K-Matrix Theory of Nuclear Form Factors*

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A method of constraining a shell-model-type nuclear bound-state calculation to give wave functions that display physical asymptotic behavior is derived and subsequently used to get an expression for the nuclear form factor. In this method one seeks the poles of the collision matrix rather than the eigenvalues of the Hamiltonian. We compare our method with that of Lane and Robson, that of Pinkston and Satchler, and that of Nagarajan and Tobocman.

I. INTRODUCTION

The form factor of a given nucleus for a particular reaction channel is just the radial function that results from calculating the projection of the wave function of the nucleus onto the channel-state wave function of the reaction channel. The nuclear form factor plays a central role in nuclear-reaction calculations. In these applications it is important that the far asymptotic behavior of the form factor have the correct exponential form. The nuclear wave functions provided by conventional shell-model-type nuclear-structure calculations will fail to produce the required accurate representation of the nuclear surface unless a large number of configurations are used.

Tobocman and Nagarajan[1] have proposed a method by means of which shell-model-type nuclear-structure calculations may be employed so that the resulting nuclear wave function has the correct asymptotic behavior. The proposed procedure requires the calculation of the poles of the collision matrix rather than the eigenvalues of the Hamiltonian. The method represents an application of the Wigner-Peierls R-matrix theory of nuclear reactions.[2] In subsequent papers[3] the related possibility of using the R-matrix formalism as a basis for shell-model-type calculations of nuclear reactions has been explored. These investigations have led to the conclusion that the K-matrix formalism is a more effective way than the R-matrix formalism to use a shell-model-type calculation to evaluate the nuclear-reaction collision matrix.

In this note the nuclear bound-state problem is reconsidered. Following the procedure developed for nuclear reactions, the R-matrix formalism for nuclear structure is transformed into a K-matrix formalism. An explicit expression for the form factor in terms of the K-matrix operator is derived in parallel with the corresponding expression for the scattering-state radial wave function.

In Sec. II the problem is formulated and the relevant quantities defined. The R-matrix formalism is derived in Sec. III and transformed to the K-matrix formalism in Sec. IV. In Sec. V explicit expressions for the scattering-state radial wave function and the bound-state form factor are presented. The derivation of the Lippman-Schwinger equation by the K-matrix formalism is given in Sec. VI. In Sec. VII our formalism for the form factor is compared with that of Lane and Robson, that of Pinkston and Satchler, and that of Nagarajan and Tobocman.

II. FORMULATION OF THE PROBLEM

Let the Schrödinger equation for the relative motion of the system of A nucleons be

$$(E-H)\Psi=0\,. \tag{1}$$

Associated with each way α of partitioning the A nucleons into two groups is a set of channel states $\Phi_{\alpha i}$ $(i=1,2,\ldots)$. Each state αi corresponds to particular states of internal motion, $\phi_{\alpha_i'}^{J_i' M_i'}(\xi_\alpha')$ and $\phi_{\alpha_i''}^{J_i'' M_i''}(\xi_\alpha'')$, being assigned to each of the two groups of nucleons, an orbital angular momentum eigenstate $Y_{L_i}^{M_i'''}(\Omega_\alpha)$ being assigned to the relative motion of the centers of mass of the two groups, and then coupling the three wave functions into a total angular momentum eigenstate,

$$\Phi_{\alpha i}(\xi_\alpha', \xi_\alpha'', \Omega_\alpha)=[\phi_{\alpha_i'}^{J_i' M_i'}(\xi_\alpha')\phi_{\alpha_i''}^{J_i'' M_i''}(\xi_\alpha'')Y_{L_i}^{M_i'''}(\Omega_\alpha)]^{J_i M_i}\,. \tag{2}$$

The form factor or radial wave function for channel αi is then

$$u_{\alpha i}(r)=\langle\Phi_{\alpha i}(\xi_\alpha', \xi_\alpha'', \Omega_\alpha)r_\alpha^{-1}\delta(r_\alpha-r)|\Psi\rangle \tag{3a}$$

$$\equiv\langle\alpha i; r|\Psi\rangle\,, \tag{3b}$$

where r_α is the separation of the centers of mass of the two groups of nucleons constituting one of the α-channel configurations.

For each partition we can decompose the Hamil-

tonian into a sum of terms:

$$H = H^0_\alpha(\xi'_\alpha, \xi''_\alpha) - \frac{\hbar^2}{2m_\alpha}\nabla^2_{r_\alpha} + V^0_\alpha(\xi'_\alpha, \xi''_\alpha, \vec{r}_\alpha), \tag{4a}$$

where H^0_α is the Hamiltonian for the internal degrees of freedom of each of the two groups of nucleons of partition α,

$$(E_{\alpha i} - H^0_\alpha)\phi^{J'_i M'_i}_{\alpha'_i}\phi^{J''_i M''_i}_{\alpha''_i} = 0, \tag{4b}$$

m_α is the reduced mass of the two groups of nucleons, and V^0_α is the interaction potential between the two groups. It is often helpful to introduce an optical-model potential $\overline{V}^0_\alpha(\vec{r}_\alpha)$ for each partition α. Then we have for the Hamiltonian the following decomposition for each partition:

$$H = H_\alpha + V_\alpha = H_\beta + V_\beta = \cdots, \tag{5a}$$

$$H_\alpha = H^0_\alpha(\xi'_\alpha, \xi''_\alpha) - \frac{\hbar^2}{2m_\alpha}\nabla^2_{r_\alpha} + \overline{V}^0_\alpha(\vec{r}_\alpha), \tag{5b}$$

$$V_\alpha = V^0_\alpha(\xi'_\alpha, \xi''_\alpha, \vec{r}_\alpha) - \overline{V}^0_\alpha(\vec{r}_\alpha). \tag{5c}$$

H_α is the optical-model Hamiltonian for partition α, and V_α is the associated residual interaction.

The asymptotic behavior of a scattering-state wave function $\Psi_{\beta j}$ is specified in terms of the elements $U_{\alpha i,\,\beta j}$ of the collision matrix and the radial wave functions $u_{\alpha i,\,\beta j}$:

$$u_{\alpha i,\,\beta j}(r) = \langle \alpha i; r | \Psi_{\beta j} \rangle, \tag{6a}$$

$$u_{\alpha i,\,\beta j}(r) \to \zeta^{(-)}_{\alpha i}(r)\delta_{\alpha\beta}\delta_{ij} - \zeta^{(+)}_{\alpha i}(r)U_{\alpha i,\,\beta j}. \tag{6b}$$

$\zeta^{(-)}_{\alpha i}$ and $\zeta^{(+)}_{\alpha i}$ are unit-current ingoing and outgoing radial wave functions for channel α:

$$(E - H_\alpha)\Phi_{\alpha j}(\xi'_\alpha, \xi''_\alpha, \Omega_\alpha)\zeta^{(\pm)}_{\alpha j}(r_\alpha)/r_\alpha = 0, \tag{7a}$$

$$\zeta^{(\pm)}_{\alpha j}(r_\alpha) \to (m_\alpha/\hbar k_{\alpha j})^{1/2}\exp(\pm i k_{\alpha j} r_\alpha \mp \tfrac{1}{2} i L_j \pi). \tag{7b}$$

$k_{\alpha j}$ is the wave number for channel αj. For closed channels $k_{\alpha j} = i|k_{\alpha j}|$. $\Psi_{\beta j}$ is a scattering state with unit incident current in channel βj and only outgoing current in all other channels. Coulomb distortions have been ignored for the sake of simplicity.

For a bound-state solution Ψ_b we specify the asymptotic behavior in terms of the channel amplitudes $U_{\alpha i,\,b}$ and the form factors $u_{\alpha i,\,b}$:

$$u_{\alpha i,\,b}(r) = \langle \alpha i; r | \Psi_b \rangle \to -\zeta^{(+)}_{\alpha i}(r)U_{\alpha i,\,b}, \tag{8}$$

where all channels are closed.

III. DERIVATION OF THE R-MATRIX FORMALISM

To implement the R-matrix formalism we divide the $3A$-dimensional configuration hyperspace of our system into an inside region and outside region by means of a closed hypersurface called the channel-entrance surface. This hypersurface consists of the adjoining hyperplane segments $r_\alpha = a_\alpha$, $r_\beta = a_\beta, \ldots$ which are called the channel entrances. The channel states $\Phi_{\alpha i}$ $(i = 1, 2, 3, \ldots)$ form a complete orthonormal set on channel entrance α.

Application of Green's theorem to $\langle H\bar{\Lambda}|\Psi\rangle - \langle\bar{\Lambda}|H\Psi\rangle$, where Ψ is a solution of Eq. (1), Λ is an arbitrary state, and the bracket is understood here and below to have the configuration space integration restricted to the inside region, leads to the following result[4]:

$$\langle\bar{\Lambda}|(E - H - L)^T|\Psi\rangle = -\langle\bar{\Lambda}|L|\Psi\rangle, \tag{9a}$$

$$L = \sum_\alpha\sum_i |\alpha i; r\rangle\,\delta(r - a_\alpha)\frac{\hbar^2}{2m_\alpha}\frac{d}{dr}\langle\alpha i; r|. \tag{9b}$$

The transpose is defined by

$$\langle\bar{\Lambda}|F^T|\Psi\rangle = \langle\bar{\Psi}|F|\Lambda\rangle, \tag{10a}$$

where the adjoint state $\bar{\Psi}$ is related to Ψ by complex conjugation of the radial wave functions,

$$u_{\alpha i}(r)^* = \langle\alpha i; r|\bar{\Psi}\rangle. \tag{10b}$$

Equation (9) can be rewritten to read

$$\Psi = G(L^T - L)\Psi \quad r_\alpha \leq a_\alpha, \tag{11a}$$

$$G = (E - H^T)^{-1}. \tag{11b}$$

This is one of the basic results of the R-matrix formalism which is so named because it is usually formulated in terms of $\langle\alpha i; a_\alpha|G|\beta j, a_\beta\rangle = R_{\alpha i;\,\beta j}$, the R matrix.

Equation (11) relates the value of Ψ at any point in the inside region to the values and normal gradients of Ψ on the channel-entrance surface. This equation provides a way of constraining the wave function Ψ to have the correct asymptotic behavior. Equation (6) or (8) is used for Ψ on the right-hand side of Eq. (11). Then, to the extent we provide an accurate representation of G and of U, the resulting expression for Ψ in the inside region has the correct asymptotic behavior as the channel-entrance surface is approached.

IV. TRANSITION TO THE K-MATRIX FORMALISM

We will next modify Eq. (11) so as to eliminate the explicit dependence on the channel radii $a_\alpha, a_\beta, \ldots$ contained in the operator L. The partition α optical-model Green's function G_α is introduced:

$$G_\alpha = (E - H^T_\alpha)^{-1}. \tag{12a}$$

For this Green's function we use the representation

$$\langle \xi'_\alpha, \xi''_\alpha, \Omega_\alpha, r_\alpha | G_\alpha | \bar{\xi}'_\alpha, \bar{\xi}''_\alpha, \bar{\Omega}_\alpha, \bar{r}_\alpha \rangle = \sum_{i=1}^{\infty} \Phi_{\alpha i}(\xi'_\alpha, \xi''_\alpha, \Omega_\alpha) \Gamma_{\alpha i}(r_\alpha, \bar{r}_\alpha) \Phi_{\alpha i}(\bar{\xi}'_\alpha, \bar{\xi}''_\alpha, \bar{\Omega}_\alpha)^* , \tag{12b}$$

$$\Gamma_{\alpha i}(r_\alpha, \bar{r}_\alpha) = -\frac{2}{\hbar} f_{\alpha i}(r_{\alpha<}) g_{\alpha i}(r_{\alpha>}) , \tag{12c}$$

$$f_{\alpha i}(r) = \frac{1}{2i} [e^{i\delta_{\alpha i}} \zeta^{(+)}_{\alpha i}(r) - e^{-i\delta_{\alpha i}} \zeta^{(-)}_{\alpha i}(r)]$$

$$\to (m_\alpha/\hbar k_{\alpha i})^{1/2} \sin(k_{\alpha i} r + \delta_{\alpha i} - \tfrac{1}{2} L_i \pi) , \tag{12d}$$

$$g_{\alpha i}(r) = \frac{1}{2i} [(i - s_\alpha) e^{i\delta_{\alpha i}} \zeta^{(+)}_{\alpha i}(r) + (i + s_\alpha) e^{-i\delta_{\alpha i}} \zeta^{(-)}_{\alpha i}(r)]$$

$$\to (m_\alpha/\hbar k_{\alpha i})^{1/2} [\cos(k_{\alpha i} r + \delta_{\alpha i} - \tfrac{1}{2} L_i \pi) - s_\alpha \sin(k_{\alpha i} r + \delta_{\alpha i} - \tfrac{1}{2} L_i \pi)] . \tag{12e}$$

The quantity $\Phi_{\alpha i} f_{\alpha i}(r_\alpha)/r_\alpha$ is the regular solution of Eq. (7a); the $\delta_{\alpha i}$'s are the optical-model phase shifts. Since the channel states $\Phi_{\alpha i}$ $(i = 1, 2, \ldots)$ are normalized on the finite hypersurface segment channel entrance α, we must write

$$G_\alpha G_\alpha^{-1} = G_\alpha^{-1} G_\alpha = P_\alpha , \tag{13a}$$

$$\langle \xi'_\alpha, \xi''_\alpha, \Omega_\alpha, r_\alpha | P_\alpha | \bar{\xi}'_\alpha, \bar{\xi}''_\alpha, \bar{\Omega}_\alpha, \bar{r}_\alpha \rangle = \sum_{i=1}^{\infty} \Phi_{\alpha i}(\xi'_\alpha, \xi''_\alpha, \Omega_\alpha) \frac{\delta(r_\alpha - \bar{r}_\alpha)}{r_\alpha \bar{r}_\alpha} \Phi_{\alpha i}(\bar{\xi}'_\alpha, \bar{\xi}''_\alpha, \bar{\Omega}_\alpha)^* . \tag{13b}$$

The quantity P_α is a projection operator onto a cylinder in configuration hyperspace whose cross section is channel entrance α.

From the fact that

$$G^{-1} = G_\alpha^{-1} - V_\alpha \tag{14a}$$

and

$$G G^{-1} = G^{-1} G = 1 \tag{14b}$$

and by using Eq. (13a) we find that

$$P_\alpha G = G_\alpha + G_\alpha V_\alpha G , \tag{15a}$$

$$G P_\beta = G_\beta + G V_\beta G_\beta . \tag{15b}$$

Use of Eq. (15) imposes the requirement on G that its asymptotic behavior at channel entrance α be exactly the same as that of G_α. This asymptotic behavior is controlled by the arbitrary coefficients s_α introduced in Eq. (12e).

Now we use Eqs. (15b), (12), and (6) in Eq. (11). The result is

$$\Psi_{\beta j} = \sum_\gamma \sum_i (1 + G V_\gamma) \phi_{\gamma i, \beta j} , \tag{16a}$$

$$\phi_{\gamma i, \beta j} = \Phi_{\gamma i}(\xi'_\gamma, \xi''_\gamma, \Omega_\gamma) f_{\gamma i}(r_\gamma) r_\gamma^{-1} \nu_{\gamma i, \beta j} , \tag{16b}$$

$$\nu_{\gamma i, \beta j} = (s_\gamma - i) e^{i\delta_{\gamma i}} \delta_{\gamma\beta} \delta_{ij} - (s_\gamma + i) e^{-i\delta_{\gamma i}} U_{\gamma i, \beta j} . \tag{16c}$$

Had we used Eq. (8) in place of Eq. (6), $\nu_{\gamma i, \beta j}$ would be replaced by $\nu_{\gamma i, b}$, and $\Psi_{\beta j}$ by Ψ_b:

$$\nu_{\gamma i, b} = -(s_\gamma + i) e^{-i\delta_{\gamma i}} U_{\gamma i, b} . \tag{16d}$$

Equation (16) gives the wave function in the interior in terms of the asymptotic behavior. Dependence on the channel radii $a_\alpha, a_\beta, \ldots$ is only implicit, no longer explicit.

V. RADIAL WAVE FUNCTION AND FORM FACTOR

Substituting Eq. (15a) for G in Eq. (16a) we proceed to the evaluation of the raidal wave function:

$$u_{\alpha i, \beta j}(r) = \sum_\gamma \sum_l \langle \alpha i; r | 1 + G_\alpha X_{\alpha\gamma} | \phi_{\alpha l, \beta j} \rangle , \tag{17a}$$

$$X_{\alpha\gamma} = V_\gamma + V_\alpha G V_\gamma . \tag{17b}$$

We substitute Eq. (12) for G_α in Eq. (17a) and find

$$u_{\alpha i, \beta j}(r) = \sum_\gamma \sum_l \left(\langle \alpha i; r | \phi_{\gamma l, \beta j} \rangle - \frac{2}{\hbar} \int_0^{a_\alpha} dr' f_{\alpha i}(r_<) g_{\alpha i}(r_>) \langle \alpha i; r' | X_{\alpha\gamma} | \phi_{\gamma l, \beta j} \rangle \right) , \tag{18}$$

which is understood to be valid for $r \leq a_\alpha$. For application to bound states the subscript βj is replaced by b.

Equations (16) and (17) are the principal results of this paper. For the choice $s_\alpha = s_\beta = \cdots = 0$ the $X_{\alpha\beta}$ operator becomes the K-matrix operator. That is why we call this the K-matrix formalism.

Now suppose we set $r = a_\alpha$ so that the asymptotic form, Eq. (6b), can be used for $u_{\alpha i,\beta j}$. Then

$$\zeta^{(-)}_{\alpha i}(a_\alpha)\delta_{\alpha\beta}\delta_{ij} - \zeta^{(+)}_{\alpha i}(a_\alpha)U_{\alpha i,\beta j} = \sum_\gamma \sum_l \left(\delta_{\alpha\gamma}\delta_{il} f_{\alpha i}(a_\alpha) - \frac{2}{\hbar} g_{\alpha i}(a_\alpha) X_{\alpha i,\gamma l}\right) \nu_{\gamma l,\beta j}, \tag{19a}$$

$$X_{\alpha i,\gamma l} = \left\langle \Phi_{\alpha i} \frac{f_{\alpha i}(r_\alpha)^*}{r_\alpha} \middle| X_{\alpha\gamma} \middle| \Phi_{\gamma l} \frac{f_{\gamma l}(r_{\gamma l})}{r_\gamma} \right\rangle . \tag{19b}$$

Substituting Eq. (16c) for $\nu_{\gamma l,\beta j}$ gives

$$Z^{(-)}_{\alpha i,\beta j} = \sum_\gamma \sum_l Z^{(+)}_{\alpha i,\gamma l} U_{\gamma l,\beta j}, \tag{20a}$$

$$Z^{(\pm)}_{\alpha i,\beta j} = g_{\alpha i}(a_\alpha)\left[\delta_{\alpha\beta}\delta_{ij} + \frac{2}{\hbar} X_{\alpha i,\beta j}(s_\beta \pm i)\right] e^{\mp i\delta_{\beta j}}. \tag{20b}$$

The corresponding equation for the channel amplitude is

$$0 = \sum_\gamma \sum_l Z^{(+)}_{\alpha i,\gamma l} U_{\gamma l,b}. \tag{20c}$$

To calculate radial wave functions for $r < a_\alpha$ we use Eq. (18) after having solved Eq. (20a) to determine the elements of the collision matrix. To calculate form factors for $r < a_\alpha$ we use Eq. (18) with subscript βj replaced by b after having solved Eq. (20c) for the channel amplitudes. Being homogeneous, Eq. (20c) can be solved only at those energies E for which $\det Z^{(+)} = 0$. These energies are the positions of the poles of the collision matrix. Then the resulting channel amplitudes will have an over-all undetermined multiplicative factor which is fixed by normalizing Ψ_b. For $r > a_\alpha$ the asymptotic expressions Eq. (6b) or (8) are used.

When we choose $s_\alpha = s_\beta = \cdots = 0$, then $X_{\alpha i,\beta j}$ will be the elements of the K matrix (reactance matrix); when $s_\alpha = s_\beta = \cdots = -i$, then the $X_{\alpha i,\beta j}$ are the elements of the T matrix (transition matrix). The formalism presented here hinges on the X operators $X_{\alpha\beta}$ defined in Eq. (17c). If our derivation had been carried out in such a manner that Eq. (15a) had been used before Eq. (15b), then we would find $X_{\alpha\gamma}$ replaced by $X'_{\alpha\gamma}$, where

$$X'_{\alpha\gamma} = V_\alpha + V_\alpha G V_\gamma . \tag{17c}$$

The equivalence of $X_{\alpha\gamma}$ and $X'_{\alpha\gamma}$ in Eq. (17a) and the equations that follow it is an expression of post-prior equivalence.

The evaluation of the operators $X_{\alpha\beta}$ which seems feasible is the use of a shell-model-type calculation to approximate G by inverting $E - H$ in a finite dimensional basis of configurations. The asymptotic behavior of the resulting G then depends on the configurations used. In order to impose the appropriate asymptotic behavior on G we make the transformation[5]

$$\begin{aligned} X_{\alpha\gamma} &= V_\gamma + V_\alpha(G_\gamma{}^{-1} - V_\gamma)^{-1}V_\gamma \\ &= V_\gamma + V_\alpha(V_\gamma - V_\gamma G_\gamma V_\gamma)^{-1}V_\gamma G_\gamma V_\gamma \end{aligned} \tag{21}$$

so that we invert $V_\gamma - V_\gamma G_\gamma V_\gamma$ instead of $E - H$. This in effect forces GP_γ to have the same asymptotic behavior as G_γ independent of the asymptotic behavior of the basis states used to calculate the matrix elements of $V_\gamma - V_\gamma G_\gamma V_\gamma$.

Note that the factor $g_{\alpha i}(a_\alpha)$ can be dropped from the definition of $Z^{(\pm)}_{\alpha i,\beta j}$ given in Eq. (20b) because it occurs as a common factor on both sides of Eqs. (20a) and (20c). Thus the only vestige of dependence on the channel radii $a_\alpha, a_\beta, \ldots$ in our formalism arises from the fact that the configuration space integration for the $X_{\alpha i,\beta j}$ is restricted to the inside region. This dependence is negligible for sufficiently large values of the channel radii provided we restrict ourselves to two-body channels.

VI. LIPPMAN-SCHWINGER EQUATION

It is an interesting exercise to see how our formalism leads to the Lippman-Schwinger equation for the wave function. Write Eq. (16a) in the form

$$\Psi_{\beta j} = \sum_\gamma \sum_i G(G^{-1} + V_\gamma)\phi_{\gamma i,\beta j} . \tag{22}$$

Then by use of Eq. (14a)

$$\Psi_{\beta j} = \sum_\gamma \sum_i G G_\gamma{}^{-1} \phi_{\gamma i,\beta j} . \tag{23}$$

Making use of Eq. (14a) again, we make the following transformation:

$$\begin{aligned} \sum_\gamma \sum_i G_\alpha G_\gamma{}^{-1} \phi_{\gamma i,\beta j} &= G_\alpha G^{-1}\Psi_{\beta j} = G_\alpha(G_\alpha{}^{-1} - V_\alpha)\Psi_{\beta j} \\ &= (P_\alpha - G_\alpha V_\alpha)\Psi_{\beta j} . \end{aligned}$$

Thus

$$P_\alpha\Psi_{\beta j} = G_\alpha \sum_\gamma \sum_i G_\gamma{}^{-1}\phi_{\gamma i,\beta j} + G_\alpha V_\alpha \Psi_{\beta j} . \tag{24}$$

Now if we choose $s_\alpha = s_\beta = \cdots = -i$, then Eq. (16c) becomes

$$\nu_{\gamma i,\beta j} = -2i\, e^{i\delta_{\gamma i}}\delta_{\gamma\beta}\delta_{ij} \quad (s_\gamma = -i) \tag{25}$$

so that

$$\phi_{\gamma i,\beta j}=\Phi_{\gamma i}f_{\gamma i}(r_\gamma)r_\gamma^{-1}\frac{2}{i}e^{i\delta_{\gamma i}}\delta_{\gamma\beta}\delta_{ij}$$
$$=\phi_{\gamma i}\delta_{\gamma\beta}\delta_{ij}. \tag{26}$$

Then Eq. (16a) becomes

$$\Psi_{\beta j}=(1+GV_\beta)\phi_{\beta j}, \tag{27}$$

and Eq. (24) becomes

$$P_\alpha\Psi_{\beta j}=G_\alpha G_\beta^{-1}\phi_{\beta j}+G_\alpha V_\alpha\Psi_{\beta j}, \tag{28a}$$

$$P_\beta\Psi_{\beta j}=\phi_{\beta j}+G_\beta V_\beta\Psi_{\beta j}. \tag{28b}$$

For the bound-state case $\nu_{\gamma i,\beta j}$ is replaced by $\nu_{\gamma i,b}$ with the result that

$$\phi_{\gamma i,b}=0 \quad (s_\gamma=-i) \tag{29a}$$

and

$$P_\beta\Psi_b=G_\beta V_\beta\Psi_b. \tag{29b}$$

Except for the presence of the projection operators, Eqs. (28) and (29b) are identical in form with the Lippman-Schwinger equation, but the interpretation of these equations is slightly different from what is customary. These equations refer to the finite inside region of configuration hyperspace. Therefore all states are normalizable. The limits of all quantities as the size of the inside region is allowed to become infinite are well defined in the sense of box normalization.

Eq. (24) is a generalization of the Lippman-Schwinger equation to the case where the Green's function operators G_α, G_β, ... fulfill arbitrary asymptotic boundary conditions.

VII. COMPARISON WITH OTHER FORMALISMS FOR THE FORM FACTOR

Finally, it might be well to compare our formalism with other procedures that have been proposed for evaluating the form factor. The Lane-Robson comprehensive formalism[6] is based on Eq. (9) with L replaced by

$$L'=\sum_a\sum_i|\alpha i;r\rangle\delta(r-a_\alpha)\frac{\hbar^2}{2m_\alpha}\left(\frac{d}{dr}-b_{\alpha i}\right)\langle\alpha i;r|, \tag{30}$$

where the $b_{\alpha i}$ are arbitrary. Then Eq. (9) is regarded as an inhomogeneous equation for Ψ, with the right-hand-side inhomogeneity reflecting the asymptotic boundary conditions. For a bound state Ψ_b, one can set $b_{\alpha j}=ik_{\alpha j}$ so that the right-hand side of Eq. (10a) vanishes:

$$(E-H-L')^T\Psi_b=0. \tag{31}$$

Equation (31) is then solved by diagonalization of $(H+L')^T$ in a finite dimensional basis with configuration-space integrations restricted to the inside region. This then provides representations for Ψ_b in the inside region which join smoothly with the asymptotic forms given by Eq. (8).

This method is very direct. It is easy to use for the type of problem where there is only one channel entrance. That would correspond to a case where inelastic channels are included but not rearrangement channels. The difficulty with including rearrangement channels is caused by the fact that the explicit dependence on the channel radii caused by L' makes it impractical to allow the channel radii to become infinite. Then the inclusion of rearrangement channels causes the channel-entrance surface to become so complex that it is difficult to restrict configuration-space integrations to the inside region.

The formalism of Pinkston and Satchler[7](PS) for the form factor uses an inhomogeneous differential equation derived from the Schrödinger equation:

$$\langle\alpha i;r|E-H_\alpha|\Psi_b\rangle=\langle\alpha i;r|V_\alpha|\Psi_b\rangle$$
$$=\left[E-E_{\alpha i}+\frac{\hbar^2}{2m_\alpha}\left(\frac{1}{r}\frac{d}{dr}r^2\frac{d}{dr}\frac{1}{r}-\frac{L_i(L_i+1)}{r^2}\right)-\bar{V}_\alpha^0(r)\right]u_{\alpha i,b}(r). \tag{32}$$

The formal solution to this equation can be constructed from Eq. (18) by replacing $X_{\alpha\gamma}$ by $V_\alpha(1+GV_\gamma)$ and using Eq. (16):

$$u_{\alpha i,b}(r)=\sum_\gamma\sum_l\langle ai;r|\phi_{\gamma l,b}\rangle-\frac{2}{\hbar}\int_0^{a_\alpha}dr'f_{\alpha i}(r_<)g_{\alpha i}(r_>)\langle\alpha i;r'|V_\alpha|\Psi_b\rangle. \tag{33}$$

The first term of Eq. (33) vanishes for the asymptotic boundary condition choice $s_\alpha=s_\beta=\cdots=-i$. To implement the PS method one carries out a conventional shell-model calculation for the bound-state wave function Ψ_b. The form factor is then calculated from Eq. (32) or (33) using this Ψ_b to calculate the inhomogeneity $\langle\alpha i;r|V_\alpha|\Psi_b\rangle$.

Thus the PS method is seen to be based on essentially the same expression for the form factor as our K-matrix method. Where they use a shell-model-type calculation to evaluate the bound-state wave function

Ψ_b [for use in Eq. (32) or (31)], we use a shell-model-type calculation to evaluate $(V_\gamma - V_\gamma G_\gamma V_\gamma)^{-1}$ in the expression for the operator

$$X_{\alpha\gamma} = V_\gamma + V_\alpha (V_\gamma - V_\gamma G_\gamma V_\gamma)^{-1} V_\gamma G_\gamma V_\gamma = V_\gamma + V_\alpha G V_\gamma \tag{34}$$

[(for use in Eq. (18)].

The PS method appears to be somewhat easier to use than ours. On the other hand, we think it may be possible that our method will give superior results. This possibility is suggested by the fact that the bound-state energy levels we get from the solution of $\det Z^{(+)} = 0$ required by our method will be shifted somewhat from those resulting from the diagonalization of H required by the PS method. The shift results from the boundary-condition constraints being present already at that stage of our calculation.

We use $\sum_\gamma \sum_i X_{\alpha\gamma} \phi_{\gamma i,b}$ to generate the form factor $u_{\alpha i,b}(r)$ in place of the mathematically equivalent $V_\alpha \Psi_b$ used by PS. In both cases the required quantities can be generated by a shell-model-type calculation. However, in our case the shell-model-type calculation is constrained to conform to the required asymptotic behavior. In the PS method it is not.

We conclude by comparing our K-matrix method with the R-matrix method of Ref. 1. The R-matrix formalism results when we work directly from Eq. (11) without making the substitutions derived in Sec. IV. Thus if we substitute the asymptotic forms of Eqs. (6) and (8) directly into Eq. (11) we find

$$Z^{(-)}_{\alpha i,\beta j} = \sum_\gamma \sum_l Z^{(+)}_{\alpha i,\gamma l} U_{\gamma l,\beta j}\,, \tag{34a}$$

$$0 = \sum_\gamma \sum_l Z^{(+)}_{\alpha i,\gamma l} U_{\gamma l,b}\,, \tag{34b}$$

$$Z^{(\pm)}_{\alpha i,\beta j} = \xi^{(\pm)}_{\alpha i}(a'_\alpha)\delta_{\alpha\beta}\delta_{ij} - \frac{\hbar^2}{2m_\beta}\left(\xi^{(\pm)}_{\beta j}(a_\beta)\frac{d}{da_\beta} R_{\alpha i,\beta j}(a'_\alpha, a_\beta) - R_{\alpha i,\beta j}(a'_\alpha, a_\beta)\frac{d}{da_\beta}\xi^{(\pm)}_{\beta j}(a_\beta)\right), \tag{34c}$$

$$R_{\alpha i,\beta j}(a'_\alpha, a_\beta) = \langle \alpha i; a'_\alpha | G | \beta j; a_\beta \rangle\,, \quad a'_\alpha \lesssim a_\alpha\,. \tag{34d}$$

These equations are to be solved for the collision matrix elements $U_{\alpha i,\beta j}$ or channel amplitudes $U_{\alpha i,b}$. Then these are to be used to calculate the radial wave functions or form factors in the inside region from the expressions provided by Eqs. (11), (8), and (6):

$$u_{\alpha i,\beta j}(r) = \sum_\gamma \sum_l R_{\alpha i,\gamma l}(r, a_\gamma)\frac{\hbar^2}{2m_\gamma}\left(\vec{\frac{d}{da_\gamma}} - \vec{\frac{d}{da_\gamma}}\right)\left[\xi^{(-)}_{\gamma l}(a_\gamma)\delta_{\gamma\beta}\delta_{lj} - \xi^{(+)}_{\gamma l}(a_\gamma)U_{\gamma l,\beta j}\right], \tag{35a}$$

$$u_{\alpha i,b}(r) = \sum_\gamma \sum_l R_{\alpha i,\gamma l}(r, a_\gamma)\frac{\hbar^2}{2m_\gamma}\left(\vec{\frac{d}{da_\gamma}} - \vec{\frac{d}{da_\gamma}}\right)\left[-\xi^{(+)}_{\gamma l}(a_\gamma)U_{\gamma l,b}\right]. \tag{35b}$$

Eqs. (34) and (35) are to be compared with Eqs. (20) and (17). By eliminating the R matrix in favor of $X_{\alpha\gamma}$ operator, the K-matrix formalism succeeds in eliminating explicit dependence on the channel radii.

In Ref. (1) it was not exactly Eq. (34b) which was derived for the bound-state case but rather

$$\begin{aligned} 0 &= \sum_\gamma \sum_l \frac{\hbar^2}{2m_\alpha}\left[\xi^{(+)}_{\alpha i}(a'_\alpha)\frac{d}{da'_\alpha} - \left(\frac{d}{da'_\alpha}\xi^{(+)}_{\alpha i}(a'_\alpha)\right)\right] Z^{(+)}_{\alpha i,\gamma l}(a'_\gamma, a_\alpha) U_{\gamma l,b}\,, \\ &= \sum_\gamma \sum_l \frac{\hbar^2}{2m_\alpha}\xi^{(+)}_{\alpha i}(a'_\alpha)\left[\vec{\frac{d}{da'_\alpha}} - \vec{\frac{d}{da'_\alpha}}\right] R_{\alpha i,\gamma l}(a'_\alpha, a_\gamma)\left[\vec{\frac{d}{da_\gamma}} - \vec{\frac{d}{da_\gamma}}\right]\frac{\hbar^2}{2m_\gamma}\xi^{(+)}_{\gamma l}(a_\gamma)U_{\gamma l,b}\,, \end{aligned} \tag{36}$$

which is more symmetric.

It is clear that Eqs. (35b) and (36) cannot be used for the outgoing wave boundary-condition choice $s_\alpha = s_\beta = \cdots = i$, since that makes the right-hand side vanish identically independent of the values chosen for the channel amplitudes. The same restriction applies to Eqs. (18) and (20c), because then $s_\beta + i = 0$, $\phi_{\gamma l,b} = 0$, and $X_{\alpha\gamma} = T_{\alpha\gamma} = \infty$ at bound-state energies.

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$(p, 2p)$ Reactions With Divers Potentials*

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The p-p cross sections relevant to a factorized-impulse-approximation treatment of $(p,2p)$ reactions from 150 to 350 MeV are calculated. Several on-shell prescriptions are used and compared with the half-off-shell prescription suggested by the Faddeev-Watson multiple-scattering series. Four phenomenological potentials, Hamada-Johnston, Bryan-Scott III, Reid hard core, and Reid soft core, are investigated as well as three potentials which are phase-shift equivalent to the Reid soft-core potential. We find significant differences between the various prescriptions for all of the potentials. We also observe a wide range in the predictions on the various phenomenological potentials for each prescription. However, the ratio of the half-shell cross section to the on-shell cross section is remarkably insensitive to the choice of potential, especially where the half-shell prescription is needed. This suggests a simple method for extrapolating from the elastic scattering data to the appropriate half-shell cross section.

I. INTRODUCTION

Most analyses of $(p, 2p)$ reactions make use of a factorized impulse approximation in which the reaction amplitude is written as a product of a proton-proton scattering amplitude and an integral over distorted waves.[1-5] In a previous paper[6] the plane-wave impulse approximation was employed to display the effects of various prescriptions for the choice of variables in the proton-proton amplitude. The conclusion of that work, in which the Reid soft-core potential[7] was taken as a model of the proton-proton interaction, was that the use of the half-off-shell scattering amplitude suggested by the Faddeev-Watson multiple-scattering theory makes a marked difference for incident proton energies below 200 MeV. The purpose of this paper is to explore (a) the extent to which that conclusion is model independent, and (b) the extent to which $(p, 2p)$ reactions may be used to distinguish among models of the proton-proton interaction by probing their off-shell cross sections.

To investigate these points, we have calculated the p-p cross sections relevant to the knock out of a proton bound by 45 MeV by an incident proton with energies ranging from 150 to 350 MeV. We have calculated these cross sections with the half-off-shell prescription as well as various on-shell prescriptions for a number of different potentials.

The potentials studied include four phenomenological potentials which are fitted to nucleon-nucleon scattering data with some theoretical constraints on their functional forms. They are the Reid soft core (RSC),[7] the Reid hard core (RHC),[7] the Hamada-Johnston (HJ),[8] and the Bryan-Scott III (BS).[9] In order to focus on purely off-shell differences, we also investigate three potentials which are phase-shift equivalent to the RSC but yield very different results when used in nuclear-matter calculations.[10]