLE I. Relative percentage of $|M_V|_L{}^2$ in several nuclei. A few slight differences from the corresponding table of Ref. 6 (1%) come from the addition of some minor configurations.

$0^+ \to J^\pi$	^{40}Ca	^{60}Ni	^{88}Sr	^{114}Sn	^{140}Ce	^{208}Pb
$0^+ \to 0^+$	7	6	15	13	19	28
$0^+ \to 1^-$	77	69	51	44	34	11
$0^+ \to 2^+$	14	22	28	35	39	51
$0^+ \to 3^-$	2	3	6	7	7	8
$0^+ \to 4^+$	0	0	0	1	1	2

clei, are decreased both by retardation and neutron-excess effects which are adding up. Finally, twice-forbidden ($0^+ \to 2^+$) transitions, which would correspond in a pure harmonic-oscillator model to a two-shell jump are less hindered by the presence of neutron excess. They are thus predominant in ^{208}Pb.

The values in Table I were obtained from Eq. (7′) and more specifically with the following prescription:

(i) The energy of the outgoing neutrino is defined as in FW, i.e.,

$$h\nu_{ab}\, c = m_\mu c^2 - E_B - E_{ba}\,,$$

with $m_\mu c^2$ the muon rest mass, and E_B the binding energy of the muon on the muonic $1s$ shell.

(ii) The wave functions are those of an harmonic-oscillator well with splitting between two major shells given by[9]:

$$h\omega = 41A^{-1/3}\,.$$

It is realized that such a model does not describe heavy nuclei so well. However, calculations based on a Saxon-Woods potential yield the same trends.

(iii) Transitions taken into account are those where a proton located in the two major shells under the Fermi level is promoted to a neutron state within the two major shells above the Fermi level.

(iv) The nuclear excitation E corresponds to the energy gap between the final neutron E_n and the initial proton E_p. Harmonic-oscillator parameters identical for neutrons and protons yield the energies $\tilde{E}_n(b)$ and $\tilde{E}_p(a)$, so that

$$E_{ba} = E_n(b) - E_p(a) + (m_p - m_n)c^2$$
$$= \tilde{E}_n(b) - \tilde{E}_p(a) + U - \Delta_c + (m_n - m_p)c^2\,.$$

Δ_c is the Coulomb shift, U is the symmetry energy defined in Ref. 9 ($U_{\text{MeV}} \approx 100\ T/A$).

C. Dipole Case

Once-forbidden electromagnetic excitations ($L=1$) have been investigated more extensively than other multipole excitations so that more data are available to undergo a fruitful application of Eq. (7′). The dipole photoabsorption cross section σ_γ is proportional to:

$$\overline{\sum_a}\sum_{b'}(E_{b'} - E_a)\,|\langle b'|\sum_i \tau_i^3 \vec{x}_i|a\rangle|^2\,\delta(E_{b'} - E_a - E)\,. \tag{8}$$

The unrelated dipole (UD) matrix element involved is $|M_V|^2_{L=1,\,\nu_{ab}|\vec{x}_i|\to 0} = |M_V|_{\text{UD}}{}^2$ which is then simply related to the quantity:

$$\sigma_{-1} = \int_0^\infty \frac{\sigma_\gamma(E)}{E}\,dE\,.$$

The above expressions are quite familiar, in particular the similarity between Eq. (1)[2] and Eq. (8) which shows how the two processes (muon and photon absorption) are related. For nuclei with extra neutrons, we expect some modifications coming from the splitting of the gdr into two fragments,[4] each with a definite isospin, the upper fragment of isospin $T=(N-Z)/2+1$ is related to the dipole collective state obtained by muon absorption. The equations below are useful guides to estimate the relevant quantities.

The energy interval between the components E_T and E_{T+1} taken as two fragments of a dipole collective state:

$$E_{T+1} - E_T = \tilde{U}(T+1)/T = 60(T+1)/A\,, \tag{9}$$

where $\tilde{U} = (3/5)\ U$; U being the symmetry energy.[10]

The ratio of the dipole reduced matrix elements σ_{-1}^{T+1} and σ_{-1} given by[4]:

$$\sigma_{-1}^{T+1}/\sigma_{-1} = \frac{1}{T+1}\left[1 - \tfrac{3}{4}(N-Z)/A^{2/3}\right]. \tag{9′}$$

Equation (9) was obtained in a schematic-model type of the gdr, where the energy splitting between the two isospin components, which amounts to $U(T+1)/T$ in the weak-coupling limit, is reduced to $\tilde{U}(T+1)/T$ due to more numerous particle-hole interactions acting to push up the T component. Equation (9′), obtained from simple shell-model considerations (however not standard harmonic-oscillator model) does not take into account the mixing of each isospin component with the T compound states and the continuum channels.

Despite their simplicity, Eqs. (9) and (9′) reproduce experimental data for large groups of nuclei[11] rather well. They also give results in over-all agreement with more sophisticated nuclear models for definite nuclei.[12] It is thus possible to take over the method initiated by FW who picked up some relevant information from experimental data on σ_{-1} to use them in order to evaluate Λ_μ. Their method has been extended to $N>Z$ nuclei in Ref. 6. In the next few lines, we want to recall and point out some of the approximations done in this reference.

As already mentioned, Eq. (7) refers to photoexcitation of the $T+1$ fragment, so that

$$|M_V|_{\text{UD}}{}^2 = \frac{\nu_\mu{}^2}{2\pi^2\alpha}\left(\frac{\nu_m}{\nu_\mu}\right)^2(T+1)\int_0^{E_M}\left(\frac{E_M - E}{E_M}\right)^4 \sigma_{-1}^{T+1}(E)\,dE\,, \tag{10}$$

where $E_M = h\nu_m c$ is the maximum energy of the outgoing neutrino. With Eqs. (9) and (9′) in mind, it is reasonable to mention that for light nuclei where neutron excess is small, the energy of the $T+1$ fragment is high, but its relative strength is large enough to be directly measured, exactly as in the FW case. In sufficiently heavy nuclei, the $T+1$ peak is less prominent; on the other hand, its relative narrowness allows one to take the energy factor out of the integral so that

$$\int_0^{E_M}\left(\frac{E_M-E}{E_M}\right)^4\sigma_{-1}^{T+1}(E)dE \simeq \left(\frac{E_M-E_{T+1}}{E_M}\right)^4\int_0^{E_M}\sigma_{-1}^{T+1}(E)dE\,. \tag{11}$$

An estimate of such an approximation, representing the $T+1$ fragment by a Lorentzian curve yields an error of less than 2% for a width smaller than 1 MeV, as is the case for every medium-heavy nucleus,

$$|M_V|_{\mathrm{UD}}{}^2=\frac{\nu_\mu{}^2}{2\pi^2\alpha}\left(\frac{E_M-E_{T+1}}{E_M}\right)^4(1-\tfrac{3}{4}(N-Z)/A^{2/3})\sigma_{-1}\,. \tag{12}$$

A numerical application is shown at the end of Sec. III in the case of ^{88}Sr.

II. RELATIONS BETWEEN THE $|M_{V,A,P}|^2$

A. Preliminary Remarks

Similarity between the three expressions $|M_{V,A,P}|^2$ led various authors to correlate them. Those expressions have been proved to be equal for a nucleus the Hamiltonian of which is invariant under SU_4.[2,13] However, studies on the breaking of these equalities have been carried out only for light $N=Z$ doubly-closed-shell nuclei. The present investigation is to try to extend those previous results to $N>Z$ nuclei. Before beginning, differences between $|M_{V,A,P}|_L{}^2$ or $|M_{V,A,P}|_{J^\pi}{}^2$ connected with transitions to definite states and $|M_{V,A,P}|^2$ must also be kept in mind; possible differences between the $|M_{V,A,P}|^2$ come from different angular momentum properties and from the neutrino energy factors; subsection C concentrates upon this later factor.

The starting point is the result found in the work of Luyten, Rood, and Tolhoek[14] (LRT) who proved the $|M_{V,A,P}|^2$ equality in the j-j model provided that, if one disregards spin-orbit energy splitting, in the $p(n_p, l_p, j_p)\to n(n_n, l_n, j_n)$ transition, *either* the proton leaves a doubly closed shell, *or* the neutron comes to a doubly empty shell. Such a result is then valid *a fortiori* in a transition from a doubly filled to a doubly empty shell.

This property can be directly applied to a total muon-capture-rate calculation, and not for a transition to a final state with a definite angular momentum; however, the LRT results still hold when $\exp(-i\vec{\nu}\cdot\vec{x}_i)$ is replaced by $\omega_L(i)$, that is, when $|M_{V,A,P}|^2$ is replaced by $|M_{V,A,P}|_L{}^2$.

B. Sum Rules and LRT Transitions

Sandwiching of Eq. (5′) between nuclear initial and final states $|a\rangle$ and $|b\rangle$ yields:

$$\langle b|A_\lambda^-(\vec{\nu})|a\rangle=-\tfrac{1}{2}\sum_{b'}\langle b|Y^-|b'\rangle\langle b'|V^3(\vec{\nu})|a\rangle +\tfrac{1}{2}\sum_{b''}\langle b|V^3(\vec{\nu})|b''\rangle\langle b''|Y^-|a\rangle\,. \tag{13}$$

The assumption that this equality involves only LRT transitions imposes a definite structure for the initial state $|a\rangle$ which necessarily contains a doubly closed last shell either in protons or in neutrons; this implies, together with the condition $N>Z$, that LRT transitions involve a change of orbital angular momentum; then the second term on the right-hand side of Eq. (13) cancels out and a derivation similar to the one given by LRT yields:

$$\begin{aligned}|M_A|^2 &=\overline{\sum_a}\sum_b\left(\frac{\nu_{ab}}{\nu_\mu}\right)^2\int\frac{d\hat{\nu}}{4\pi}\left|\left\langle b\left|\sum_i\tau_i^-\frac{\vec{\sigma}_i}{\sqrt{3}}e^{-i\vec{\nu}_{ab}\cdot\vec{x}_i}\right|a\right\rangle\right|^2\\ &=\overline{\sum_a}\sum_b\left(\frac{\nu_{ab}}{\nu_\mu}\right)^2\int\frac{d\hat{\nu}}{4\pi}\left|\left\langle b'\left|\sum_i\tau_i{}^3e^{-i\vec{\nu}_{ab}\cdot\vec{x}_i}\right|a\right\rangle\right|^2\left(\frac{T+1}{2}\right)\\ &=|M_V|^2\,.\end{aligned} \tag{14}$$

As in Sec. I, the derivation does not depend on $e^{-i\vec{\nu}_{ab}\cdot\vec{x}_i}$ which may be replaced by $\omega_L(i)$ as well. For $T=0$, FW obtained such an equality, using the supermultiplet formalism, $|a\rangle$ being a scalar representation ($S=T=0$) of SU_4 and $|b\rangle, |b'\rangle, |b''\rangle$ belonging to a same supermultiplet.

C. Shell-Model Analysis

In this subsection, equalities between $|M_{V,A,P}|_L{}^2$ are rederived through standard angular momentum algebra. Although some of the steps are known from previous muon-capture calculations,[15] it illustrates relations between the $|M_{V,A,P}|^2$, $|M_{V,A,P}|_L{}^2$ and other quantities and also shows how the LRT conditions enter the equations.

For a transition $0^+ \to (n_n\, l_n j_n m_n, n_p\, l_p j_p m_p)JM$ the following expressions are useful:

$$\int \Theta^{V,A,P} = \sum_{m_n m_p} (-1)^{j_p - m_p} \langle j_n j_p m_n - m_p | JM \rangle \langle n_n\, l_n j_n m_n | e^{-i\vec{\nu}\cdot\vec{x}} | n_p\, l_p j_p m_p \rangle , \tag{15}$$

$$\int \Theta^{V} = \sum_L i^J (-1)^{j_p+\frac{1}{2}} \hat{j}_n \hat{j}_p \hat{l}_n \hat{l}_p \hat{J} \begin{pmatrix} l_n & l_p & J \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} J & j_n & j_p \\ \frac{1}{2} & l_p & l_n \end{Bmatrix} I_{npL}\, \delta_{JL} , \tag{16}$$

$$\int \Theta^{A}_{\mu} = \sum_L (-i)^L (-1)^{l_p - 1 - M} \hat{j}_p \hat{j}_n \hat{l}_n \hat{l}_p \hat{J} \hat{L}^2 \sqrt{6} \begin{pmatrix} L & 1 & J \\ 0 & \mu & -M \end{pmatrix} \begin{pmatrix} l_n & l_p & L \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} l_n & l_p & L \\ \frac{1}{2} & \frac{1}{2} & 1 \\ j_n & j_p & J \end{Bmatrix} I_{npL} , \tag{16'}$$

with

$$\hat{x} = (2x+1)^{1/2}$$

and

$$I_{npL} = \int_0^\infty R_{n_n l_n j_n}(r)\, j_l(\nu_{np} r) R_{n_p l_p j_p}(r)\, r^2 dr$$

in which ν_{np} is the energy of the outgoing neutrino corresponding to the $(n_p\, l_p j_p \to n_n\, l_n j_n)$ transition. Spin-dependent forces affect $|M_{V,A,P}|^2$ in two ways: The radial parts $R_{nlj}(r)$ depend slightly on the value of j. We neglect this source of variation throughout this paper. The weakness of this effect is illustrated for example in Ref. 9 for ^{208}Pb where it is supposed at its highest value. Second, the neutrino energy comes in the expressions of the $|M_{V,A,P}|^2$ in the factor $(\nu_{np})^2$ and in the $\exp(-i\vec{\nu}_{np}\cdot\vec{x})$. In a first step, we neglect the variations of ν_{np} with spin-orbit coupling; later on, changes brought in the results by this effect will be investigated. Using

$$|M_{V,A,P}|^2 = \sum_J \left(\frac{\nu_{np}}{\nu_\mu}\right)^2 \int \frac{d\hat{\nu}}{4\pi} \left| \int \Theta^{V,A,P} \right|^2 , \tag{17}$$

we get:

$$|M_V|^2 = \sum_L \sum_{Jnp} \left(\frac{\nu_{np}}{\nu_\mu}\right)^2 (\hat{j}_n \hat{j}_p \hat{l}_n \hat{l}_p \hat{J})^2 \begin{pmatrix} l_n & l_p & J \\ 0 & 0 & 0 \end{pmatrix}^2 \begin{Bmatrix} J & j_n & j_p \\ \frac{1}{2} & l_p & l_n \end{Bmatrix}^2 I_{npL}{}^2 \delta_{LJ} , \tag{18}$$

$$|M_A|^2 = \tfrac{1}{3} \sum_L \sum_{Jnp} \left(\frac{\nu_{np}}{\nu_\mu}\right)^2 (\hat{j}_n \hat{j}_p \hat{l}_n \hat{l}_p \hat{J} \hat{L})^2\, 6 \begin{pmatrix} l_n & l_p & L \\ 0 & 0 & 0 \end{pmatrix}^2 \begin{Bmatrix} l_n & l_p & L \\ \frac{1}{2} & \frac{1}{2} & 1 \\ j_n & j_p & J \end{Bmatrix}^2 I_{npL}{}^2 , \tag{18'}$$

$$|M_P|^2 = \sum_{Jnp} \left(\frac{\nu_{np}}{\nu_\mu}\right)^2 (\hat{j}_n \hat{j}_p \hat{l}_n \hat{l}_p \hat{J})^2\, 6 \left| \sum_L (-i)^L \hat{L}^2 \begin{pmatrix} J & 1 & L \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_n & l_p & L \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} l_n & l_p & L \\ \frac{1}{2} & \frac{1}{2} & 1 \\ j_n & j_p & J \end{Bmatrix} I_{npL} \right|^2 . \tag{18''}$$

$|M_P|^2$ contains cross terms in L and L'; assuming the LRT conditions (e.g. a doubly closed proton shell) leads to cancellation of these cross terms (see Appendix I), so that:

$$|M_P|^2 = \sum_L \sum_{Jnp} \left(\frac{\nu_{np}}{\nu_\mu}\right)^2 \hat{L}^4 (\hat{j}_n \hat{j}_p \hat{l}_n \hat{l}_p \hat{J})^2\, 6 \begin{pmatrix} J & 1 & L \\ 0 & 0 & 0 \end{pmatrix}^2 \begin{pmatrix} l_n & l_p & L \\ 0 & 0 & 0 \end{pmatrix}^2 \begin{Bmatrix} l_n & l_p & L \\ \frac{1}{2} & \frac{1}{2} & 1 \\ j_n & j_p & J \end{Bmatrix}^2 I_{npL}{}^2 . \tag{19}$$

Thus,

$$|M_{V,A,P}|^2 = \sum_L |M_{V,A,P}|_L{}^2 \tag{20}$$

and, again if the LRT conditions are filled, orthogonality of $6j$ and $9j$ coefficients, together with the equality of the ν_{np}'s associated with $|M_{V,A,P}|^2$, give rise to:

$$|M_V|_L{}^2 = |M_A|_L{}^2 = |M_P|_L{}^2 ,$$

$$|M_V|_L{}^2 = \sum_{l_n l_n j_n} \left(\frac{\nu_{np}}{\nu_\mu}\right)(\hat{L}\,\hat{l}_p\,\hat{j}_n)^2 \begin{pmatrix} l_n & l_p & L \\ 0 & 0 & 0 \end{pmatrix}^2 I_{npL}{}^2 . \tag{21}$$

Disappearance of $\sum_{jp}$ corresponds to the fact that a doubly closed proton shell is involved. Finally,

$$|M_V|_{l_n l_p L}{}^2 = \sum_{j_n} (2j_n+1)\left[\left(\frac{\nu_{np}}{\nu_\mu}\right)^2 (2L+1)(2l_p+1)\begin{pmatrix} l_p & l_n & L \\ 0 & 0 & 0 \end{pmatrix}^2 I_{npL}{}^2\right]$$
$$= |M_A|_{l_n l_p L}{}^2 = |M_P|_{l_n l_p L}{}^2 . \tag{22}$$

Thus, the expressions $|M_{V,A,P}|^2$ are sums of $|M_{V,A,P}|_L{}^2$ which are themselves, sums of $|M_{V,A,P}|_{L l_n l_p}{}^2$.

Furthermore, some equalities are due to summations on both j_n and j_p, i.e., they are typical of doubly closed-shell nuclei, in particular, the following relations between $|M_{V,A,P}|_{L,J}{}^2$

$$|M_A|_{L,L+1}{}^2/|M_A|_{L,L}{}^2/|M_A|_{L,L-1}{}^2 = 2L+3/2L+1/2L-1 , \tag{23}$$

$$|M_V|_{L,L}{}^2 = 3|M_A|_{L,L}{}^2 . \tag{24}$$

The non-LRT transitions, if there are any, come from surface shells in nuclei. At first sight, the largest deviations from equality of $|M_{V,A,P}|^2$ should come in nuclei with half-closed shells in both protons and neutrons which yield non-LRT $L=0$ transitions, for instance ($1p_{3/2}\to 1p_{1/2}$) in ^{12}C. Also, $1f_{7/2}\to 1f_{5/2}$ transitions in the Co region were investigated in the past to try to differentiate $|M_V|^2$ and $|M_A|^2$ out of the total muon-capture rate.[16] Table II displays the quantities $|M_{V,A,P}|^2$ for several nuclei. ^{40}Ca and ^{140}Ce involve only LRT transitions so that $|M_V|^2 = |M_A|^2 = |M_P|^2 = |M_{V,A,P}|_{\rm LRT}{}^2$. In ^{88}Sr, the only non-LRT transition is $2p_{3/2}\to 1g_{7/2}$ ($\Delta l=3$). Non-LRT effects are expected to be large in ^{60}Ni and ^{114}Sn which have half-filled surface shells both in protons and neutrons. The table indicates that deviations from equality are minor for $|M_{V,A,P}|^2$ including non-LRT transitions with change of orbital momentum $\Delta l>1$ ($\Delta l=2$ in ^{114}Sn) and more significant for $|M_{V,A,P}|^2$ including a non-LRT transition with $\Delta l\leq 1$ ($\Delta l=0$ in ^{60}Ni).

Although these results do not apply to transitions to states of definite angular momentum J^π, it has been shown in Ref. 6 that they are useful for investigating equalities between $|M_{V,A,P}|_{0^-,1^-,2^-}{}^2$. Such quantities are almost equal to $|M_{V,A,P}|_{L=1}{}^2$ in light $N=Z$ nuclei previously studied in the frame of the Tamm-Dancoff method.[17,18] For heavier nuclei, Table III shows that the $|M_{V,A,P}|_{0^-,1^-,2^-}{}^2$ still reflect properties from the equalities of $|M_{V,A,P}|_{L=1}{}^2$. We remember that for LRT transitions,

$$|M_{V,A,P}|_{0^-,1^-,2^-}{}^2 = |M_{V,A,P}|_{L=1}{}^2 + |M_{V,A,P}|_{L=3}{}^2 \tag{25}$$

and for non-LRT transitions,

$$|M_{V,A,P}|_{0^-,1^-,2^-}{}^2 = |M_{V,A,P}|_{L=1}{}^2 + |M_{V,A,P}|_{L=3(2^-)}{}^2 + (L, L'=1,3) , \tag{25'}$$

TABLE II. Fluctuations from equalities between $|M_{V,A,P}|^2$, without spin-orbit energy difference. The last line yields the contribution to $|M_{V,A,P}|^2$ from LRT transitions only; for instance, in ^{208}Pb, 3.599 means $|M_{V,A,P}|^2$ summed over all transitions but $1h_{11/2}\to 1i_{11/2}$. In this table as well as in Tables III and IV, the 4 place precision is obviously not physically founded but is a useful test of the accuracy of equalities between $|M_{V,A,P}|^2$.

	^{40}Ca	^{60}Ni	^{88}Sr	^{114}Sn	^{140}Ce	^{208}Pb
$\|M_V\|^2$		4.623	4.548	5.853		3.633
$\|M_A\|^2$		5.580	4.576	6.044		4.696
$\|M_P\|^2$		5.111	4.585	6.075		4.662
$\|M_{V,A,P}\|_{\rm LRT}{}^2$	3.987	4.558	4.535	5.848	5.137	3.599

TABLE III. Comparison of the expressions $|M_{V,A,P}|_{0^-,1^-,2^-}{}^2$ over several nuclei. A similar table has been given in Ref. 6 with less nuclei and with more details on the effect of $|M_{A,P}|_{L=3(2^-)}{}^2$ on the breaking of equalities of the $|M_{V,A,P}|^2$.

0^-, 1^-, 2^-	^{40}Ca	^{60}Ni	^{88}Sr	^{114}Sn	^{140}Ce	^{208}Pb
$\|M_V\|^2$	3.063	3.169	2.284	2.564	1.724	0.415
$\|M_A\|^2$	3.088	3.211	2.374	2.668	1.835	1.482
$\|M_P\|^2$	3.108	3.246	2.447	2.751	1.922	1.552

where $|M_{V,A,P}|_{L(J^-)}{}^2$ is for a given L and j and $(L, L'=1,3)$ is an interference term coming from $|M_P|^2$.

D. Investigations on the Spin-Orbit Energy Splitting

In the preceding subsection, variations of ν_{np} due to spin-orbit coupling were ignored. The physical basis for expecting small variations in $|M_{V,A,P}|^2$ due to this effect is that, within a given l orbit, $\Delta E < 5$ MeV so that $[\Delta\nu_{np}]/[\nu_{np}] \le 10\%$. The spin-orbit energy coupling $\Delta E \simeq 20\, \vec{l}\cdot\vec{s}/A^{2/3}$ MeV[9] so that in heavier nuclei the presence of configurations with large l's increase the spin-orbit effect relative to the shell spacing. However, its net effect on the outgoing neutrino energy is decreased by its $A^{-2/3}$ dependence.

Some investigations have been carried out on $N=Z$ light nuclei. For transitions involving doubly closed shells in protons and neutrons, Walker[19] shows that the average effect of doubly-closed- to doubly-empty-shell summations cancels the first-order term of an expansion of the square of the outgoing neutrino energy in terms of the spin-orbit splitting, the second order term giving a 1% effect. On less restrictive grounds, Rho[20] considering light $N=Z$ nuclei found a 13% upper limit for fluctuations from equality between the $|M_{V,A,P}|^2$. For $N>Z$ nuclei, in the set of various shell to shell transitions, the derivation of Walker based on angular momentum algebra is still valid for doubly-closed- to doubly-empty-shell transitions. Furthermore, it is possible to take over the method of Rho for LRT transitions as shown below.

The Eq. (5″) is sandwiched between the initial and final nuclear states $|a\rangle$ and $|b\rangle$ where $|a\rangle$ has at least one doubly closed shell in protons (or neutrons) and then is described by $\vec{S}_p$ (or $\vec{S}_n$) = 0.

$$\begin{aligned}\langle b|A_\chi^-(\vec{\nu})|a\rangle &= -\tfrac{1}{2}\langle b|[Y^3, V^-(\vec{\nu})]|a\rangle \\ &= \tfrac{1}{2}\langle b|[\sum_i \vec{\sigma}_i - 2\sum_i[(1+\tau_i{}^3)/2]\vec{\sigma}_i, \sum_i e^{-i\vec{\nu}\cdot\vec{x}}\tau_i^-]|a\rangle \\ &= -\langle b|[\sum_i[(1+\tau_i{}^3)/2]\vec{\sigma}_i, \sum_i e^{-i\vec{\nu}\cdot\vec{x}}\tau_i^-]|a\rangle \\ &= -\langle b|\vec{S}_p V^-(\vec{\nu})|a\rangle .\end{aligned} \tag{26}$$

From now on, the Rho treatment of the perturbation due to the factor $(\nu_{ab})^2$ in $|M_{V,A,P}|^2$ may be exactly followed,[20] so that

$$(1-x)^2 \le \frac{M_A'^2}{M_V'^2} = \frac{\bar{\sum_a}\sum_b \left(\frac{\nu_{ab}}{\nu_\mu}\right)^2 \langle b|A^-(\vec{\nu}_{av})|a\rangle^2}{\sum_a\sum_b \left(\frac{\nu_{ab}}{\nu_\mu}\right)^2 \langle b|V^-(\vec{\nu}_{av})|a\rangle^2} \le (1+x)^2 . \tag{27}$$

TABLE IV. Quantities $|M_{V,A,P}|^2$ with spin-orbit effects.

	^{40}Ca	^{60}Ni	^{88}Si	^{114}Sn	^{140}Ce	^{208}Pb
$\lvert M_V\rvert^2$		4.609	4.515	5.867		3.625
$\lvert M_A\rvert^2$		5.344	4.458	5.889		4.512
$\lvert M_P\rvert^2$		4.902	4.472	5.936		4.480
$\lvert M_V\rvert_{\rm LRT}{}^2$	3.991	4.556	4.505	5.863	5.106	3.592
$\lvert M_A\rvert_{\rm LRT}{}^2$	4.004	4.311	4.425	5.707	5.046	3.444
$\lvert M_P\rvert_{\rm LRT}{}^2$	4.002	4.304	4.431	5.723	5.062	3.420

$|M'_{V,A,P}|^2$ is $|M_{V,A,P}|^2$ with $\vec{\nu}_{ab}$ replaced by an average $\vec{\nu}_{av}$ in $V^-(\vec{\nu})$ and $A^-(\vec{\nu})$, and x is the averaged value $\langle|\Delta E|\rangle/\nu_{av}$. Rho gets a maximum fluctuation of 13%, assuming $\langle|\Delta E|\rangle \simeq 5$ MeV. Now, $\langle|\Delta E|\rangle$ is smaller in heavier nuclei.

Variations of $\exp(-i\nu r)$ which are essentially negligible in light nuclei are still very small when higher retardation effects are considered. Such a point of view is illustrated by Fig. 1 which shows $\int\mathcal{O}^\nu$ as a function of ν. Table IV illustrates this weak over-all effect of spin-orbit energy coupling since the numerical values displayed are extremely close to the results of Table II.

III. EFFECT OF RESIDUAL INTERACTIONS

The two-nucleon interaction induces a coupling between the possible particle-hole excitations considered in Sec. II so that the final state after muon absorption is a linear combination of neutron-particle–proton-hole states. It is known, too, that these residual interactions contain a spin-dependent part which, breaking the SU_4 invariance of the

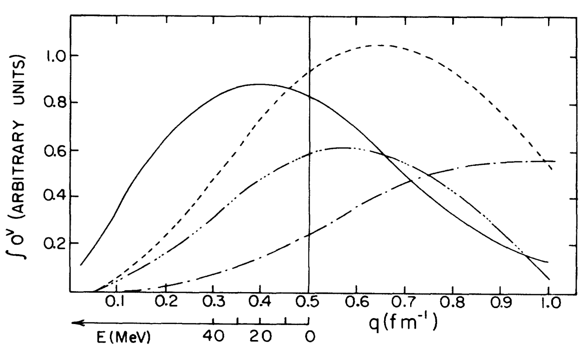

FIG. 1. $\int\mathcal{O}^\nu$ as the ordinate plotted versus momentum transfer q as the abscissa for a few hole-particle transitions in ^{88}Sr. The lower abscissa corresponds to the energy of the residual nucleus. The symbols are as follows: —··—, $1f_{7/2}\to 2f_{7/2}, \Delta l=0$; ——, $2p_{3/2}\to 2d_{5/2}, \Delta l=1$; ------, $1f_{7/2}\to 1h_{11/2}, \Delta l=2$; —·—·—, $1d_{3/2}\to 1h_{9/2}, \Delta l=3$.

nuclear Hamiltonian, may affect the equality between the $|M_{V,A,P}|^2$. Previous investigations have been carried out for $N=Z$ nuclei in the frame of the Tamm-Dancoff approximation, with the main result that configuration mixing did not affect appreciably the pure shell-model result concerning $|M_{V,A,P}|_{L=1}^2$.[17,18] Such a calculation is presented below for a medium-heavy nucleus ($N>Z$):

$$\mu^- + {}^{88}\mathrm{Sr} \rightarrow \nu_\mu + {}^{88}\mathrm{Rb}^*(0^-, 1^-, 2^-)$$

in order to investigate to what extent conclusions upon light $N=Z$ nuclei keep their validity.

Proton shells are supposed filled up to the $2p_{3/2}$ shell, and the 12 excess neutrons fill the $2p_{1/2}$ and $1g_{9/2}$ shells. Eleven 1p-1h configurations span the excited levels space (8, 11, and 3 configurations correspond, respectively, to 1^-, 2^-, and 0^- excitations). Table V displays these neutron-particle-proton-hole excitations and their unperturbed energies taken from Zamischa and Werner.[21] The wave functions are harmonic-oscillator states with a parameter $f=2.12$ fm. The residual interaction is a Soper-type interaction[22] which has the following expression:

$$V(1,2) = V_0[V_{CTE}\pi^T\pi^E + V_{CTO}\pi^T\pi^O + V_{CSE}\pi^S\pi^E + V_{CSO}\pi^S\pi^O]\, e^{-\beta^2 r_{12}^2}, \tag{28}$$

where π^T, π^S, π^E, π^O, are, respectively, the projection operators on triplet, singlet, even, and odd states:

$$V_{CTE}=1.0, \quad V_{CTO}=0.14, \quad V_{CSO}=-0.4,$$
$$V_{CSE}=0.46, \quad V_O=-45 \text{ MeV}, \quad \beta^2=0.3086 \text{ fm}^{-2}.$$

Numerical calculations have been performed with the SMERCH program set up by Hughes.[22] The resulting states $|i\rangle$ and their components a_k^i are listed in Table V. For each of the angular momenta involved ($0^-, 1^-, 2^-$) the highest energy state has a strong ($1f_{7/2}$-$1g_{7/2}$) component. The 1^- states at 10.3 and 9.5 MeV are essentially composed of ($1f_{5/2}$-$1g_{7/2}$), ($1f_{5/2}$-$1d_{3/2}$), ($1f_{7/2}$-$1d_{5/2}$), and ($1f_{7/2}$-$1g_{7/2}$) components. The $1f_{7/2}$ proton-hole energy being obtained from simple estimates and not directly from experiments, this induces some uncertainty on the numerical value of the energy of the highest level but does not alter the remarks mentioned above. Indeed, this calculation is not performed in view of a detailed comparison with experimental results but rather in order to appreciate the modifications brought about by a reasonable particle-hole interaction. This same calculation done with the set of unperturbed energies taken from Ref. 22 which involves a deeper $1f_{7/2}$ hole energy yields the three highest 1^- states at, respectively, 15.1, 12.3, and 10.2 MeV.

A. Distribution of $|M_{V,A,P}|^2$

Table VI shows the same features as similar tables previously published on $N=Z$ nuclei,[17,18] i.e., a concentration of the dipole strength at the highest energy levels for ^{88}Sr, most of the dipole strength corresponding to $|M_A|^2$ is robbed by the highest energy 1^- state, in obvious connection with the presence of the "spin-flip" transition ($1f_{7/2}$-$1g_{7/2}$). $|M_V|^2$ strength is concentrated at the second and third highest energy levels, corresponding to roughly "non-spin-flip" transitions. Near equality for

$$|M_{V,A,P}|_{0^-,1^-,2^-}^2 \text{ i.e.}$$
$$|M_V|_{1^-}^2/|M_A|_{0^-,1^-,2^-}^2/|M_P|_{0^-,2^-}^2 = 1.00/1.01/1.03 \tag{29}$$

reflects not so much the fact that the supermultiplet scheme is very close to the j-j shell model, but rather the fact that the energy shifts on "spin-

TABLE V. Energies and wave functions for the $T=5$, $J^\pi=1^-$ states in ^{88}Rb. $J^\pi=0^-$ and 2^- states are not shown in this table in order to simplify the presentation. Unperturbed configuration energies are labeled under the basis states. Energies of the $J^\pi=0^-$ states are given with origin at the ground state of ^{88}Sr, energies found after diagonalization were shifted in order to get the lowest 2^- state at the ground state of ^{88}Rb (5.2 MeV).

Energy (MeV) $\|k\rangle$	$2p_{3/2}\rightarrow 2d_{5/2}$	$1f_{5/2}\rightarrow 2d_{5/2}$	$2p_{3/2}\rightarrow 3s_{1/2}$	$2p_{3/2}\rightarrow 2d_{3/2}$	$1f_{5/2}\rightarrow 2d_{3/2}$	$1f_{5/2}\rightarrow 1g_{7/2}$	$1f_{7/2}\rightarrow 2d_{5/2}$	$1f_{7/2}\rightarrow 1g_{7/2}$
$i\rangle$	3.75	4.05	4.95	6.55	6.85	6.85	6.90	9.70
12.83	0.002	−0.020	0.080	−0.187	0.028	0.428	0.115	−0.872
10.33	−0.161	−0.042	−0.164	0.100	−0.443	0.524	−0.564	−0.382
9.51	−0.050	0.042	0.030	0.114	0.402	−0.682	0.528	−0.275
8.37	0.034	0.149	−0.067	0.158	0.771	0.058	−0.587	−0.067
8.02	0.025	−0.015	−0.090	0.949	−0.120	0.222	0.136	−0.089
6.82	−0.319	0.014	−0.917	−0.118	0.068	0.118	0.157	0.021
6.11	0.919	0.157	−0.329	−0.049	−0.081	−0.094	0.009	−0.068
5.48	−0.153	0.974	0.068	0.004	−0.143	0.024	0.043	−0.003

flip" and "non-spin-flip" excitations induced by the residual interaction do not affect significantly the conclusions of Sec. II. Such a near equality was also found for ^{28}Si and ^{32}S which are not doubly-closed shells.[17] The mechanism of compensation of the various effects breaking the equality of $|M_{V,A,P}|_{0^-,1^-,2^-}{}^2$ in the framework of the Tamm-Dancoff approximation will be published elsewhere.

On the other hand, some equalities found for doubly-closed-shell $N=Z$ nuclei and originating from Eqs. (23) and (24) are no longer valid. This is the case of the relationships between the expressions D and S_J defined below.[2, 23]

$$|M_V|_{\text{UD}}{}^2 = \sum_{1^- \text{ states}} \left(\frac{\nu_{ab}}{\nu_\mu}\right)^2 D\,, \tag{30}$$

$$|M_A|_{\text{UD}}{}^2 = \frac{1}{3}\sum_{0^-}\left[\left(\frac{\nu_{ab}}{\nu_\mu}\right)^2 S_0 + \sum_{1^-}\left(\frac{\nu_{ab}}{\nu_\mu}\right)^2 S_1 + \sum_{2^-}\left(\frac{\nu_{ab}}{\nu_\mu}\right)^2 S_2\right], \tag{30'}$$

where

$$D = D_{\nu_{ab}R\ll 1} = 4\pi\left|\left\langle b_{T+1,\,1^-}\left\|\sum_{i=1}^{A}\tau_i^-\mathfrak{M}_1(\Omega_i)\right\|a_{T,\,0^+}\right\rangle\right|^2, \tag{31}$$

$$S_J = (S_{\nu_{ab}R\ll 1})_J = 4\pi\left|\left\langle b_{T+1,\,J}\left\|\sum_{i=1}^{A}\tau_i^-[\mathfrak{M}_1(\Omega_i)\otimes\vec{\sigma}_i]_J\right\|a_{T,\,0^+}\right\rangle\right|^2. \tag{31'}$$

Then, introducing the quantity $\eta=\nu_{ab}f$,

$$\sum_{0^-} S_0/\eta^2 \Big/ \sum_{1^-} S_1/\eta^2 \Big/ \sum_{2^-} S_2/\eta^2 = 1/2.4/2.15\,, \tag{32}$$

$$\sum D/\eta^2 \Big/ \sum S_1/\eta^2 = 1.18/1.0\,. \tag{32'}$$

The sequence 1/3/5 is not satisfied in Eq. (32), indeed the presence of extra neutrons in $1g_{9/2}$ shell does not quench the $0^+\to 0^-$ transitions but decreases the $0^+\to 1^-$ and $0^+\to 2^-$. The amount of this shell effect depends on each specific case.

B. Interpretation in the $\vec{L}\cdot\vec{S}$ Coupling

The SU_4 model for $N=Z$ nuclei leads to $L=1$ resonant states, members of the same supermultiplet $L=1$, $S=0$, $S=1$; the various members of such a supermultiplet are degenerate and well separated from members of other $(L\neq 1)$ supermultiplets. Introduction of a spin-dependent force: (a) introduces mixing of supermultiplets and (b) splits the degeneracy within the same supermultiplet. The squared amplitudes $|A_{LSJ}|^2$ corresponding to the $(L=1, S=0, 1, J=1^-)$ and $(L=2, S=1, J=1^-)$ modes have been calculated (see Appendix II) and displayed in Table VII; we may see that, although the $|A_{LSJ}|^2$ are spread over the various 1^- states, the highest % of $|A_{L=1,\,S=1}|^2$ corresponds to the highest energy state and highest $|M_A|^2$, the highest % of $|A_{L=1,\,S=0}|^2$ correspond to a large concentration of $|M_V|^2$ in the second and third levels, which is not the case for $|A_{L=2,\,S=1}|^2$, so the simple picture of spin-flip and non-spin-flip states is still quite reasonable, even for a nucleus like ^{88}Sr, as it is in lighter $N=Z$ nuclei.[21] The calculation done in this section was also done with a different set of single-particle energies from Ref. 22 which give rise to the same trend; However, differences produced in the individual levels show that the results presented in this section describe the overall features of a strength distribution and not its details. The somewhat unexpected relevance of the $\vec{L}\cdot\vec{S}$ picture in a medium-heavy nucleus recalls that there is not a direct conflict between the j-j and $\vec{L}\cdot\vec{S}$ model; it has been mentioned already in Sec. IID that spin-orbit energy splittings $(\Delta E = E_{j=l-\frac{1}{2}} - E_{j=l+\frac{1}{2}})$ get higher values for higher l relative to shell spacing; this effect is not critical for muon-capture transitions. On the other hand,

TABLE VI. Numerical values of the quantities of the expressions $|M_{V,A,P}|^2$ for the various 0^-, 1^-, 2^- states of ^{88}Rb. Energies are given following the same convention as in Table V.

J^π	Energy (MeV)	$\|M_V\|^2$	$\|M_A\|^2$	$\|M_P\|^2$
0^-	13.45	...	0.368	1.103
	8.31	...	0.036	0.109
	5.91	...	0.009	0.028
1^-	12.83	0.576	0.772	...
	10.33	0.985	0.000	...
	9.51	0.235	0.016	...
	8.37	0.005	0.036	...
	8.02	0.001	0.044	...
	6.82	0.105	0.005	...
	6.11	0.249	0.103	...
	5.48	0.000	0.004	...
2^-	11.47	...	0.269	0.197
	11.18	...	0.004	0.014
	8.95	...	0.159	0.404
	8.45	...	0.019	0.031
	8.15	...	0.010	0.000
	7.88	...	0.051	0.016
	7.70	...	0.011	0.016
	6.37	...	0.011	0.000
	6.32	...	0.096	0.127
	5.31	...	0.134	0.191
	5.20	...	0.016	0.006
		2.157	2.174	2.228

$j = l \pm \frac{1}{2} \to l' \pm \frac{1}{2}$ transitions describe more closely a non-spin-flip transition (notion exact in the $\vec{L}\cdot\vec{S}$ model) when higher l's are involved. This is illustrated in the following sequence giving ratios of angular parts[24] of $|M_V|^2$, which decrease significantly when l increases:

$$\left\{\frac{l+\frac{1}{2}\to l'-\frac{1}{2}}{l+\frac{1}{2}\to l'+\frac{1}{2}}\right\} = \left\{\frac{3s_{1/2}\to 2p_{1/2}}{3s_{1/2}\to 2p_{3/2}}\right\} \Big/ \left\{\frac{2d_{3/2}\to 2p_{3/2}}{2d_{5/2}\to 2p_{3/2}}\right\} \Big/ \left\{\frac{2d_{5/2}\to 1f_{5/2}}{2d_{5/2}\to 1f_{7/2}}\right\} \Big/ \left\{\frac{1g_{9/2}\to 1h_{9/2}}{1g_{9/2}\to 1h_{11/2}}\right\}$$
$$=0.70/0.33/0.21/0.14 . \quad (33)$$

C. Comparison with Other Results

The Tamm-Dancoff method, with use of the constants given in the Introduction and without the relativistic corrections leads to $\Lambda_\mu^{\rm TD}(0^+\to 0^-, 1^-, 2^-) = 71.8\times 10^5\ {\rm sec}^{-1}$. This can be compared with the FW method, where $|M_V|_{\rm UD}{}^2$ is taken from Leprêtre *et al.*[11] photoneutron measurements ($\sigma_{-1}=80$ mb, $E_{T+1}=20$ MeV) and the elastic form factor from a shell-model calculation $|F_{\rm el}(\nu=94)|^2=0.22$. Thus,

$$\Lambda_\mu^{\rm FW}(0^+\to 0^-, 1^-, 2^-) = 315\frac{(Z_{\rm eff}=25.15)^4}{2}|F_{\rm el}(E_M-E_{T+1})|^2|M_V|_{\rm UD}{}^2 = 40\times 10^5\ {\rm sec}^{-1}, \quad (34)$$

with $|M_V|_{\rm UD}{}^2$ given by Eq. (12).

Equation (34), based in fact upon the approximate Eq. (9′), yields, as for $N=Z$ nuclei, a once-forbidden muon-absorption rate significantly smaller than the shell model and the Tamm-Dancoff estimate. It is of interest to remember the total muon-capture rate by ^{88}Sr as measured by Eckhause *et al.*[25] to be $(66.1\pm 2.7)\times 10^5\ {\rm sec}^{-1}$, so that the ratio of {dipole/total} muon absorption (≃60%) is still roughly equal to the number given in Table I by the pure shell model (≃50%).

To end this section, Table VIII yields some muon-capture rates calculated in the pure shell model and the corresponding experimental values, in spite of the wide spectrum of nuclei considered, the ratios $\Lambda_{\rm th}/\Lambda_{\rm exp}$ stay near 2 as the relative importance of multipole strength is greatly changed (Table I). This is an indication that the renormalization effects stay approximately the same for various multipoles. It might be interesting to investigate this point further in the future.

TABLE VII. Probabilities $|A_{L,S}|^2$ of finding an orbital momentum L and a spin S for a 1^- excited state of ^{88}Rb. The $|A_{LS}|^2$ are normalized so that for each level the sum of the three probabilities is equal to unity.

E (MeV)	$\|A_{L=1,S=0}\|^2$	$\|A_{L=1,S=1}\|^2$	$\|A_{L=2,S=1}\|^2$
12.83	0.17	0.70	0.13
10.33	0.64	0.02	0.35
9.51	0.61	0.02	0.37
8.37	0.01	0.61	0.38
8.02	0.08	0.55	0.37
6.82	0.64	0.35	0.01
6.11	0.60	0.26	0.14
5.48	0.03	0.38	0.59

IV. CONCLUSION

This work was concerned with muon-capture transitions in nuclei with excess neutrons and more specifically with the investigation of the quantities $|M_{V,A,P}|^2$; the purpose was to extend various results and assumptions previously obtained for $N=Z$ relaxing the equality between the number of protons and neutrons. The essence of the method may be summed up as follows: The relations between the current densities defined in the Introduction follow a $SU_2\otimes SU_2$ current algebra which gives rise to sum rules by sandwiching the commutation relations of this algebra between adequate states; then, relations connecting the $|M_{V,A,P}|^2$ between themselves and other related quantities are obtained by putting in the neutrino energy factor $(\nu_{ab}/\nu_\mu)^2$ and the integration over $\hat{\nu}$. Previous investigations on muon capture dealing with doubly-closed-shell ($N=Z$) nuclei were simplified by the cancellation of the second term of the commutator. Indeed, this second term is still zero for a new class of nuclei where T and S are not necessarily zero.

In Sec. I, a relation between $|M_V|^2$ representing the polar-vector part of muon capture and the analog electromagnetic matrix element is shown to hold for any stable $N>Z$ nucleus, this quality being separately valid for each multipole; the results of increasing retardation effects and the number of

TABLE VIII. Comparison of pure shell-model calculations and experimental total capture rates. Units are 10^{-5} sec. Λ_0, Λ_1, Λ_2 are, respectively, the leading v/c and $(v/c)^2$ terms.

	^{40}Ca	^{60}Ni	^{88}Sr	^{114}Sn	^{140}Ce	^{208}Pb
Λ_0	43.7	117.8	148.6	258.9	246.7	226.0
Λ_1	2.9	6.6	13.5	18.8	22.7	23.9
Λ_2	1.2	3.2	4.8	6.7	6.9	7.3
Λ_T	47.8	127.6	166.9	284.4	276.3	257.2
$\Lambda_{\rm exp}$	24.5	59.2	66.1	106.8	114.4	130.2
$\Lambda_T/\Lambda_{\rm exp}$	1.95	2.16	2.52	2.66	2.42	1.98

extra neutrons upon the importance of once-forbidden ($L=1$) transitions is shown in Table I. The case of dipole transitions is considered in more detail, particularly the extension of the FW method to $N>Z$ nuclei. Section II investigates the equalities between the $|M_{V,A,P}|^2$ in the frame of a pure j-j shell model; the breaking of these equalities comes from the presence of a spin-dependent part in the nuclear Hamiltonian so that the one-particle states corresponding to definite (n, l) quantum numbers are split by a spin-orbit coupling and do not have a good spin projection. In a first step where energy splittings are not considered, application of Eq. (13) and of the notion of LRT transition defined in Sec. II A leads to equalities between the $|M_{V,A,P}|^2$ and $|M_{V,A,P}|_L{}^2$ for a nucleus with the highest shell closed either in protons or neutrons. A more conventional shell-model analysis shows that such equality is correct for each partial quantity $|M_{V,A,P}|_{l_p l_n L}{}^2$; then examples are given to illustrate the effects of non-LRT transitions in $|M_{V,A,P}|^2$. It is finally shown that these notions are still quite useful to analyze the quantities $|M_{V,A,P}|_{0^-,1^-,2^-}{}^2$ corresponding to excitations of parent analog of the $T+1$ component of the electromagnetic giant dipole resonance by $L=1$ photonuclear reaction. The influence of spin-orbit coupling on $|M_{V,A,P}|^2$ is investigated, starting from previous results on $N=Z$ nuclei, and remembering that the fluctuations of the outgoing neutrino energy ν_{ab} due to spin-orbit coupling in the final nucleus do not go beyond a few percent. A result worked out before by Rho for light $N=Z$ nuclei is extended to $N>Z$ nuclei with one doubly closed shell in protons or neutrons. The effect of residual interactions which are known to contain a spin-dependent part is investigated with a definite example; muon absorption by ^{88}Sr is calculated by the Tamm-Dancoff method, using a Soper mixture already used elsewhere for other purposes. The result is a near equality of $|M_{V,A,P}|_{0^-,1^-,2^-}{}^2$ (Table VI) and a nonnegligible amount of spin-flip ($S=1$, $L=1$) strength for the highest levels and of non-spin-flip ($S=0$, $L=1$) strength for levels characterized by high $|M_V|^2$. Now, to explain the somewhat surprising result of the validity of the $\vec{L}\cdot\vec{S}$ description, it is useful to remember that for increasing orbital angular momentum transfers, nucleons have a better defined spin transition. Thus, the general trend is that the kinematic structure of the neutrino-energy factor in $|M_{V,A,P}|^2$ minimizes the effects of the breaking of SU_4 symmetry. It is also noticed that the conventional shell-model techniques reproduce, for $N>Z$, as well as for $N=Z$ nuclei, the factor of 2 discrepancy between theory and experiment.

This brings us to the limitations of this work which amount to those of the standard shell model under its present form. Although the inadequacy of the shell model to reproduce the experimental $|M_V|^2$ may cast some doubts on the near equality of $|M_{V,A,P}|^2$ based on this same model, it is reminded that a decrease of $|M_{A,P}|^2$ to arrive at the muon-capture experimental result would not solve the problem of $|M_V|^2$ itself which is directly connected to the gdr; the FW method, taking $|M_V|_{L=1}{}^2$ essentially from experimental data and letting the $|M_{V,A,P}|^2$ be equal, yields a result much closer to the experimental total-absorption rate; for $N>Z$ nuclei, where the increasing importance of other than dipole transitions does not allow a direct comparison between theory and experiment, our results at the end of Sec. III indicate that a similar renormalization of the shell-model results affects the various multipoles $|M_V|_L{}^2$, with the quantities $|M_{V,A,P}|_L{}^2$ staying nearly equal.

A last aspect of this extension to heavier nuclei is the use of the connection between dipole states obtained by muon capture and by photoabsorption; location of parents of the analog dipole states seems experimentally confirmed for a few $N=Z$ nuclei.[26] Such an identification for $N>Z$ nuclei may help to elucidate the analog character of the upper component of the gdr. These parent states obtained by muon capture stand higher than the particle threshold so that their identification necessitates the introduction of nuclear-reaction-theory methods. Forthcoming pion factories should put muon-capture physics on the same footing as photoabsorption. To complement photonuclear reactions, the new field of muonuclear reaction is opening soon.

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APPENDIX I

This Appendix shows that the LRT condition leads to the cancellation of the cross term of $|M_P|^2$. In Eq.

(18″), use of the equality[27]

$$\begin{Bmatrix} l_n & l_p & L \\ \frac{1}{2} & \frac{1}{2} & 1 \\ j_n & j_p & J \end{Bmatrix}_{L\neq J} = (-1)^{L+J} \begin{Bmatrix} j_n & j_p & J \\ \frac{1}{2} & \frac{L+J}{2} & l_p \end{Bmatrix} \begin{Bmatrix} l_n & l_p & L \\ \frac{1}{2} & \frac{L+J}{2} & j_n \end{Bmatrix} \begin{Bmatrix} L & J & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{L+J}{2} \end{Bmatrix}^{-1} \qquad \text{(AI 1)}$$

leads to the following expression for the cross term $(L, L' = J-1, J+1)$:

$$\sum_{Jnp} \left(\frac{\nu_{np}}{\nu_\mu}\right)^2 (\hat{j}_n \hat{j}_p \hat{l}_n \hat{l}_p \hat{J}) 6 \begin{pmatrix} J & 1 & J-1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} J & 1 & J+1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_n & l_p & J-1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_n & l_p & J+1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\times \frac{1}{9} \begin{Bmatrix} j_n & j_p & J \\ \frac{1}{2} & \frac{2J-1}{2} & l_p \end{Bmatrix} \begin{Bmatrix} j_n & j_p & J \\ \frac{1}{2} & \frac{2J+1}{2} & l_p \end{Bmatrix} \begin{Bmatrix} l_p & l_n & J-1 \\ \frac{1}{2} & \frac{2J-1}{2} & j_n \end{Bmatrix} \begin{Bmatrix} l_p & l_n & J+1 \\ \frac{1}{2} & \frac{2J+1}{2} & j_n \end{Bmatrix}$$

$$\times \begin{Bmatrix} J & J-1 & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{2J-1}{2} \end{Bmatrix}^{-1} \begin{Bmatrix} J & J+1 & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{2J+1}{2} \end{Bmatrix}^{-1} I_{n,p,J-1} I_{n,p,J+1} . \qquad \text{(AI 2)}$$

Assuming that the proton shells are doubly closed, the orthogonality relations of the two $6j$ symbols containing j_p cancels the expression (AI 2), i.e., the cross term of $|M_p|^2$.

APPENDIX II

This appendix gives the probability that a state $|(j_n j_p)J\rangle$ is an eigenstate of the total orbital angular momentum. Thus, the expression to calculate is $\langle (j_n' j_p')J | (j_n j_p) L_0 J\rangle$

$$|(j_n j_p)J\rangle = \sum_{\substack{LS \\ M_L M_S}} \hat{j}_n \hat{j}_p \hat{S} \hat{L} \begin{Bmatrix} l_n & l_p & L \\ \frac{1}{2} & \frac{1}{2} & S \\ j_n & j_p & J \end{Bmatrix} \langle LSM_L M_S | JM\rangle \, | LSM_L M_S\rangle ,$$

$$|(j_n j_p) L_0 J\rangle = \sum_{\substack{S \\ M_{L_0} M_S}} \hat{j}_n \hat{j}_p \hat{S} \hat{L}_0 \begin{Bmatrix} l_n & l_p & L_0 \\ \frac{1}{2} & \frac{1}{2} & S \\ j_n & j_p & J \end{Bmatrix} \langle L_0 S M_{L_0} M_S | JM\rangle \, | L_0 S M_{L_0} M_S\rangle$$

so that

$$\langle (j_n' j_p')J | (j_n j_p) L_0 J\rangle = \sum_{\substack{L'S'S \\ M_{L_0}' M_{L_0}}} \hat{j}_n' \hat{j}_p' \hat{S} \hat{L}' \hat{j}_n \hat{j}_p \hat{S} \hat{L} \begin{Bmatrix} l_n' & l_p' & L' \\ \frac{1}{2} & \frac{1}{2} & S' \\ j_n' & j_p' & J \end{Bmatrix} \begin{Bmatrix} l_n & l_p & L_0 \\ \frac{1}{2} & \frac{1}{2} & S \\ j_n & j_p & J \end{Bmatrix}$$

$$\times \langle L' S' M_{L'} M_{S'} | JM\rangle \langle L_0 S M_L M_S | JM\rangle \, \delta_{l_n' l_n} \delta_{l_p' l_p} \delta_{L' L_0} \delta_{SS'} \delta_{M_{L'} M_{L_0}} \delta_{M_S' M_S} .$$

Orthogonality properties of Clebsch-Gordan coefficients yield:

$$\langle (j_n' j_p')J | (j_n j_p)_{L_0} J\rangle = \sum_S \hat{j}_n' \hat{j}_p' \hat{S}^2 \hat{L}_0^2 \hat{j}_n \hat{j}_p \begin{Bmatrix} l_n' & l_p' & L_0 \\ \frac{1}{2} & \frac{1}{2} & S \\ j_n' & j_p' & J \end{Bmatrix} \begin{Bmatrix} l_n & l_p & L_0 \\ \frac{1}{2} & \frac{1}{2} & S \\ j_n & j_p & J \end{Bmatrix} .$$

The squared amplitude $\langle nJM | nL_0 JM\rangle^2$ for $|nJM\rangle = \sum_\alpha \chi_\alpha^n \, | j_{\alpha n} j_{\alpha p} JM\rangle$ is:

$$|A^n(L_0)|^2 = \sum_{\substack{\alpha\alpha' \\ S}} \chi_{\alpha'}^{n*} \chi_\alpha \hat{j}_{\alpha'}^n \hat{j}_{\alpha'}^p \hat{S}^2 \hat{L}_0^2 \hat{j}_\alpha^n \hat{j}_\alpha^p \begin{Bmatrix} l_{\alpha_n'} & l_{\alpha_p'} & L_0 \\ \frac{1}{2} & \frac{1}{2} & S \\ j_{\alpha_n'} & j_{\alpha_p'} & J \end{Bmatrix} \begin{Bmatrix} l_{\alpha_n} & l_{\alpha_p} & L_0 \\ \frac{1}{2} & \frac{1}{2} & S \\ j_{\alpha_n} & j_{\alpha_p} & J \end{Bmatrix} \delta(l_{n_{\alpha'}} l_{n_\alpha}) \delta(l_{p_\alpha} l_{p_{\alpha'}}) .$$

The $|A_{L,S}|^2$ of Table VII are then obtained by fixing S in the above expression.

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[1]R. Feynman and M. Gell-Mann, Phys. Rev. 109, 193 (1958).

[2]L. L. Foldy and J. D. Walecka, Nuovo Cimento 34, 1026 (1964). In this article, they are referred to as FW.

[3]The most recent and comprehensive review on this matter is given by J. D. Walecka, Semi-leptonic Weak Interactions in Nuclei, in Muon Physics, edited by W. W. Hughes and C. S. Wu (Academic, to be published).

[4]S. Fallieros and B. Goulard, Nucl. Phys. A147, 593 (1967).

[5]R. Leonardi and M. Rosa-Clot, Riv. Nuovo Cimento 1, 1 (1971); J. V. Noble, Ann. Phys. (N.Y.) 67, 98 (1971).

[6]B. Goulard, J. Joseph, and F. Ledoyen, Phys. Rev. Letters 27, 1238 (1971).

[7]See for example, B. Goulard, G. Goulard, and H. Primakoff, Phys. Rev. 133, B186 (1964).

[8]See for example, J. D. Bjorken and M. Nauenberg, Ann. Rev. Nucl. Sci. 18, 228 (1968).

[9]A. Bohr and B. R. Mottelson, in *Nuclear Structure* (Benjamin, New York, 1969).

[10]R. O. Akyüz and S. Fallieros, Phys. Rev. Letters 27, 1016 (1971). Equation (9) established by these two authors, was quoted erroneously in Ref. 6, the rest of which was not affected by this misprint. The relation between the isodoublet interval and the symmetry energy ($E_{T+1}-E_T=[(T+1)/T]U$) was given in the weak coupling limit by S. Fallieros, B. Goulard, and R. H. Venter, Phys. Letters 19, 398 (1965). Modifications of this relation by a residual interaction were investigated in the Tamm-Dancoff method on ^{88}Sr by B. Goulard, T. A. Hughes, and S. Fallieros, Phys. Rev. 176, 1345 (1968). See also R. Leonardi, Phys. Rev. Letters 28, 836 (1972) who relates U to a few physical parameters for several nuclei.

[11]Among recent work, see for example, P. Paul, J. F. Amman, and K. A. Snover, Phys. Rev. Letters 27, 1013 (1971) who use (p,γ) reactions; H. Arenhovel and J. M. Maison, Nucl. Phys. A147, 305 (1970), with photon elastic scattering; A. Leprêtre, H. Beil, R. Bergère, P. Carlos, A. Veyssière, and M. Sugawara, Nucl. Phys. A175, 609 (1971) with (γ,n) reactions. Other references are mentioned in Ref. 4 of this work.

[12]C. Ngo-Trong and D. J. Rowe, Phys. Letters 36B, 553 (1971). See also the last paper mentioned in Ref. 10.

[13]FW were concerned with $N=Z$ nuclei; for $N>Z$ nuclei see P. Hrasko, Phys. Letters 28B, 470 (1969); B. Goulard and H. Primakoff, Phys. Rev. 135, B1139 (1964) (Appendix of this paper).

[14]J. R. Luyten, H. P. C. Rood, and H. A. Tolhoek, Nucl. Phys. 41, 236 (1963).

[15]See for example, H. P. C. Rood, Ph.D. thesis, University of Groningen, 1964 (unpublished); A. Bogan, Nucl. Phys. B5, 431 (1968).

[16]H. A. Tolhoek and J. R. Luyten, Nucl. Phys. 3, 679 (1957); G. Goulard and B. Goulard, J. Phys. Rad. 21, 452 (1960).

[17]L. Hill, Ph. D. thesis, The Catholic University of America, 1967 (unpublished); Phys. Letters 25B, 169 (1967); see also FW.

[18]G. E. Walker and T. W. Donnelly, Ann. Phys. (N.Y.) 60, 209 (1970).

[19]G. E. Walker, Phys. Rev. 157, 854 (1967).

[20]M. Rho, Phys. Letters 16, 161 (1965).

[21]D. Zamischa and E. Werner, Nucl. Phys. A125, 383 (1969).

[22]T. A. Hughes, private communication. A Soper mixture for $A=88$ nuclei was also used by S. Ramavataram, Phys. Rev. 135, 1288 (1964); T. A. Hughes and S. Fallieros, in *Nuclear Isospin*, edited by J. D. Anderson, S. D. Bloom, J. Cerny, and W. W. True (Academic, New York, 1969), p. 109.

[23]The quantities S_J/η^2, D/η^2 with $\eta=\nu_{ab}b$, are respectively proportional to the quantities $|\int(\vec{\sigma}\otimes\vec{r})_J|^2$ and $|\int\vec{r}|^2$ defined in the Ref. 18.

[24]A. R. Edmond, in *Angular Momentum in Quantum Mechanics* (Princeton U. P., Princeton, New Jersey, 1960), p. 113.

[25]M. Eckhause, R. T. Siegel, R. E. Welsh, and T. A. Filippas, Nucl. Phys. 81, 575 (1966).

[26]T. Bressani, Rev. Nuovo Cimento 1, 268 (1971).

[27]M. Rotenberg, R. Bivins, N. Metropolis, and J. K. Wooten, Jr., *The 3j and 6j Symbols* (Crosby-Lockwood, London, 1959), p. 22.