Elastic Scattering of Pions on ^{16}O near the (3-3)-Resonance Energy: A Self-Consistent Calculation*

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An optical potential for elastic scattering of mesons on a nucleus is developed theoretically which is self-consistent at all distances from the center of the nucleus. Application is made to the case of scattering of pions on ^{16}O in the (3-3)-resonance energy region for the Fermi-gas model and the shell model of the nucleus. Pair correlations are included. Energy-dependent empirically fitted parameters are used as inputs for the pion-nucleon interaction. For the shell model a downward energy shift of 30–45 MeV, as well as considerable broadening, is predicted for the (3-3) pion-nucleus resonance. Graphs of the differential cross-section for various energies are included, with discussion of some significant features. Effects of inclusion and exclusion of the self-consistency requirement are considered.

INTRODUCTION

Recent experiments[1] on the scattering of pions with laboratory kinetic energies near the (3-3)-resonance energy on ^{12}C have renewed theoretical interest in pion-nucleus scattering,[2] a topic discussed primarily for lower energies in the 1950's.[3] The problem is often discussed in the framework of an optical model,[4] and one often uses techniques appropriate, at first glance, only to high-energy scattering.[5] This paper reports on calculations for scattering of pions with energies near the (3-3) resonance on ^{16}O described by the shell model and the Fermi-gas model. An optical potential is used which is self-consistent at every pion-nucleus distance. The energy-dependent fit of Roper and Wright[6] for the pion-nucleon scattering amplitudes and phase shifts is used as a given input. In this article the theoretical basis for the optical potential employed and the argument for requiring a self-consistent energy for evaluation of the pion-nucleon scattering parameters are given in the first section. Also included in this section are the derivation of density and pair correlation functions for the shell model and the Fermi-gas model of ^{16}O, a short discussion of the calculation of scattering amplitudes, and the procedure used for evaluating the shell-model parameter ν. In the second section results of the calculations are discussed, with particular reference to differences between predictions based on the shell model and the Fermi-gas model. The Appendix investigates the approximations used and attempts to evaluate the probability of their causing significant error.

Experiments, of which Ref. 1 is representative, have shown that the energy at which the (3-3) resonance produces a resonance in ^{12}C is considerably lower than the pion-proton (3-3)-resonance energy. By using the excellent fit of Ref. 6 and the self-consistent approach, the author has obtained what appears to be a reasonable shift for the ^{16}O case, at least for the shell model.

The present calculations (self-consistent results of which were first recorded in 1968[7]) differ from the later ones of Ref. 2 in their use of the exact fit to the pion-nucleon scattering and in the extensive calculations imposed by the point-by-point imposition of self-consistency. Further discussion of the resonance shift can be found in the section on results.

BASIC THEORY

From many-body multiple scattering theory[8] one can derive the following expression for an "optical potential" V_{c00} which describes the elastic scattering of an incident particle from a target system assuming that at most two-body correlations in the target are important:

$$V_{c00} = t_c + \Delta_{00} + \sum_{\alpha_2 \neq \alpha_1} \left(0 \left| I_{\alpha_1} \frac{1}{a - V_c + i\eta} I_{\alpha_2} \right| 0 \right) .$$

We now define the symbols:

$$t_c \equiv \left(0 \left| \sum_{\alpha=1}^{N} t_\alpha \right| 0 \right) ,$$

where t_α is the transition operator for scattering of the incident particle on the bound αth particle of the N-particle target system,

$$t_\alpha = V_\alpha + V_\alpha \frac{1}{W_A + \epsilon_q + i\eta - H_0 - V_\alpha} V_\alpha ,$$

and t_c is the average of the sum of these operators over the ground state of the target. Δ is an operator which represents the effects of true absorption.

I_α is an inelastic scattering operator. It has only those matrix elements of t_α which refer to inelastic scatterings (i.e., those which correspond to a change in nuclear excitation energy). We define operators $\langle t_\alpha \rangle$ which have matrix elements only for elastic scattering. Thus the matrix elements of t_α are decomposed into those of I_α and $\langle t_\alpha \rangle$.

$$a \equiv W_0 + \epsilon_0 - H_n - h + i\eta ,$$

where W_0 is the energy of the ground state of the target; ϵ_0 is the energy of the incident particle (when it is far from the target system); H_n is the Hamiltonian for the target system; h is the Hamiltonian for the incident particle; and η is the customary parameter, which we allow to approach zero at the end of the calculation.

Throughout the calculation (until the end) the Coulomb force is "turned off." The effects of the Coulomb force have been included by using an approximate method which is an extension of Bethe's work.[9]

Since all operators in the "energy denominator" term of V_{c00} are diagonal with respect to the unperturbed target states, one can write

$$V_{c00} = \sum_\alpha \langle 0 | t_\alpha | 0 \rangle + \Delta_{00} + \sum_{\alpha_1 \neq \alpha_2;\, \lambda \neq 0} \langle 0 | t_{\alpha_1} | \lambda \rangle \times \frac{1}{W_0 + \epsilon_q - W_\lambda - h + i\eta - V_{c\lambda\lambda}} \langle \lambda | t_{\alpha_2} | 0 \rangle .$$

Consider an expansion of the double sum in terms of a set of wave functions of the WKB form normalized in a box of volume V:

$$|njo\rangle = \frac{[q_{nj}(\infty)]^{1/2}}{[Vq_{nj}(\vec{r})]^{1/2}} \exp\left[i \int_{-\infty \text{ along } \sigma_j}^{s(\vec{r})} q_{nj}(s)\, ds \right] \chi_j(\vec{\xi}),$$

where $q_{nj}^2(\vec{r}) = E_j - V_{cnn}(\vec{r})$ (E_j is the energy of the jth state of the incident particle); σ_j denotes the dynamical path corresponding to the jth state; $s(\vec{r})$ is the path-length variable; and χ_j contains information about spin, isospin, etc. The states are normalized as follows:

$$\langle nj\sigma | nl\sigma' \rangle = \delta_{jl}\delta_{\sigma\sigma'};$$

furthermore,

$$h\,|nj\sigma\rangle = q_{nj}^2\,|nj\sigma\rangle .$$

It will be convenient to define a vector function $\vec{Q}_{nj\sigma}(\vec{r})$ such that

$$\vec{Q}_{nj\sigma}(\vec{r}) \cdot \vec{r} \equiv \int_{-\infty}^{s(\vec{r})} q_{nj}(s)\, ds$$

and the direction of $\vec{Q}_{nj\sigma}$ is that of the dynamical path at $\vec{r}$. The factor

$$\frac{[q_{nj}(\infty)]^{1/2}}{[Vq_{nj}(\vec{r})]^{1/2}}$$

will often be suppressed in writing $|nj\sigma\rangle$. It is useful to expand the expression for V_{c00} in terms of the states $|0j\rangle$, where the zero for n refers to the ground-state energy. Reasonable arguments can be made for the ideas that the most significant terms in the double sum are those for which $V_{c\lambda\lambda} \simeq V_{c00}$ and $W_\lambda \simeq W_0$. It is also possible to neglect true absorption (thus allowing one to set $\Delta = 0$). If there are several definable types of particles (e.g., neutrons and protons), it is productive to write

$$t_\alpha = \sum_I t_I \Lambda_I(\alpha),$$

where t_I is a transition operator for particles of the Ith type; $\Lambda_I(\alpha)$ is a projection operator for type-I particles [i.e., $\Lambda_I(\alpha)$ operating on a product wave function yields 1 if the αth particle is of type I and zero otherwise]; and I varies over the number of types of particles. Let

$$t_{JJ'I\alpha} \equiv \langle j\sigma | t_I \Lambda_I(\alpha) | j'\sigma' \rangle .$$

Then

$$V_{c00} = \sum_{joj'\sigma'\alpha I} |j\sigma\rangle\langle j\sigma|\langle 0|t_I\Lambda_I(\alpha)|0\rangle|j'\sigma'\rangle\langle j'\sigma'| + \sum_{J,J',J'',L,\alpha_1\neq\alpha_2} |j\sigma\rangle \frac{1}{\epsilon_0 - \epsilon_{j'} + i\eta} \times \{\langle 0|t_{JJ'I\alpha_1}t_{J'J''L\alpha_2}|0\rangle - \langle 0|t_{JJ'I\alpha_1}|0\rangle\langle 0|t_{J'J''L\alpha_2}|0\rangle\}\langle j''\sigma''| \equiv V_{c00}^{(1)} + V_{c00}^{(2)} .$$

First consider $V_{c00}^{(1)}$. It consists of many terms, each of which represents the scattering of the incoming meson with a nucleon in the nuclear ground state. Furthermore, it often happens that the vast bulk of the contributions to this scattering come from the region in the immediate neighborhood of the nucleon doing the scattering. As is shown in the section on approximations, if q_{0j} and $\vec{Q}_j$ vary slowly over the region near each nucleon in which the scattering contribution is significant, in $|nj\sigma\rangle$ one can use the approximation

$$e^{-i\vec{Q}_{j\sigma}(\vec{r})\cdot\vec{r}} \simeq e^{-i\vec{Q}_{j\sigma}(\vec{r}_\alpha)\cdot\vec{r}_\alpha} e^{-i\vec{q}_{j\sigma}(\vec{r}_\alpha)\cdot(\vec{r}-\vec{r}_\alpha)},$$

when $t_I\Lambda_I(\alpha)$ operates on the wave function. Keeping in mind that the matrix elements of t_I between plane-

wave states are just the scattering amplitudes $-4\pi f$, letting $\vec{\lambda}_{JJ'} \equiv \vec{q}_{j\sigma} - \vec{q}_{j'\sigma'}$, $\lambda_{JJ'} = |\vec{\lambda}_{JJ'}|$, and $\vec{\Delta}_{JJ'} \equiv \vec{Q}_{j\sigma} - \vec{Q}_{j'\sigma'}$, one obtains under the stated conditions

$$V_{c00}^{(1)} = \sum_{JJ'} \sum_{I} |j\sigma\rangle N_I \int d^3x\, \rho_{I0}(x) e^{-i\vec{\Delta}_{JJ'}(\vec{x})\cdot\vec{x}} (-4\pi) f_{IE}(\Delta_{JJ'}) \langle j'\sigma'| , \tag{1}$$

where

$$\rho_{I0}(\vec{x}) \equiv \frac{1}{N_I} \sum_{\alpha} \langle 0|\Lambda_I(\alpha)\delta^3(\vec{x} - \vec{r}_\alpha)|0\rangle , \tag{2}$$

and E = pion energy near bound nucleon $= \epsilon_q - V_{c00}(\vec{x})$ (as will be shown in the discussion of $V_{c00}^{(2)}$).

Under the same assumptions about q_j and $\vec{Q}_{j\sigma}$, and adding the terms in $T^{(2)}$ for which $\alpha_1 = \alpha_2$ (which are, at most, of order $1/N_I$ or $1/N_L$ compared with the entire sum), one obtains

$$V_{c00}^{(2)} = \sum_{JJ'J''} |j\sigma\rangle \frac{1}{\epsilon_0 - \epsilon_{j'} + i\eta} (T_{JJ'J''}^{(1)} - T_{JJ'J''}^{(2)}) \langle j''\sigma''| ,$$

where

$$T_{JJ'J''}^{(1)} = (4\pi)^2 \int d^3x\, d^3x'\, e^{-i\vec{\Delta}_{JJ'}(\vec{x})\cdot\vec{x}}\, e^{-i\vec{\Delta}_{J'J''}(\vec{x}')\cdot\vec{x}'}$$

$$\times \left\{ \sum_I f_I(\lambda_{JJ'}(x)) f_I(\lambda_{J'J''}(x'))\, N_I(N_I - 1) P_{I0}^{(2)}(\vec{x}, \vec{x}') + \sum_I \sum_{L \neq I} f_I(\lambda_{JJ'}(\vec{x})) f_L(\lambda_{J'J''}(\vec{x}'))\, N_I N_L \rho_{I0}(\vec{x}) \rho_{L0}(\vec{x}') \right\}$$

and

$$T_{JJ'J''}^{(2)} = (4\pi)^2 \sum_{I,L} \int d^3x\, d^3x'\, e^{-i\vec{\Delta}_{JJ'}(\vec{x})\cdot\vec{x}} e^{-i\vec{\Delta}_{J'J''}(\vec{x}')\cdot\vec{x}'} f_I(\lambda_{JJ'}(\vec{x})) f_L(\lambda_{J'J''}(\vec{x}'))\, N_I N_L \rho_{I0}(\vec{x}) \rho_{L0}(\vec{x}') ,$$

with the $P^{(2)}$ relating to two-nucleon correlations and defined by

$$P_{IL}^{(2)}(\vec{x}, \vec{x}') \equiv \begin{pmatrix} \frac{1}{N_I(N_I - 1)} \\ \frac{1}{N_I N_L} \end{pmatrix} \sum_{\alpha, \beta \neq \alpha} \delta(\vec{r}_\alpha - \vec{x}) \delta(\vec{r}_\beta - \vec{x}') \Lambda_I(\alpha) \Lambda_L(\beta) , \tag{3}$$

when

$$\begin{pmatrix} I = L \\ I \neq L \end{pmatrix}$$

and

$$P_{I0}^{(2)}(\vec{x}, \vec{x}') \equiv \langle 0|P_{II}^{(2)}|0\rangle .$$

It is customary to define a function $g_I(\vec{x}, \vec{x}')$ via

$$N_I(N_I - 1) P^{(2)}(\vec{x}, \vec{x}') - N_I^2 P_{I0}^{(1)}(\vec{x}) P_{I0}^{(1)}(\vec{x}') \equiv N_I^2 \rho_{I0}(\vec{x}) \rho_{I0}(\vec{x}') [g_I(\vec{x}, \vec{x}') - 1] . \tag{4}$$

If g_I is a function only of $|\vec{x} - \vec{x}'|$, if the radius of the matter described by $\rho_{I0}(\vec{x})$ is much larger than the distance over which $(g_I - 1)$ is large, if f_I is a slowly varying function of $\vec{x}$, and assuming that the functions $q_{j'\sigma'}(\vec{x})$ vary little within the target, one can recall the expression for $|j\sigma\rangle$ in terms of $\vec{Q}$ to write

$$V_{c00}^{(2)} = \sum_I N_I^2 (4\pi)^2 \sum_{j'} \frac{1}{\epsilon_0 - \epsilon_{j'} + i\eta} \sum_{JJ''} |j\sigma\rangle\langle j\sigma| f_I(\lambda_{JJ'}) f_I(\lambda_{J'J''}) \rho_{I0}^2 C_I(\vec{\lambda}_{J'J''}) |j''\sigma''\rangle\langle j''\sigma''| ,$$

where

$$C_I(\vec{\lambda}_{JJ'}) \equiv \int [g_I(\vec{d}) - 1]\, e^{-i\vec{\lambda}_{J'J''}(\vec{x})\cdot\vec{d}}\, d^3d . \tag{5}$$

Consider the dependence on $j''\sigma''$. To discuss this, one should recall that $V_{c00}^{(2)}$ operates on ϕ, the wave function for the incident particle. For reasonably high-energy scattering, we expect ϕ to be of essentially WKB nature, say $\phi \simeq |j_0\sigma_0\rangle$; thus $\langle j''\sigma''|\phi\rangle$ is a "nearly-δ" function, eliminating all $j''\sigma''$ except $j''\sigma'' \simeq j_0\sigma_0$.

Furthermore, in several cases of interest (nuclear scattering, particularly) the product $f_I f_I C_I \rho_{I0}$ is a relatively slowly varying (i.e, nearly constant) function of $\vec{x}$. One therefore expects its "WKB-Fourier" transform, which appears above, to include a "nearly-δ" function which eliminates all $j\sigma$ except $j\sigma \simeq j''\sigma''$.

The presence of the combination of these two "nearly-δ" functions leads to the approximation

$$V_{c00}^{(2)} = \sum_I N_I^2 (4\pi)^2 \times \sum_{J'} \frac{1}{\epsilon_0 - \epsilon_{J'} + i\eta} f_I(\lambda_{J'J_0}) f_I(\lambda_{J_0J'}) \rho_{I0}{}^2 C_I(\vec{\lambda}_{J'J_0}) .$$

Using the standard transformation

$$\sum_{J'} \to \frac{1}{(2\pi)^2} \int \tfrac{1}{2}\sqrt{E}\, dE d\Omega ,$$

giving the Dirac[10] meaning to the energy denominator, and neglecting the principal part integral, one obtains

$$V_{c00}^{(2)} = \sum_I N_I^2 \rho_I^2(\vec{x}) \sqrt{\epsilon_0}\, (-i) \int d\Omega\, C_I(\vec{\lambda}_{J'J}) f_I^2(\lambda_{J'J}) . \tag{6}$$

Note that the energy δ function yields an interesting result for the energy at which f_I should be evaluated, for it tells us that $\epsilon_{J'} = \epsilon_0$ throughout. But

$$q_{J'\sigma'}{}^2(\vec{r}) = \epsilon_{J'} - V_{c00}(\vec{r}) .$$

Thus for a particular value of $\vec{r}$, only those $q_{J'\sigma'}$ enter for which $\epsilon_{J'} = \epsilon_0$. This implies that at each position $\vec{r}$, one must evaluate the two-body scattering amplitudes f_I at an energy such that

$$\epsilon_0 = q^2(r) + V_{c00} . \tag{7}$$

It is interesting to examine the high-energy behavior of this expression. In the high-energy limit, the WKB wave functions approximate exponentials. Thus,

$$C_I(\vec{\lambda}_{J'J_0}(x_0)) \to \int e^{-i\vec{\lambda}\cdot\vec{r}} [g-1]\, d^3r , \tag{8}$$

where

$$\vec{\lambda} = \vec{k}_0 - \vec{k}_{J'} ,$$

and

$$|\vec{k}_0| \simeq |\vec{k}_{J'}| = \sqrt{\epsilon_0} .$$

In this case

$$d\Omega = \frac{2\pi\lambda d\lambda}{\epsilon_0}$$

and

$$V_{c00}^{(2)} = \sum N_I^2 \rho_I^2 \frac{-2\pi i}{\sqrt{\epsilon_0}} \int C_I(\lambda) f_I^2(\lambda) \lambda\, d\lambda .$$

Under certain common conditions one can further simplify the expression for $V_{c00}^{(1)}$ of Eq. (1), particularly since it acts on nearly WKB state ϕ. If $|\phi\rangle \simeq |j_0\sigma_0\rangle$ and ρ_{I0} varies slowly with $\vec{r}$, one can set $q_{j\sigma}/q_{j'\sigma'} \simeq 1$ and replace $|j\sigma\rangle$ in Eq. (1) as follows:

$$|j\sigma\rangle \simeq |j''\sigma''\rangle\, e^{-i\vec{\Delta}_{J_0J'}\cdot\vec{r}} .$$

If $f_I(\lambda_{JJ'}(\vec{r}))$ is a very slowly varying function of $\vec{r}$, one can then obtain

$$V_{c00}^{(1)} = \sum_I \sum_J (-4\pi) N_I\, e^{-i\vec{\Delta}_{J_0J}\cdot\vec{r}} \langle j\sigma|\rho_{I0}|j_0\sigma_0\rangle f_I(\lambda_{JJ_0}(r)) . \tag{9}$$

The behavior of $V_{c00}^{(1)}$ at high energies is interesting. In this case $\vec{Q}_{j\sigma}$ and $\vec{q}_{j\sigma}$ approach some $\vec{k}_{j\sigma}$ which is independent of $\vec{r}$. The dependence on j, σ is through

$$\vec{\lambda}_{JJ_0} = \vec{k}_{j_0\sigma_0} - \vec{k}_{j\sigma} .$$

Using this in Eq. (9), one obtains

$$V_{c00}^{(1)} = -\frac{1}{2\pi^2} \sum_I N_I \int S_I(\lambda) f_I(\lambda) e^{-i\vec{\lambda}\cdot\vec{r}} d^3\lambda , \tag{10}$$

where

$$S_I(\lambda) = \int d^3x \rho_{I0} e^{-i\vec{\lambda}\cdot\vec{x}} .$$

Combining Eqs. (6) and (10), one obtains at high energies (with no true absorption)

$$V_{c00} = \sum_I N_I \left[-\frac{1}{2\pi^2} \int S_I(\lambda) f_I(\lambda) e^{-i\vec{\lambda}\cdot\vec{r}} d^3\lambda - 2\pi i N_I \rho_I^2(r) \frac{1}{\sqrt{\epsilon_0}} \int C_I(\lambda) f_I^2(\lambda) \lambda d\lambda \right] . \tag{11}$$

Throughout the calculations, this equation will be the basic one used. f_I is evaluated at the self-consistent energy of Eq. (7). Although various authors have attempted to justify use of the convenient high-energy formula at energies which seem too low for obvious applicability, the basic factor in its favor seems to be that it, to an unexpected degree, works.

DENSITY AND PAIR DISTRIBUTION FUNCTIONS FOR DETERMINANTAL WAVE FUNCTIONS

If the wave function for the target can be written as a Slater determinant,

$$|\phi\rangle = \frac{1}{\sqrt{N!}} \sum_Q \epsilon_Q |\omega_{Q(1)}(1)\rangle \cdots |\omega_{Q(N)}(N)\rangle ,$$

where the sum extends over all the permutations, Q, of the numbers 1, ..., N, and

$$\epsilon_Q = (\pm 1) \text{ if } Q \text{ is an } \begin{pmatrix}\text{even}\\ \text{odd}\end{pmatrix} \text{permutation},$$

ω_I represents particles of type I for $\nu_{I-1}+1 \leqq I \leqq \nu_I$, where $\nu_i = \sum_{j=1}^{i} N_j$, and the orthogonality relation

$$\langle \omega_i(a) | \omega_j(a) \rangle = \delta_{ij}$$

holds. It can be readily shown that

$$P_I^{(1)}(\vec{\mathbf{X}}) = \frac{1}{N_I} \sum_{a=\nu_{I-1}+1}^{\nu_I} \langle \omega_a(1) | \delta^3(\vec{\mathbf{r}}_1 - \vec{\mathbf{X}}) \omega_a(1) \rangle .$$

If one can write ω_a in the form $\omega_a(\vec{\mathbf{r}}, \vec{\mathbf{S}}) = \Omega_a(\vec{\mathbf{r}}) \chi_a(\vec{\mathbf{S}})$, one obtains

$$P_I^{(1)}(\vec{\mathbf{X}}) = \frac{1}{N_I} \sum_{\nu_{I-1}+1}^{\nu_I} |\Omega_a(\vec{\mathbf{X}})|^2 .$$

Under the same conditions, one can show that

$$P_I^{(2)}(\vec{\mathbf{X}}, \vec{\mathbf{X}}') = \frac{1}{N_I(N_I-1)} \sum_{a=\nu_{I-1}+1}^{\nu_I} \sum_{\substack{b=\nu_{I-1}+1 \\ b \neq a}}^{\nu_I} [\,|\Omega_a(\vec{\mathbf{X}})|^2 |\Omega_b(\vec{\mathbf{X}}')|^2 - \Omega_a^*(\vec{\mathbf{X}}) \Omega_b^*(\vec{\mathbf{X}}') \Omega_b(\vec{\mathbf{X}}) \Omega_a(\vec{\mathbf{X}}') \langle \chi_b | \chi_a \rangle \langle \chi_a | \chi_b \rangle] .$$

To proceed further in evaluating $P_I^{(1)}$ and $P_I^{(2)}$, one must know the specific form of the ω_a.

Consider a (Fermi-gas) model of a nucleus, for which the individual particle wave functions have the form:

$$\omega_a(\vec{\mathbf{X}}, \vec{\mathbf{S}}) = \frac{1}{\sqrt{V}} e^{i \vec{\mathbf{K}}_a \cdot \vec{\mathbf{X}}} S_a(\vec{\mathbf{S}}) \tau_a(\vec{t}\,) ,$$

where normalization is in a box of volume V, S_a contains the spin dependence, and τ_a contains the isospin dependence. One can show that use of such wave functions leads to

$$P_{I\,\mathrm{FG}}^{(1)}(\vec{\mathbf{X}}) = \frac{1}{N_I} \sum \frac{1}{V} = \frac{1}{V} = \frac{2}{N_I} \int_0^{K_F} \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d^3K = \frac{1}{N_I} \frac{1}{3\pi^2} K_F{}^3 , \tag{12}$$

where K_F = maximum permitted value of K. Similarly, use of these wave functions in the expression for $P_I^{(2)}$ leads to

$$P_{I\,\mathrm{FG}}^{(2)}(\vec{\mathbf{X}}, \vec{\mathbf{X}}') = \frac{1}{V^2} \left[\sum_{a=1}^{N_I} \sum_{\substack{b=1 \\ b \neq a}}^{N_I} 1 \cdot 1 - \sum_a \sum_b e^{-i(\vec{\mathbf{K}}_a - \vec{\mathbf{K}}_b) \cdot (\vec{\mathbf{X}} - \vec{\mathbf{X}}')} |\langle \chi_a | \chi_b \rangle|^2 \right] .$$

With the standard conversion to an integral (keeping in mind the $|\langle \chi_a | \chi_b \rangle|^2$ factor), one obtains for large N_I

$$P_{I\,\mathrm{FG}}^{(2)} = \frac{N_I^2}{V^2} \left[1 - \tfrac{9}{4}\pi \frac{1}{(K_F |\vec{\mathbf{X}} - \vec{\mathbf{X}}'|)^3} [J_{3/2}(K_F |\vec{\mathbf{X}} - \vec{\mathbf{X}}'|)]^2 \right] .$$

For a simple shell model of the nucleus the calculation of $P_I^{(1)}$ and $P_I^{(2)}$ is somewhat more complicated.

In the work which follows ^{16}O is considered as an example. A simple choice for the single-particle functions in the shell-model wave function is obtained by taking the eigenfunctions of the isotropic 3-dimensional harmonic oscillator. (This choice neglects spin-orbit forces.) The radial functions which enter are

$$R = \frac{2}{\pi^{1/4}} (\nu)^{3/4} e^{-\nu r^2}$$

and

$$R_{11} = \frac{1}{\pi^{1/4}} \left(\tfrac{8}{3}\right)^{1/2} (\nu)^{5/4} r e^{-\nu r^2} .$$

If one combines these with the spherical harmonics and uses them, one obtains the same expressions for

protons as for neutrons:

$$P^{(1)}(r,\theta,\phi)=\frac{1}{4}\left(\frac{2\nu}{\pi}\right)^{3/2}(1+4\nu r^2)e^{-2\nu r^2}, \tag{13a}$$

$$P^{(2)}=\frac{1}{28}\left(\frac{2\nu}{\pi}\right)^{3}e^{-2\nu(r^2+r'^2)}[1+8\nu(r^2+r'^2)+32\nu^2r^2r'^2-8\nu\vec{r}\cdot\vec{r}'-16\nu^2(\vec{r}\cdot\vec{r}')^2],$$

and

$$P^{(1)}(r)P^{(1)}(r')[g(\vec{r},\vec{r}')-1]=-\frac{1}{32}\left(\frac{2\nu}{\pi}\right)^{3}(1+4\nu\vec{r}\cdot\vec{r}')^2e^{-2\nu(r^2+r'^2)}.$$

Using the expression for $P^{(1)}$, one obtains

$$g(\vec{r},\vec{r}')-1=-\frac{1}{4}\frac{(1+4\nu\vec{r}\cdot\vec{r}')^2}{(1+4\nu r^2)(1+4\nu r'^2)}. \tag{13b}$$

For these calculations the Fourier components of the density and correlation functions are required.

For the shell model one can show that

$$S_{\text{shell}}(\vec{\kappa})=\frac{1}{V}\left(1-\frac{\kappa^2}{16\nu}\right)e^{-\kappa^2/8\nu}. \tag{14}$$

For the Fermi-gas model, the calculation is trivial:

$$S_{\text{FG}}(\vec{\kappa})=\frac{1}{V}\,\delta^3(\vec{\kappa}) \tag{15}$$

(this disregards the finite dimensions of the nucleus).

For a Fermi gas the calculation of $C(\vec{\kappa})$ is relatively straightforward. The result is

$$C(\kappa)=\begin{cases}\dfrac{9\pi^2}{2K_F^{\,3}}\left[\dfrac{2}{3}-\dfrac{1}{2}\dfrac{\kappa}{K_F}+\dfrac{1}{24}\left(\dfrac{\kappa}{K_F}\right)^3\right] & \text{if } 0<\kappa\le 2K_F\\ 0 & \text{if } \kappa>2K_F\end{cases}. \tag{16}$$

For the proposed shell model, calculation of $C(\kappa)$ is considerably trickier. Since $g(\vec{r},\vec{r}')$ is not a function of $\vec{r}-\vec{r}'$ only, the Fourier transform of $\rho(r)\rho(r')[g-1]$ does not reduce to $C(\vec{\kappa})(\rho^2)_{KK''}$. For the particular case of interest, nevertheless, one can write an effective $C(\kappa)$ which should lead to approximately correct results when inserted into the expression for V_{c00}. Under the conditions that the scattering amplitude for angles less than, say, $\frac{1}{3}\pi$ be peaked near 0 and that the magnitude of the square of the incident momentum be considerably greater than 8ν, it can be shown (Ref. 7, pp. 66–67) that a reasonable $C(\kappa)$ can be obtained by requiring that

$$C(\vec{K}'-\vec{K})\int d^3K''\langle K|\rho^2|K''\rangle=\int d^3K''[N(N-1)\langle K|\langle K'|P^{(2)}|K''\rangle|K'\rangle-N^2\langle K|P^{(1)}|K'\rangle\langle K'|P^{(1)}|K''\rangle]. \tag{17}$$

Use of the expressions for $P^{(2)}$ and $P^{(1)}$ from the proposed ^{16}O shell model leads to

$$C(\kappa)=\tfrac{1}{2}(\pi/2\nu)^{3/2}e^{-\kappa^2/8\nu}. \tag{18}$$

CALCULATIONS OF SCATTERING AMPLITUDES

With the condition indicated by Eq. (7) in mind, one can now calculate the optical potential of Eq. (11) using the density functions and their transforms and the correlation function transforms of Eqs. (12)–(18).

Equation (7) expresses an intuitively plausible notion. If one regards the nuclear matter as producing a nuclear potential describing an averaged effect of the nucleons, it seems reasonable that within the range of this potential the "effective total energy" of the meson should be its energy outside the range of the potential plus the averaged potential energy.

The procedure adopted for calculating the optical potential is as follows:

(1) For a given $\vec{r}$ regard V_{c00} as a function of ϵ, the energy at which the single-particle scattering amplitudes are evaluated. Calculate V_{c00} as a function of ϵ.

(2) Proceed by iteration to find a value $\epsilon_{\text{evaluation}}$, such that $\epsilon_0+V_{c00}(\vec{r},\epsilon_{\text{evaluation}})=\epsilon_{\text{evaluation}}$.

There is some difficulty with this equation be-

cause V_{c00} has both a real and an imaginary part. Although it might be possible to use dispersion relations to estimate the scattering amplitudes at complex energies, such a procedure would probably not be very accurate. For $\epsilon_0 \gg \mathrm{Im}V_{c00}$, one would not expect large errors from neglecting the imaginary part in iterating the equation. Since $V_{c00} \geqq \epsilon_0$ would lead to many other difficulties, this requirement probably adds little new to the limitations of this calculation.

From scattering theory with WKB wave functions (Ref. 7, pp. 71–76) or Glauber's work, one can show that in the high-energy limit

$$f(\theta) = \frac{K}{i} \int J_0(2Kb \sin\tfrac{1}{2}\theta)(e^{2i\delta_l} - 1) b\, db\,, \qquad (19a)$$

where

$$\delta_l = -\frac{1}{2\hbar v} \int_{-\infty}^{\infty} V_{c00}(\vec{b} + \hat{K}z')dz'\,, \qquad (19b)$$

$l = Kb - \frac{1}{2}$, θ is the laboratory scattering angle, $\hat{K}$ is a unit vector in the initial direction of the incoming particle, $\vec{b}$ is a vector orthogonal to K and such that $\vec{r} = \vec{b} + \hat{K}z$, v is the speed of the incoming particle with respect to the center of mass of the target, and J_0 is the zeroth-order Bessel function.

Equation (19a), with (19b), was derived for small-angle scattering. Nevertheless, there are several cases for which it can be expected to give reasonable results for relatively large angles.[11] Because of its simplicity this equation is used throughout this paper. Work is currently in progress on better treatment of the larger-angle scattering. Equations (19) have been discussed by many authors. In particular, Glauber[5] has discussed some of the errors and limitations involved.

In this paper Eqs. (19) have been used only to obtain the purely nuclear part of the scattering. As previously noted, effects of the Coulomb force have been included by an approximate method[12] based on extension of work by Bethe.

EVALUATION OF THE SHELL-MODEL PARAMETER

In the calculations the value

$$\nu = \frac{5/2}{r_0^2(16)^{2/3}}$$

has been used throughout for the parameter in the ^{16}O shell model. r_0 is the familiar parameter such

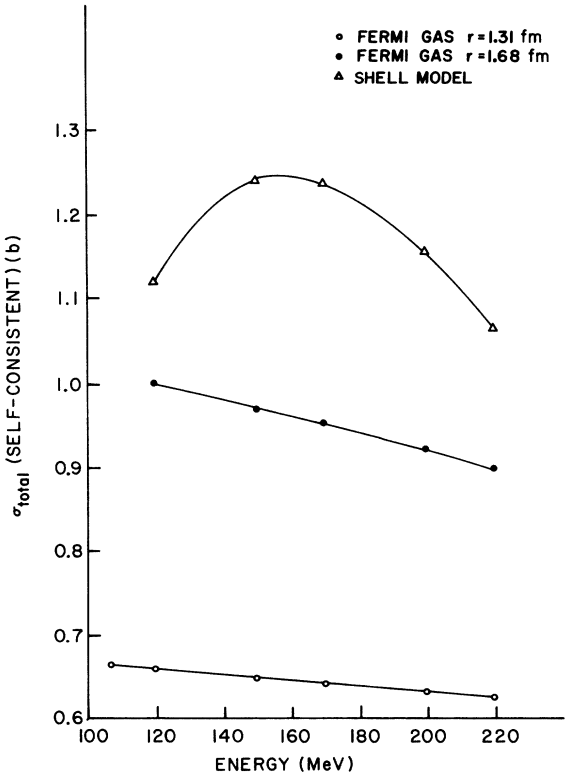

FIG. 1. Self-consistent total cross sections of Fermi-gas model and shell model as a function of pion incident laboratory energy.

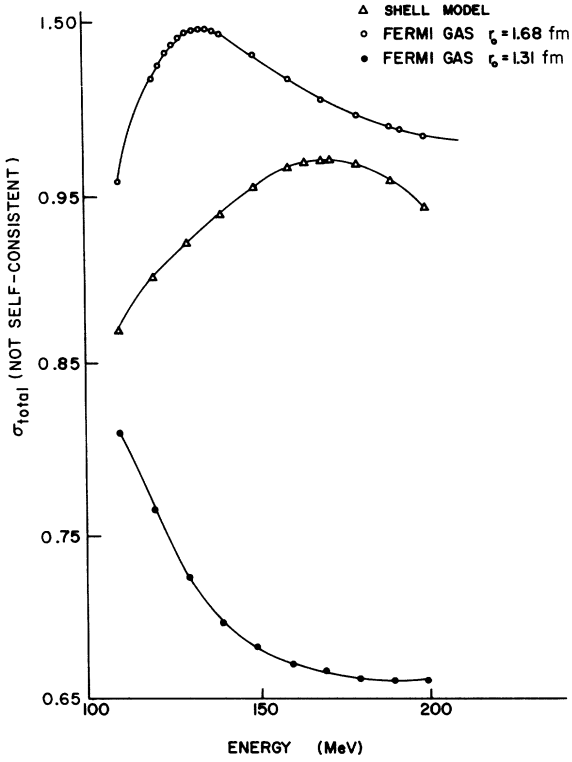

FIG. 2. Non-self-consistent total cross sections as a function of pion incident lab energy.

that $r_0 A^{1/3}$ is the approximate radius of a nucleus of A nucleons. This expression results from equating the expectation value of the oscillator potential energy with half the total energy of the outermost nucleons.[13] We have used $r_0 = 1.31 \times 10^{-13}$ cm.[14] An alternative method suggested by Mayer and Jensen, requiring that rms radius implied by the shell-model density function equal the measured rms radius for ^{16}O, was rejected because the unrealistically long "tail" on the shell-model distribution tends to make ν evaluated by this method considerably smaller than would seem warranted.

RESULTS OF THE CALCULATION

Examination of the results of the detailed calculations shows that, if the choice were limited to the two models considered here, one could determine the correct nuclear model from reasonably straightforward experiments, performable with available equipment. Although the choice of models is not so limited, it is useful to examine the differences.

Total Cross Section

Figure 1 shows a plot of the total cross section for the two models as a function of the initial value of the pion kinetic energy in the laboratory system. There are two major features to discuss: first, the placement of the peaks relative to one another and relative to the position of the peak for the (3-3) resonance in the pion-proton system, and, second, the width of the peak.

The peak in the shell-model total cross section appears to be at about 145–160-MeV laboratory kinetic energy. If there is a peak in the Fermi-gas-model cross section it must be at or below 120 MeV. Both the full self-consistent calculation and non-self-consistent calculations (see Fig 2) show a shift of the resonance energy such that the resonance in the nucleus appears at a lower incident pion energy than that for the (3-3) pion-proton resonance. It is noteworthy, furthermore, that for a given model the shift is greater for the self-consistent calculation than it is when self-consistency was not imposed. [In the non-self-consistent case the resonance appears at about

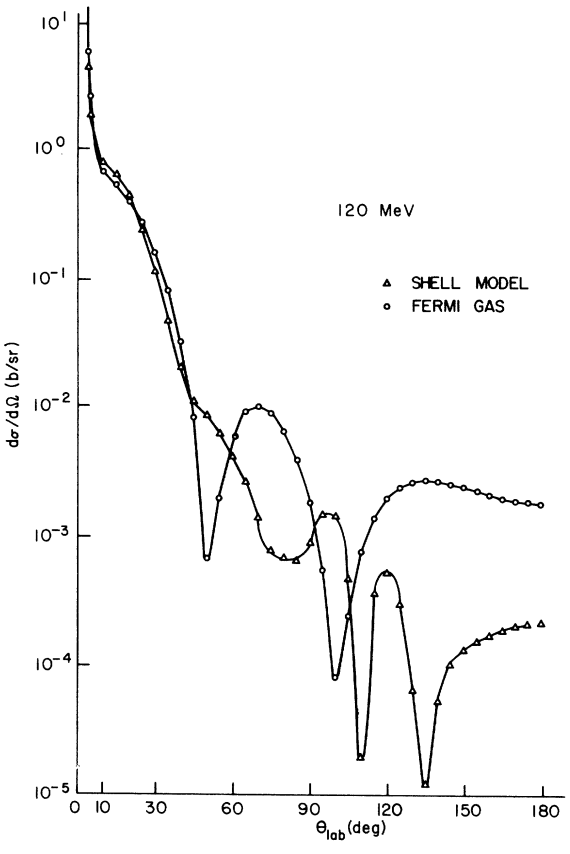

FIG. 3. Pion on ^{16}O. Differential cross sections – Fermi-gas model and shell model. 120 MeV.

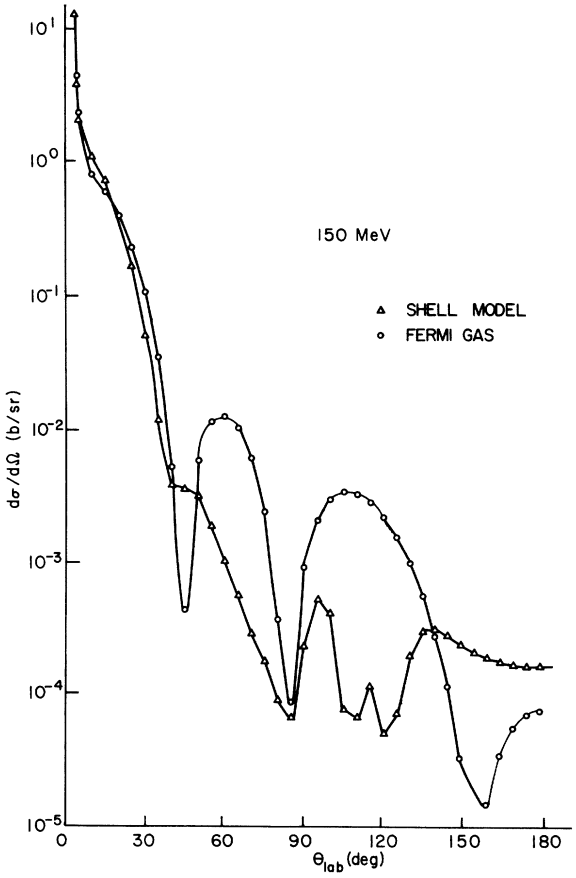

FIG. 4. Pion on ^{16}O. Differential cross sections 150 MeV.

172 MeV for the shell model, representing a shift of about 20 MeV down from (3-3) resonance; does not appear for the Fermi-gas model with nuclear radius 1.31 F $A^{1/3}$; and appears at about 135 MeV for the Fermi-gas model with nuclear radius 1.68 F $A^{1/3}$.

In a recent paper,[2] Ericson and Hüfner analyze the downward shift in the position of the (3-3) resonance in pion scattering by nuclei in an approximate, but relatively general, way. Using a p-wave dependence on momentum and an energy-dependent resonance function, and noting that a carbon nucleus is strongly absorbing near resonance energy, they conclude that one can attribute almost all (according to them, within 10% to 30%) of the downward shift to the momentum and energy dependence of the index of refraction. This is equivalent to the requirement of self-consistency imposed in the calculations reported here. Locher *et al.*[2] deal with the pion-nucleus resonance shift from the viewpoint of the sequential multiple-scattering form of Glauber approximations. Noting that near resonance the first terms of the Glauber series do not converge, they rearrange and sum the series to include contributions from all terms. Correlations among nucleons are not considered. Simplifying assumptions for the density and for the angular dependence of the pion-nucleon scattering amplitudes lead then to a more or less readily differentiable expression for the total cross section. Using the fit of a slightly later version of Ref. 6, they obtain a shift which appears from their graph to be a reasonable 30 MeV or so for ^{16}O. They do not seem to deal in any transparent way with the question of a self-consistent energy at which to evaluate the pion-nucleon data for use in calculating pion-nucleus scattering. (Though this problem may not be in the spirit of the adopted treatment, it appears to be significant for this scattering.) Glauber himself[15] has noted (p. 327) that the difference between his treatment and the standard optical-model approach used here is that between $(1+i\chi/A)^A$ and $e^{i\chi}$. In the case of ^{16}O with $A=16$ and χ not very interesting when greater than about unity, the difference is quite small (around 10% for $\chi=1$, less than 1% for $\chi=0.1$). By summing their Glauber series, Locher *et al.* have included the kind of multiple scattering implicit in the exponentiation of χ of this paper. Their assumptions, however, seem more drastic, and they do not appear to include self-consistency conditions.

From the results of the present calculations it seems clear that two mechanisms each producing significant effects are responsible for the fact that the resonance energy is lower in pion-nucleus scattering than it is for a pion on a single proton. These mechanisms are multiple scattering as expressed through exponentiation of the Born term and the requirement of self-consistency (which might be interpreted as the effect of the potentials of more distant nucleons on the energy of the incident pion). Certainly, for the case in which nucleons are sufficiently far apart that

$$\left| -\frac{1}{\hbar v} \int V_{\text{opt}}\, dz \right| \ll 1$$

for all incident energies of interest, one can readily obtain that the total cross section for a pion on a nucleus would be just the sum of the cross sections of the pion on the individual nucleons of the nucleus. In such a case, the nuclear resonance would be essentially at the energy for the (3-3) pion-nucleon resonance. (Self-consistency requirements would not significantly affect this because the optical potential would be small everywhere.) Even without the requirement of self-consistency, these calculations showed a significant lowering of the resonance energy when

$$\left| -\frac{1}{\hbar v} \int V_{\text{opt}}\, dz \right|$$

is not small compared to unity. In this case $\exp[-(i/\hbar v)\int V_{\text{opt}}\, dz]$ includes sequential multiple-scattering terms. Even without elaborate analysis one can see how the self-consistency requirement results in a lowering of the resonance energy for the present calculation, if one keeps in mind the fact that the scattering cross section for pions on nucleons is skewed when plotted against energy. When iteration is completed, one discovers that a plot of the real part of the optical potential versus the pion incident energy is nearly a straight line. [This reflects the fact that, to first approximation, V_{opt} is proportional to $f(0)$, the real part of which goes through zero in a reasonably linear way near the resonance.] Thus the energy at which phase shifts are evaluated to obtain the optical potential employed differs from the incident laboratory energy by an amount linear in energy and going through zero near the resonance. But the below-resonance side of a plot of the scattering cross section for pions on nucleons has a considerably greater slope upward than the above-resonance side has downward. Thus, a change in the evaluation energy by a similar amount on each side will cause a greater shift upward in the cross section on the low-energy side. This will shift the resonance downward.

This effect will be more pronounced for the Fermi-gas model because the nucleons in such a model tend to be more tightly "bunched" near the nuclear center than they would be in a shell model of the same radius. Thus contributions from re-

gions of high density (and, hence, high optical-potential energy) are greater.

The considerable broadening of the resonance peak is also understandable to a certain extent from the calculation. Calculation shows that for each incident energy the self-consistent energy at which the optical potential is evaluated is always nearer the (3-3) resonance energy than is the incident energy. Thus the pion-nucleus cross section decreases more slowly as a function of energy away from resonance.

Differential Cross Section

Figures 3–7 compare Fermi-gas model results to those for a shell model at incident pion energies from 120 to 220 MeV. To facilitate comparison the Fermi-gas radius has been adjusted so that the width of the central peak approximately matches the width for a shell model with a radius of $r_0(16)^{1/3}$, where $r_0 = 1.31$ F. [The result, corresponding to $r_0 = 1.68$ F, was calculated approximately from consideration of the black-body result in which the first minimum occurs at the first zero of $J_1(Kr\theta)$.] The outstanding difference between models is the much more complex structure of the shell-model curves. (Results were obtained for a Fermi-gas model with $r_0 = 1.31$ F, but they yield no significant differences.)

Throughout the figures, the Fermi-gas cross section is quite regular, with well-defined maxima and minima, as one would expect for a very nearly black body. For the shell model, however, not only are the maxima and minima less well defined, but there are some rather curious occurences, as one can note by looking at $\theta_{lab} \approx 40°$ and $\theta_{lab} \approx 120°$ for various energies. One can account for at least part of the differences in the differential cross sections by an argument based on the shape of the density distribution. To see this, one notes that the following correspondence is valid semiclassically between the angular momentum quantum number and the impact parameter b:

$$b = \frac{l+\frac{1}{2}}{K}.$$

As one can see from Eqs. (19), $f(\theta)$ (and hence $d\sigma/d\Omega$) will be influenced by δ_l, which reflects

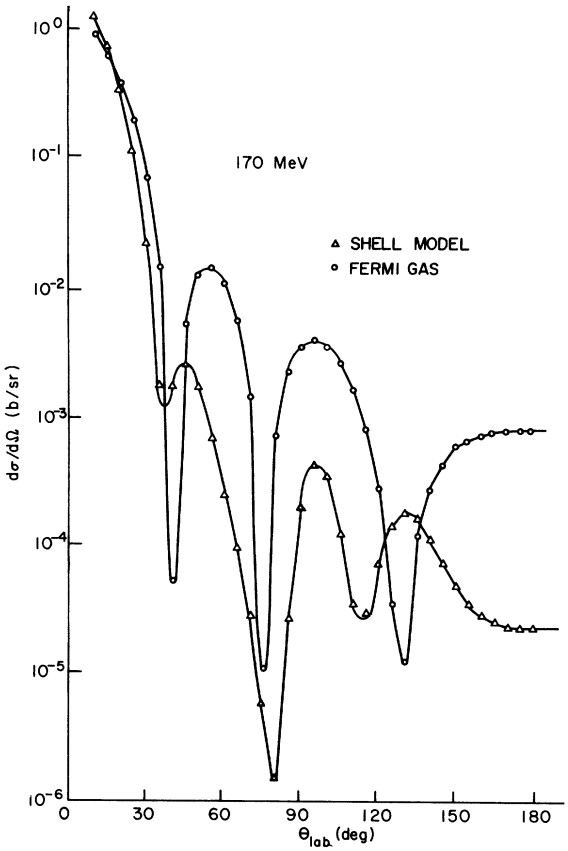

FIG. 5. Pion on ^{16}O. Differential cross sections 170 MeV.

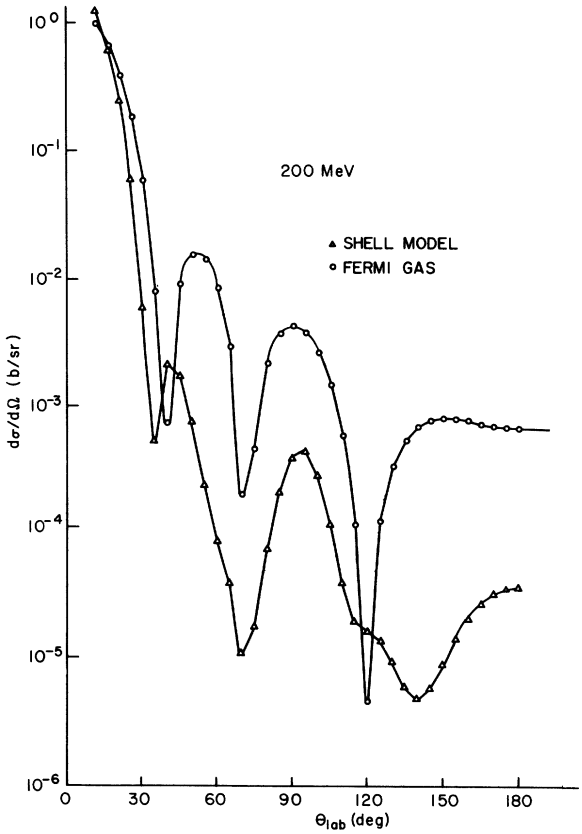

FIG. 6. Pion on ^{16}O. Differential cross sections 200 MeV.

the density via its dependence on V_{c00}. To lowest order of approximation, $V_{c00} \propto \rho$. In this lowest order of approximation, one can show that for a Fermi gas $\delta_l(b)$ is of the form $(R^2 - b^2)^{1/2}$ for $b > R$ and is zero for $b > R$. For the shell model, $\delta_l(b)$ is of the form $De^{-2\nu b^2}(1 + 2\nu b^2)$. For the values of K corresponding to energies near the (3-3)-resonance energy, $e^{2i\delta_l} - 1$ for a Fermi-gas model will vanish for $l \gtrsim 5$. Since terms corresponding to a particular l give rise to peaks of angular width π/l, one would expect no peaks with angular width $< \frac{1}{5}\pi$. The graphs of Figs. 3–7 show that this is true. For the shell model, on the other hand, one might see peaks of any width but of very small height compared to the central maximum, since b corresponding to all l are available. For example, one can consider the peak which appears at about 115° with an angular width of $\frac{1}{12}\pi$ to $\frac{1}{9}\pi$ (15–20°) in the shell-model curve at 150 MeV. It should, therefore, correspond to l between 9 and 12. One notes that the peak in the same graph centered at approximately 95° has a width which corresponds to $l \approx 6$ or 7. Although considered as a function of l the relative peak height does not decline as rapidly as one might expect from the crude approximation considered here, it should be noted that the height of the peak at 155° is considerably lower than that at 95° as one would expect from the l dependence of the approximate δ_l chosen.

Thus, the shell model used here gives rise to a more complex structure than the Fermi-gas model. This can be qualitatively understood, in part, by a simple density dependence for V_{c00}. The precise details, however, are not consistent with such a simplified treatment.

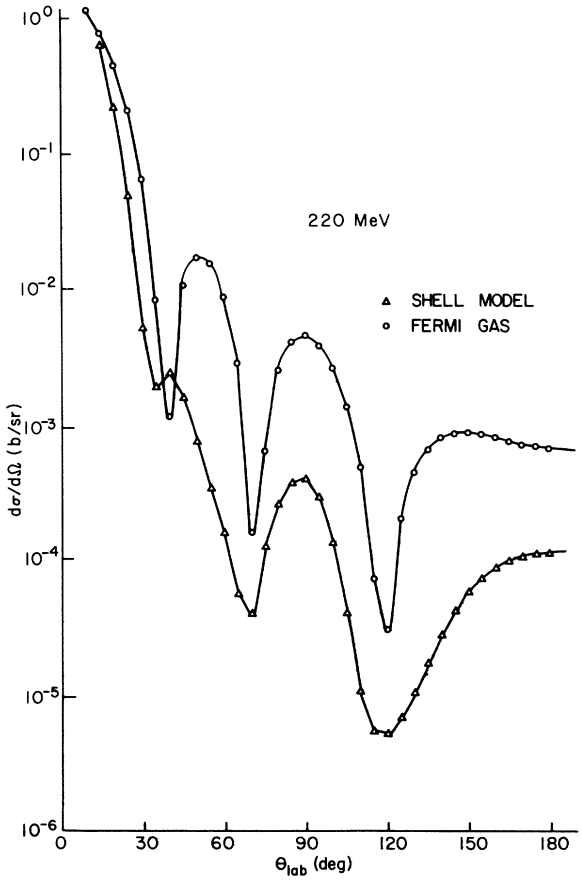

FIG. 7. Pion on ^{16}O. Differential cross sections 220 MeV.

ACKNOWLEDGMENTS

The idea of reconsidering pion-nucleus scattering was suggested by Professor Samuel Devons. The bulk of the work (including all self-consistent calculations) was done under the guidance of Professor Robert Serber. Professor Henry Foley and Professor Richard Friedberg made helpful comments. The computer centers of Columbia University, the City College of New York, and Queensborough Community College aided in calculations. Professor Victor Young provided constant encouragement during the preparation of this work.

APPENDIX: APPROXIMATIONS

In arriving at optical potentials and using these potentials to derive scattering amplitudes, the author has made a number of approximations, the validity of which can be investigated. In general, as Glauber[5] has emphasized, the accuracy of the treatment improves with increasing energy.

Possibly the most severe error comes at large scattering angles. In deriving Eq. (19a) the approximation has been made that

$$(\vec{K} - \vec{K}') \cdot \hat{K} Z \simeq 0 .$$

This clearly becomes invalid for θ such that

$$2KR \sin^2 \tfrac{1}{2}\theta > 1 .$$

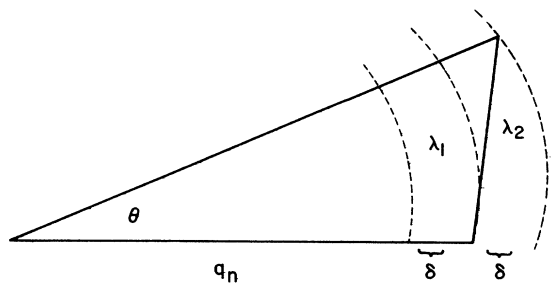

FIG. 8. Diagram indicating sense in which $f^2(\lambda)C(\lambda)$ is almost symmetric about $K' = q_n$.

For pions of 170-MeV kinetic energy, and a nuclear radius of 3.3×10^{-13} cm, this limits the region of obvious applicability to

$$\theta \ll \tfrac{1}{2}.$$

Glauber[5] (p. 345) has argued that the use of $[J_0(2l+1)\sin\frac{1}{2}\theta]$ to replace $P_l(\cos\theta)$ may increase the accuracy at large angles, but the author knows of no convincing analytic proof that this is so. Recent experiments[1] seem to indicate, nevertheless, that similar approximations work fairly well even at relatively large angles.

Another source of error is the neglect of third- and higher-order correlations. For the Fermi-gas model this certainly causes no problem, since one can readily verify that with or without higher-order correlations the nucleus will act as an essentially black body. For the shell model, on the other hand, the correlation contribution might be seen even in the near-resonance region, since the density goes to zero comparatively slowly. Now for constant or slowly varying densities, Eq. (2) suggests that a reasonable measure of the importance (P_i) of a correlation term of the ith order is

$$P_i=\left[\frac{2\pi N}{K}\,\mathrm{Im}f(0)\rho(r)\right]^{i-1}R_{ci}^{i-1}\ ,$$

where R_{ci} is a "correlation length" for the ith-order correlations (and is typically of the order of 1 F). One assumes that $\mathrm{Re}f(0)\ll\mathrm{Im}f(0)$ to obtain this. For second-order correlations, for example,

$$R_{c2}\equiv\int_0^\infty[g(r)-1]\,dr\,.$$

For our ^{16}O shell model $R_{c2}=-0.924$ F. Since the region of the ^{16}O nucleus within about 3 F of the center is effectively "black" to pions with energy near the (3-3) resonance, one need look only at $r\gtrsim3$ F to study the importance of higher-order correlations. Assuming that R_{ci} is of order unity, one can show that for the energies considered P_i is always less than $(0.2)^{i-1}$. Thus, higher-order correlations should not be very significant.

In the derivation of Eq. (2), the difference between V_{c00} and $V_{c\gamma\gamma}$, the averages of the total nucleus-pion potential over the ground and γth states, respectively, of the nucleus, has been neglected at certain points. For the case of matter extending uniformly to infinity this is exactly true. For further discussion see Ref. 7 (p. 94), where it is pointed out that only states γ reasonably similar to the ground state contribute significantly.

Use of the approximation

$$\int_{-\infty}^{s(\vec{r})}q_{nj}(s')ds'\simeq\int_{-\infty}^{s(\vec{x})}q_{nj}(s')ds'+\vec{q}_{nj}(\vec{x})\cdot(\vec{r}-\vec{x})$$

is valid under the same conditions as those required for the WKB approximation. To see this one notes that the approximation can be regarded as a Taylor expansion with respect to the path-length variable s, since

$$\frac{d}{ds}\int_{-\infty}^{s(\vec{x})}q_{nj}(s')ds'=q_{nj}(\vec{x})\,.$$

Then if higher derivatives of q_{nj} are small compared to q_{nj}, the approximation is valid. This imposes the condition

$$\left|\frac{1}{q_{nj}}\frac{dq_{ni}}{dr}\right|=\left|-\frac{1}{2(E_j-V_{c00})}\frac{dV_{c00}}{dr}\right|\ll1\,.$$

Thus, the approximation is good if the pion energy is high or if the potential varies slowly and is different in magnitude from the pion energy – the WKB restrictions.

The Fourier transform of Eq. (8) shows that this approximates the expression for $g-1$ of Eq. (4b) by $-\frac{1}{4}e^{-2\nu r^2}$. For $r=r'=0$, the two expressions agree. As $r\to\infty$ the expression of Eq. (4b) yields $\frac{3}{4}\leq g\leq\frac{5}{4}$, reflecting the influence of the exclusion principle. The approximate expression, of course, yields $g=1$ as $r\to\infty$. It has been shown (Ref. 7, pp. 66–67) that for incident wave number $K_0\gg\sqrt{8\nu}$ the $C(\kappa)$ given by Eq. (8) should be reasonable for those values of κ which are most heavily weighted by other factors in obtaining the scattering amplitudes. Since $K_0^2/8\nu$ is of the order of 2.5 for the energies considered, and since the laboratory scattering amplitude is peaked strongly forward for scattering angles $<60°$, as required from the discussion in the reference, one expects relatively little error from this approximation.

Neglect of the Principal Part Integral

In the $V_{c00}^{(2)}$ part of the V_{c00} of Eq. (2), a contribution which can be written in the form

$$(4\pi)^2N\rho^2\left[P\int d^3K'\frac{f^2(|\vec{q}_n-\vec{K}'|)C(|\vec{q}_n-\vec{K}'|)}{q_n^{\,2}-K'^2}\right]$$

has been neglected. Close examination of the integral justifies this for both the Fermi-gas model and the shell model of ^{16}O in the near-resonance region. Because the product of f^2 and C is vanishingly small otherwise, one needs only consider $\vec{K}'$ such that $|\vec{q}_n-\vec{K}'|$ is near zero. Thus the denominator in the integrand can be approximated by

$$\frac{1}{2q_n}\,\frac{1}{q_n-K'}$$

and the K'^2 in the d^3K' can be replaced by $q_n^{\,2}$.

One then has to evaluate

$$P\int_{-\infty}^{\infty} dK' \frac{f^2(|\vec{q}_n - \vec{K}'|)C(|\vec{q}_n - \vec{K}'|)}{q_n - K'}$$

(the lower limit can be extended to $-\infty$ since for regions with $K' \ll q_n$ the integrand is very small).

Let

$$\lambda(\vec{K}') \equiv |\vec{q}_n - \vec{K}'| .$$

One could prove that the integral vanishes by showing that, for each $\vec{K}'$ such that $|\vec{K}'| \approx q_n$ and $\lambda(\vec{K}') \ll q_n$, there is another wave vector $\vec{K}''$ which satisfies the same requirements *vis-a-vis* q_n and for which

$$q_n - K'' = K' - q_n$$

and

$$\lambda(\vec{K}'') = \lambda(\vec{K}') .$$

If this were true, the integral would vanish because of subtraction to zero for each $\vec{K}'$, $\vec{K}''$ pair. Although the conditions cannot be met exactly, they can be met approximately to high accuracy for $\lambda \ll q_n$. Referring to Fig. 8, one can show that for $|K' - q_n| \ll q_n$ and $\theta \ll 1$,

$$\frac{\Delta\lambda}{\lambda} \approx \frac{\delta}{q_n} \ll 1 .$$

Thus by choosing $\vec{K}''$ to be along the same direction as $\vec{K}'$ and with magnitude such that

$$|K'' - q_n| = |K' - q_n| ,$$

one sees that if the product $f^2(\lambda)C(\lambda)$ is a slowly varying function of λ for small λ, the integral will approximately vanish. Thus neglect of the principal-part integral is justifiable.

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*Supported in part by the National Science Foundation, U. S. Atomic Energy Commission, and the CUNY Research Foundation.

[1]E.g., F. Binon *et al.*, Nucl. Phys. B17, 168 (1970).

[2]For example, T. E. O. Ericson and J. Hüfner, Phys. Letters 33B, 601 (1970); M. P. Locher, O. Steinmann, and N. Straumann, Nucl. Phys. B27, 598 (1971); L. A. Charlton and J. M. Eisenberg, Ann. Phys. (N.Y.) 63, 386 (1971).

[3]For example, R. M. Frank, J. L. Gammel, and K. M. Watson, Phys. Rev. 101, 891 (1956).

[4]R. Serber, Phys. Rev. 72, 1114 (1947).

[5]R. J. Glauber, in *Lectures in Theoretical Physics*, edited by W. E. Brittin *et al.* (Interscience, New York, 1959), Vol. 1, p. 315.

[6]D. Roper and R. Wright, UCRL Report No. UCRL-7846 (unpublished). A slightly different version of their results (with B. Feld as co-author) appeared in Phys. Rev. 138, B190 (1965).

[7]D. Engelberg, Ph.D. dissertation, Columbia University, 1968 (unpublished).

[8]K. M. Watson, Phys. Rev. 89, 575 (1953). See Ref. 7 for a very direct approach which does not use the formalism of Watson and co-workers.

[9]H. A. Bethe, Ann. Phys. (N.Y.) 3, 190 (1958).

[10]P. A. M. Dirac, *The Principals of Quantum Mechanics*, (Oxford Univ. Press, 1958), p. 198, Eq. (35).

[11]R. Serber, Proc. Natl. Acad. Sci. 54, 692–696 (1965).

[12]D. Engelberg (unpublished); see also Ref. 7.

[13]M. Mayer and J. H. D. Jensen, *Nuclear Shell Structure* (Wiley, New York, 1955).

[14]R. Hofstadter [Ann. Rev. Nucl. Sci. 7, 231 (1957)] finds that the best rms value yields $r_0 = 1.35 \times 10^{-13}$ cm. The average value was chosen here in an attempt to lessen the dependence on the specific ^{16}O nucleus.

[15]R. J. Glauber, in *High Energy Physics and Nuclear Structure* (North-Holland, 1967), pp. 311–338.