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# Gel'fand-Levitan-Unitarity-Transform Formalism for Direct Extension of the Two-Nucleon $T$ Matrix off the Energy Shell*

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The unitary-transform method of Coester *et al.* is modified, for uncoupled partial waves in which there are no bound states, so that empirical phase shifts rather than a potential fitted to them may be used as the basic input. This is accomplished by invoking the Gel'fand-Levitan inverse scattering formalism to generate a complete orthonormal set of scattering wave functions from the phase shifts. The result is a convenient formal framework for analyzing the uncertainties in the off-energy-shell behavior of the two-nucleon interaction. Variations in the off-energy-shell $T$ matrix arising from changes in the phase shifts, as well as those due to different short-range nonlocalities, may be studied directly using the method presented here.

## I. INTRODUCTION

The unitary-transform method of Coester *et al.*[1] provides an elegant and straightforward procedure for studying the arbitrariness in the two-nucleon $T$ matrix off the energy shell (hereafter called the off-shell $T$) once the on-energy-shell $T$ matrix (on-shell $T$) has been specified. As such, it has already been applied in several calculations to investigate the dependence of multinucleon observables on specifics of the two-nucleon interaction.[2] However, because this scheme takes as its basic input a potential fitted to the empirical nucleon-nucleon elastic scattering phase shifts, the resulting off-shell $T$'s are related only indirectly to the available data. Moreover, reliance on a parametrized potential introduced at the outset is a disadvantage in the following practical sense: The elastic scattering phase shifts at high energies are unknown and almost certainly unknowable. It is therefore important to determine the sensitivity of the off-shell $T$ to variations in these ambiguous quantities. A calculation which adopts a particular potential commits itself to a fixed set of high-energy phase shifts, and a different potential must be introduced in order to change them. Not only does this entail cumbersome recalculation, but it also introduces additional uncertainties because it is unlikely that the second potential gives the same fit to the empirical low-energy phase shifts as the first one. Of course, since the low-energy phase shifts are not known to arbitrary accuracy, it is of interest to test the sensitivity of the off-shell $T$ to changes in these quantities as well. However, the uncontrollable differences which result from the *ad hoc* substitution of one potential for another do not seem well suited to such studies.

In this paper, we present a pedestrian remedy for the above difficulties. We eliminate the input potential by merging the unitary-transform method with the inverse scattering theory of Gel'fand and Levitan,[3] which generates a complete orthonormal set of scattering wave functions directly from the phase shifts. The resulting formalism provides a complete framework for analyzing the sources of uncertainty in the off-energy-shell behavior of the two-nucleon interaction, assuming that this interaction is well represented by an energy-independent potential (local or nonlocal) in conjunction with the nonrelativistic Schrödinger equation. Specifically, our emendation of the procedure of Ref. 1 makes it possible to:

(1) Go directly from phase shifts to off-shell $T$'s without explicit introduction of a potential.

(2) Investigate the arbitrariness of these off-shell $T$'s subject to the constraints that the long-range part of the interaction is given by a local potential suggested by meson theory and that the scattering wave functions form a complete orthonormal set (in the absence of bound states).

(3) Study in a direct way the influence of high-energy phase shifts on off-energy-shell behavior.

Our discussion is restricted to uncoupled partial waves in which there are neither bound states nor Coulomb effects.

The idea of using inverse scattering theory to study the two-nucleon interaction is by no means novel. However, previous papers[4] have been directed towards computing a local potential from the phase shifts. Since their appearance, it has been pointed out by Baranger *et al.*[5] and by Mongan[6] that it would generally be preferable to eliminate the potential in favor of the off-shell $T$, which has closer links with experiment. This is the point of view we have adopted here. To us, the local potential which is determined by the Gel'fand-Levitan procedure is a largely irrelevant by-product of an algorithm which furnishes a convenient basis of scattering wave functions from which all necessary $T$ matrix elements are easily computed. Except for its long-range part, which has been fairly well determined, we never concern ourselves with the explicit form of this local potential. The short-range part of the true interaction is probably nonlocal in any event, and it is to simulate this nonlocality that the unitary transforms are introduced.

In Secs. II and III, we assemble known but previously unrelated results into a coherent formalism for going from empirical phase shifts to off-shell $T$'s. A summary of the Gel'fand-Levitan derivation of scattering wave functions from phase shifts is given in Sec. II. Section III presents the formulas for constructing $T$ matrix elements half off the energy shell (half-shell $T$'s) from the Gel'fand-Levitan wave functions. From these Gel'fand-Levitan half-shell $T$'s, it is easy to obtain the families of half-shell $T$'s generated by various short-range unitary transforms. The subtracted Low equation[5,6] may then be used to find the corresponding $T$ matrix elements completely off the energy shell (fully off-shell $T$'s). Explicit formulas giving the response of the half-shell $T$'s to changes in the phase shifts are developed in Sec. IV. Again, the subtracted Low equation then provides the corresponding alterations in the fully off-shell $T$'s. In Sec. V, we indicate how meson-theory constraints on the long-range part of the interaction may be incorporated. We discuss the prospects for application and generalization of our approach in Sec. VI.

## II. FROM PHASE SHIFTS TO A COMPLETE ORTHONORMAL SET OF SCATTERING RADIAL FUNCTIONS

We work in the center-of-mass frame of the two nucleons, with units such that $\hbar = 1 = M$, where $M$ is the nucleon mass.

Given a set of empirical $l$-wave phase shifts, $\delta_l(k_i)$, at momentum data points $k_i$, our first task is to construct a function $\delta_l(k)$, $0 \leq k < \infty$, which interpolates smoothly between the prescribed data points and which extrapolates them to high momenta. Both interpolation and extrapolation are clearly nonunique. For the time being, we assume that the necessary curve fitting has been done and that one particular phase-shift parametrization, $\delta_l(k)$, has been chosen. (We shall show, in Sec. IV, how the effects of phase-shift ambiguities may be investigated; considerations in the choice of functional forms for $\delta_l(k)$ are discussed briefly in Sec. V.) From this function, a complete orthonormal set of scattering radial functions may be constructed by the method of Gel'fand and Levitan.[3] In order to define the terminology of this procedure, we digress momentarily to discuss the radial equation in the $l$th partial wave for a local potential.

The equation in question is

$$\left[\frac{d^2}{dr^2} + k^2 - \frac{l(l+1)}{r^2} - L(r)\right] w_l(k,r) = 0\,, \tag{1a}$$

where $L(r)$ is a local potential subject to the usual conditions that $\int_0^\infty dr\, r|L(r)| < \infty$ and $\int_0^\infty dr\, r^2|L(r)| < \infty$, and

$$w_l(k,r) \underset{r\to\infty}{\sim} \sin[kr - l\pi/2 + \delta_l(k)] \tag{1b}$$

is the "physical" radial function. Here $\delta_l(k)$ is the usual scattering phase shift generated by the potential $L(r)$. When $L(r) = 0$, $w_l(k,r) = u_l(kr)$, the $l$th Riccatti-Bessel function. It is convenient to define the Jost regular radial function $\varphi_l(k,r)$ as that solution of Eq. (1a) whose behavior at the origin is

$$\varphi_l(k,r) \underset{r\to 0}{\sim} r^{l+1}/(2l+1)!!\,. \tag{1c}$$

$\varphi_l(k,r)$ is an entire function of $k$ for fixed finite $r$, and the connection between $\varphi_l$ and $w_l$ is

$$w_l(k,r) = [k^{l+1}/|f_l(k)|\,]\varphi_l(k,r)\,, \tag{2}$$

where $f_l(k)$ is the $l$-wave Jost function.[3] The singularities in $k$ of $w_l(k,r)$ are thus isolated in the factor $|f_l(k)|^{-1}$. The free-particle regular solution

$$\varphi_l^{(0)}(k,r)=u_l(kr)/k^{l+1} \tag{3}$$

is also useful.

With these definitions, we may now pose the problem which the Gel'fand-Levitan theory answers for us: Given $\delta_l(k)$, $0\le k<\infty$, and the presumption that there are no bound states in the $l$th partial wave, construct a complete orthonormal set of functions $w_l(k,r)$ such that $w_l(k,r) \underset{r\to\infty}{\sim} \sin[kr-l\pi/2+\delta_l(k)]$, with the $\delta_l(k)$ in this asymptotic form guaranteed to be the prescribed phase shifts. The construction proceeds as follows:

First, calculate the Jost function using the representation

$$f_l(k)=\exp\left(-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{dq\,\delta_l(q)}{q-k+i\epsilon}\right), \tag{4}$$

where the continuation of $\delta_l(q)$ to negative momenta is defined by

$$\delta_l(-q)=-\delta_l(q). \tag{4$'$}$$

Next, define the spectral density $\rho_l(E)$ and its free-particle counterpart $\rho_l^{(0)}(E)$ such that

$$\frac{d\rho_l(E)}{dE}=\frac{2}{\pi}k^{2l+1}/|f_l(k)|^2 \tag{5a}$$

and

$$\frac{d\rho_l^{(0)}(E)}{dE}=\frac{2}{\pi}k^{2l+1}. \tag{5b}$$

From these, construct the Gel'fand-Levitan driving term $g_l(r,r')$ given by

$$g_l(r,r')=\int_0^\infty d[\rho_l^{(0)}(E)-\rho_l(E)]\varphi_l^{(0)}(k,r)\varphi_l^{(0)}(k,r') \tag{6}$$

or, more explicitly,

$$g_l(r,r')=(2/\pi)\int_0^\infty dk\,k^{2l+2}(1-|f_l(k)|^{-2}) \times\varphi_l^{(0)}(k,r)\varphi_l^{(0)}(k,r'). \tag{6$'$}$$

Then solve the Gel'fand-Levitan equation

$$K_l(r,r')=g_l(r,r')+\int_0^r dr''K_l(r,r'')g_l(r'',r') \tag{7}$$

for the Gel'fand-Levitan kernel $K_l(r,r')$. Finally, obtain $\varphi_l(k,r)$ according to

$$\varphi_l(k,r)=\varphi_l^{(0)}(k,r)+\int_0^r dr'K_l(r,r')\varphi_l^{(0)}(k,r'), \tag{8a}$$

which translates into

$$w_l(k,r)=|f_l(k)|^{-1}\left[u_l(kr)+\int_0^r dr'K_l(r,r')u_l(kr')\right]. \tag{8b}$$

As a by-product of the Gel'fand-Levitan analysis, it can be shown that $w_l(k,r)$ satisfies a radial equation of the form (1a) with a local potential $L(r)$ given by

$$L(r)=2\frac{d}{dr}K_l(r,r). \tag{9}$$

The resulting $w_l(k,r)$ are orthonormal,

$$\int_0^\infty dr\,w_l(k,r)w_l(k',r)=(\pi/2)\delta(k-k'), \tag{10a}$$

and complete,

$$\int_0^\infty dk\,w_l(k,r)w_l(k,r')=(\pi/2)\delta(r-r'). \tag{10b}$$

## III. FROM RADIAL FUNCTIONS TO FAMILIES OF PHASE-SHIFT-EQUIVALENT OFF-SHELL $T$'S

### A. Gel'fand-Levitan Off-Shell $T$'s

The off-shell partial-wave $T$ matrix $t_l(p,k;s^2)$ is defined by the Lippmann-Schwinger equation[7]

$$t_l(p,k;s^2)=v_l(p,k)+\frac{2}{\pi}\int_0^\infty\frac{dq\,q^2v_l(p,q)t_l(q,k;s^2)}{s^2-q^2+i\epsilon}, \tag{11}$$

where

$$v_l(p,k)=\int_0^\infty dr\,r^2j_l(pr)\int_0^\infty U(r,r')j_l(kr')r'^2dr', \tag{11$'$}$$

and $j_l(x)$ is the usual spherical Bessel function. $U(r,r')$ is in general a nonlocal potential; the local Gel'fand-Levitan potential is inserted in (11$'$) by writing $U(r,r')=\delta(r-r')L(r)/rr'$. When $p$, $k$, and $s$ are all different, $t_l$ is fully off-shell; when either $p=s$ or $k=s$, $t_l$ is half off-shell; and when $p=k=s$, $t_l$ is on-shell. $t_l(p,k;s^2)$ also satisfies the Low equation,[7] which, in the absence of bound states, reads

$$t_l(p,k;s^2)=v_l(p,k)+\frac{2}{\pi}\int_0^\infty\frac{dq\,q^2t_l(p,q;q^2)t_l^*(k,q;q^2)}{s^2-q^2+i\epsilon}. \tag{12}$$

As pointed out in Refs. 5 and 6, by taking the difference of (12) and the expression

$$t_l(p,k;k^2)=v_l(p,k)+\frac{2}{\pi}\int_0^\infty \frac{dq\,q^2 t_l(p,q;q^2)t_l^*(k,q;q^2)}{k^2-q^2+i\epsilon}\,,$$

we obtain the subtracted Low equation,

$$t_l(p,k;s^2)=t_l(p,k;k^2)+(k^2-s^2)\times\frac{2}{\pi}\int_0^\infty \frac{dq\,q^2 t_l(p,q;q^2)t_l^*(k,q;q^2)}{(s^2-q^2+i\epsilon)(k^2-q^2+i\epsilon)}\,, \tag{13}$$

which provides an explicit recipe for calculating the fully off-shell $T$ from the half-shell $T$. Thus all off-shell information is contained in the quantities $t_l(p,k;k^2)$.

It has been shown[8,9] that $t_l(p,k;k^2)$ may be evaluated without explicit use of a potential by means of the relation

$$t_l(p,k;k^2)=t_l(k)(p/k)^l+(pk)^{-1}(k^2-p^2)\times e^{i\delta_l(k)}\int_0^\infty dr\,u_l(pr)\Delta_l(k,r)\,, \tag{14a}$$

where

$$t_l(k)\equiv t_l(k,k;k^2)=-(1/k)\sin\delta_l(k)e^{i\delta_l(k)}\,, \tag{14b}$$

and the difference function $\Delta_l(k,r)$ is

$$\Delta_l(k,r)=w_l(k,r)-\cos\delta_l(k)u_l(kr)-\sin\delta_l(k)y_l(kr)\,. \tag{14c}$$

Here $y_l(x)$ is the Riccatti-Neumann function defined such that

$$y_l(x)\underset{x\to\infty}{\sim}\cos(x-l\pi/2)\,.$$

Equations (14a), (14b), and (14c) permit us to calculate a set of half-shell $T$ matrix elements directly from the parametrization of the empirical phase shifts, $\delta_l(k)$, and the Gel'fand-Levitan radial functions $w_l^{\rm GL}(k,r)$. Using (8b) for the radial function, we see that the Gel'fand-Levitan difference function $\Delta_l^{\rm GL}(k,r)$ is given by

$$\Delta_l^{\rm GL}(k,r)=|f_l(k)|^{-1}\int_0^r dr'K_l(r,r')u_l(kr')+[|f_l(k)|^{-1}-\cos\delta_l(k)]u_l(kr)-\sin\delta_l(k)y_l(kr)\,. \tag{15}$$

The off-shell $T$ matrix elements obtained from Eqs. (14a)–(15) we call the Gel'fand-Levitan $T$'s, or G-L $T$'s for short. By construction, they satisfy the constraints imposed by the empirical phase shifts and by the requirement that the scattering wave functions form a complete orthonormal set. Once $\delta_l(k)$ is specified on $0\le k<\infty$, the G-L $T$'s are uniquely determined (when there are no bound states) by the Gel'fand-Levitan algorithm. However, while the Gel'fand-Levitan potential is necessarily local, there is every reason to believe that the two-nucleon potential should be nonlocal at small distances. Consequently, the G-L $T$'s must be regarded as only one set out of an infinite class of physically plausible off-shell $T$'s satisfying identical phase-shift constraints. Until a reliable theoretical model of the short-range nonlocality is found, it is necessary to try to isolate the resultant ambiguities and to examine their practical implications for multinucleon calculations. Since we now have at our disposal a complete orthonormal set of scattering wave functions, we can study the changes induced in the off-shell $T$'s by the short-range unitary transforms of Coester *et al.*,[1] which correspond to nonlocal modifications of the potential at small distances. Although such a procedure cannot exhaust all imaginable alterations of the off-shell $T$'s compatible with phase-shift equivalence, it should provide a good indication of the importance of such variations.

### B. Families of Phase-Shift-Equivalent Half-Shell $T$'s

Following Coester *et al.*[1] and Haftel and Tabakin,[10] consider the functions $\tilde w_l(k,r)$ generated from the G-L $w_l^{\rm GL}(k,r)$ according to the rule

$$\tilde w_l(k,r)=w_l^{\rm GL}(k,r)+\int_0^\infty dr'\Lambda_l(r,r')w_l^{\rm GL}(k,r')\,. \tag{16a}$$

If

$$\lim_{r\to\infty}|r\Lambda_l(r,r')|<\infty \tag{16b}$$

and

$$w_l^{\rm GL}(k,r)\underset{r\to\infty}{\sim}\sin[kr-l\pi/2+\delta_l(k)]\,,$$

then also

$$\tilde w_l(k,r)\underset{r\to\infty}{\sim}\sin[kr-l\pi/2+\delta_l(k)]\,. \tag{16c}$$

The $\tilde w_l(k,r)$ may therefore be viewed as an alternate set of scattering radial functions satisfying the same phase-shift constraints as the G-L $w_l^{\rm GL}(k,r)$. From them, we may compute another set of off-shell $T$'s which has as much phenomenological validity as the set of G-L $T$'s. In order that the $\tilde t_l(p,k;k^2)$ calculated from these functions fulfill the requirement of half-shell unitarity,

$$\operatorname{Im}\tilde t_l(p,k;k^2)=-k\tilde t_l(p,k;k^2)\tilde t_l^*(k)\,,$$

(where "Im" denotes "imaginary part of"), $\tilde{w}_l(k,r)$, and hence also $\Lambda_l(r,r')$, must be real. Since the transformed radial functions must be sufficiently regular at the origin to be compatible with the centrifugal term in the radial equation, $\Lambda_l(r,r')$ must behave as

$$\Lambda_l(r,r') \underset{r\to 0}{\sim} r^{l+1}. \tag{16d}$$

Finally, the necessary and sufficient conditions which guarantee that the $\tilde{w}_l$ obey the same orthonormality and completeness relations as the $w_l^{\mathrm{GL}}$ are

$$\Lambda_l(r,r')+\Lambda_l(r',r)+\int_0^\infty dr''\Lambda_l(r,r'')\Lambda_l(r',r'')=0 \tag{16e}$$

and

$$\int_0^\infty dr''\Lambda_l(r,r'')\Lambda_l(r',r'')=\int_0^\infty dr''\Lambda_l(r'',r)\Lambda_l(r'',r'), \tag{16f}$$

as may be verified by direct substitution. Any real kernel $\Lambda_l(r,r')$ which satisfies (16b), (16d), (16e), and (16f) may be used to generate from the Gel'fand-Levitan radial functions another complete orthonormal set of radial functions corresponding to the same prescribed phase shifts. Particularly simple examples of suitable $s$-wave kernels have been presented in Refs. 2 and 10. Variation of the parameters in the transforms given in these papers generates families of radial functions having identical phase shifts.

The correspondingly transformed half-shell $T$'s are easily related to the G-L $T$'s by means of Eq. (14a) applied with both transformed and untransformed radial functions:

$$\tilde{t}_l(p,k;k^2)=\tilde{t}_l(k)(p/k)^l+(pk)^{-1}(k^2-p^2)e^{i\tilde{\delta}_l(k)}\int_0^\infty dr\,u_l(pr)\tilde{\Delta}_l(k,r), \tag{17}$$

where

$$\tilde{\Delta}_l(k,r)=\tilde{w}_l(k,r)-\cos\tilde{\delta}_l(k)u_l(kr)-\sin\tilde{\delta}_l(k)y_l(kr), \tag{17$'$}$$

and

$$t_l(p,k;k^2)=t_l(k)(p/k)^l+(pk)^{-1}\ (k^2-p^2)e^{i\delta_l(k)}\int_0^\infty dr\,u_l(pr)\Delta_l^{\mathrm{GL}}(k,r). \tag{18}$$

Since $\tilde{\delta}_l(k)=\delta_l(k)$, $\tilde{t}_l(k)=t_l(k)$. Subtracting (18) from (17), we find

$$\tilde{t}_l(p,k;k^2)=t_l(p,k;k^2)+(pk)^{-1}(k^2-p^2)e^{i\delta_l(k)}\int_0^\infty dr\,u_l(pr)[\tilde{\Delta}_l(k,r)-\Delta_l^{\mathrm{GL}}(k,r)]. \tag{19}$$

According to (14c), (17$'$), and (16a), we have

$$\begin{aligned}\tilde{\Delta}_l(k,r)-\Delta_l^{\mathrm{GL}}(k,r)&=\tilde{w}_l(k,r)-w_l^{\mathrm{GL}}(k,r)\\&=\int_0^\infty dr'\Lambda_l(r,r')w_l^{\mathrm{GL}}(k,r'),\end{aligned} \tag{20}$$

so that

$$\tilde{t}_l(p,k;k^2)=t_l(p,k;k^2)+(pk)^{-1}(k^2-p^2)\,e^{i\delta_l(k)}\int_0^\infty dr\,u_l(pr)\int_0^\infty dr'\Lambda_l(r,r')w_l^{\mathrm{GL}}(k,r'). \tag{21}$$

An operator version of Eq. (21), relating the half-shell $T$'s for two phase-shift-equivalent potentials, has previously been derived by Monahan, Shakin, and Thaler,[11] by means of an operator two-potential formula.

With the help of the subtracted Low equation, (13), we may also relate the transformed fully off-shell $T$'s to their Gel'fand-Levitan counterparts. Let us write

$$D_l(p,k)=\int_0^\infty dr\,u_l(pr)\int_0^\infty dr'\Lambda_l(r,r')w_l^{\mathrm{GL}}(k,r') \tag{22a}$$

and

$$\tau_l(p,k)=e^{-i\delta_l(k)}t_l(p,k;k^2). \tag{22b}$$

Then, substitution of (21) into

$$\bar{t}_l(p,k;s^2)=\bar{t}_l(p,k;k^2)+(k^2-s^2)\ \frac{2}{\pi}\int_0^\infty \frac{dq\,q^2\bar{t}_l(p,q;q^2)\bar{t}_l^*(k,q;q^2)}{(s^2-q^2+i\epsilon)(k^2-q^2+i\epsilon)}$$

yields

$$\bar{t}_l(p,k;s^2)=t_l(p,k;s^2)+(pk)^{-1}(k^2-p^2)e^{i\delta_l(k)}D_l(p,k)$$
$$+(k^2-s^2)\frac{2}{\pi}\int_0^\infty \frac{dq}{(s^2-q^2+i\epsilon)}\left[\frac{q(q^2-p^2)}{p(k^2-q^2+i\epsilon)}D_l(p,q)\tau_l(k,q)\right.$$
$$\left.-\frac{q}{k}\tau_l(p,q)D_l(k,q)+\frac{p^2-q^2}{pk}D_l(p,q)D_l(k,q)\right]. \qquad (23)$$

The above analysis shows that once the Gel'fand-Levitan equation has been solved for a particular parametrization of the empirical phase shifts, whole classes of off-shell $T$'s which are phase-shift-equivalent to the G-L $T$'s may be calculated simply by a series of quadratures. As in Refs. 1 and 10, the transform kernel $\Lambda_l(r,r')$ may be chosen to facilitate some of the integrations. Moreover, Eqs. (21) and (23) provide simple, direct relations among the various transformed off-shell $T$'s (it is clear that these equations can be rewritten to relate two transformed $T$'s, say $\bar{t}_l$ and $\bar{\bar{t}}_l$, by mere relabeling) and also show just how the transformed $T$'s differ from the $T$ matrix corresponding to the unique local potential determined by the phase-shift parametrization.

## IV. RESPONSE OF THE HALF-SHELL $T$'S TO CHANGES IN THE PHASE SHIFTS

As mentioned at the beginning of Sec. II, there is considerable latitude in the choice of a parametrization of the empirical phase shifts. Most of the freedom lies in the extrapolation beyond the range of the data – reliable phase-shift analyses do not extend to center-of-mass momenta much above 2 fm$^{-1}$ – but there is also some leeway in fitting the phase shifts at lower energies. It is clearly useful to have a systematic procedure for assessing the variations which reasonable changes in the phase-shift parametrization may produce in the off-shell $T$'s.

Suppose, then, that we vary $\delta_l(k)$ according to

$$\delta_l(k)\to\delta_l(k,\lambda)=\delta_l(k)+\lambda\eta_l(k). \qquad (24)$$

What is the concomitant variation in the G-L half-shell $T$'s? From the answer to this question, we can eventually deduce, by means of the formulas of the preceding section, the changes induced in all other off-shell $T$'s under consideration.

Let $t_l(p,k;k^2,\lambda)$ be the G-L half-shell $T$ corresponding to $\delta_l(k,\lambda)$. Equation (14a) allows us to write

$$t_l(p,k;k^2,\lambda)=t_l(k,\lambda)(p/k)^l+(pk)^{-1}(k^2-p^2)$$
$$\times e^{i\delta_l(k,\lambda)}\int_0^\infty dr\,u_l(pr)\Delta_l^{\rm GL}(k,r;\lambda), \qquad (25a)$$

where

$$t_l(k,\lambda)=-(1/k)\sin\delta_l(k,\lambda)e^{i\delta_l(k,\lambda)} \qquad (25b)$$

is the new on-shell $T$ and $\Delta_l^{\rm GL}(k,r;\lambda)$ is the new Gel'fand-Levitan difference function determined by $\delta_l(k,\lambda)$. Subtracting (14a) from (25a), we have the change induced in the G-L half-shell $T$ by the variation (24):

$$t_l(p,k;k^2,\lambda)-t_l(p,k;k^2)$$
$$=[t_l(k,\lambda)-t_l(k)](p/k)^l+(pk)^{-1}(k^2-p^2)$$
$$\times e^{i\delta_l(k)}\int_0^\infty dr\,u_l(pr)\mathfrak{D}_l(k,r;\lambda), \qquad (26)$$

where

$$\mathfrak{D}_l(k,r;\lambda)=e^{i\lambda\eta_l(k)}\Delta_l^{\rm GL}(k,r;\lambda)-\Delta_l^{\rm GL}(k,r). \qquad (26')$$

The only quantity on the right-hand side of (26) which is not immediately obtainable in terms of $\delta_l(k,\lambda)$ and $\delta_l(k)$ is $\mathfrak{D}_l(k,r;\lambda)$. According to Eq. (14c),

$$\mathfrak{D}_l(k,r;\lambda)=e^{i\lambda\eta_l(k)}w_l^{\rm GL}(k,r;\lambda)-w_l^{\rm GL}(k,r)$$
$$-[e^{i\lambda\eta_l(k)}\cos\delta_l(k,\lambda)-\cos\delta_l(k)]u_l(kr)$$
$$-[e^{i\lambda\eta_l(k)}\sin\delta_l(k,\lambda)-\sin\delta_l(k)]y_l(kr),$$

where $w_l^{\rm GL}(k,r;\lambda)$ is the Gel'fand-Levitan radial function constructed from $\delta_l(k,\lambda)$. Since

$$\cos\delta u_l(x)+\sin\delta y_l(x)=e^{-i\delta}[u_l(x)-ktz_l(x)], \qquad (27a)$$

where

$$t=-(1/k)\sin\delta e^{i\delta} \qquad (27b)$$

and

$$z_l(x)=y_l(x)+iu_l(x) \qquad (27c)$$

is a Riccatti-Hankel function, the above may be written in slightly more compact form as

$$\mathfrak{D}_l(k,r;\lambda)=e^{i\lambda\eta_l(k)}w_l^{\mathrm{GL}}(k,r;\lambda)-w_l^{\mathrm{GL}}(k,r)+ke^{-i\delta_l(k)}[t_l(k,\lambda)-t_l(k)]z_l(kr)\,. \tag{28}$$

Finally, $w_l^{\mathrm{GL}}(k,r;\lambda)$ may be generated from the initial $w_l^{\mathrm{GL}}(k,r)$ by a variant of the procedure of Sec. II.[3]

From $\delta_l(k,\lambda)$ compute a new Jost function

$$f_l(k,\lambda)=\exp\left[-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{dq\;\delta_l(q,\lambda)}{q-k+i\epsilon}\right], \tag{29a}$$

which may also be written as

$$f_l(k,\lambda)=f_l(k)F_l(k,\lambda)\,, \tag{29b}$$

where

$$F_l(k,\lambda)=\exp\left[-\frac{\lambda}{\pi}\int_{-\infty}^{\infty}\frac{dq\;\eta_l(q)}{q-k+i\epsilon}\right]. \tag{29c}$$

Define a new spectral density $\rho_l(E,\lambda)$ such that

$$\frac{d\rho_l(E,\lambda)}{dE}=\frac{2}{\pi}\frac{k^{2l+1}}{|f_l(k,\lambda)|^2}=|F_l(k,\lambda)|^{-2}\frac{d\rho_l(E)}{dE}. \tag{30}$$

Then construct a driving term

$$\begin{aligned}\Delta g_l(r,r';\lambda)&=\int_0^\infty d[\rho_l(E)-\rho_l(E,\lambda)]\varphi_l(k,r)\varphi_l(k,r)\\&=\frac{2}{\pi}\int_0^\infty dk\,(1-|F_l(k,\lambda)|^{-2})w_l^{\mathrm{GL}}(k,r)w_l^{\mathrm{GL}}(k,r')\,,\end{aligned} \tag{31}$$

where $\varphi_l(k,r)$ and $w_l^{\mathrm{GL}}(k,r)$ are the regular and physical radial functions constructed from $\delta_l(k)$ by means of Eqs. (4)–(8b). Next, find the kernel $\Delta K_l(r,r';\lambda)$ which solves

$$\Delta K_l(r,r';\lambda)=\Delta g_l(r,r';\lambda)\;+\int_0^r dr''\Delta K_l(r,r'';\lambda)\Delta g_l(r'',r';\lambda)\,. \tag{32}$$

The new regular function $\varphi_l(k,r;\lambda)$ is then given by

$$\varphi_l(k,r;\lambda)=\varphi_l(k,r)+\int_0^r dr'\Delta K_l(r,r';\lambda)\varphi_l(k,r')\,. \tag{33}$$

Replacing regular radial functions by physical ones in (33), we have the desired expression for $w_l^{\mathrm{GL}}(k,r;\lambda)$ in terms of $w_l^{\mathrm{GL}}(k,r)$,

$$w_l^{\mathrm{GL}}(k,r;\lambda)=|F_l(k,\lambda)|^{-1}\left[w_l^{\mathrm{GL}}(k,r)+\int_0^r dr'\Delta K_l(r,r';\lambda)w_l^{\mathrm{GL}}(k,r')\right]. \tag{34}$$

In view of this result, we may rewrite Eq. (28) as

$$\begin{aligned}\mathfrak{D}_l(k,r;\lambda)=[F_l(-k,\lambda)]^{-1}&\left\{[1-F_l(-k,\lambda)]w_l^{\mathrm{GL}}(k,r)+\int_0^r dr'\Delta K_l(r,r';\lambda)w_l^{\mathrm{GL}}(k,r')\right\}\\&+ke^{-i\delta_l(k)}[t_l(k,\lambda)-t_l(k)]z_l(kr)\,,\end{aligned} \tag{35}$$

where we have used the fact that, for real $k$,

$$F_l(-k,\lambda)=|F_l(k,\lambda)|e^{-i\lambda\eta_l(k)}, \tag{35$'$}$$

which may be deduced from Eq. (29c).

Equations (26) and (35), together with the rules embodied in Eqs. (29a)–(34), provide an explicit prescription for calculating the change in the G-L half-shell $T$'s when the phase-shift parametrization is altered according to Eq. (24). It is quite clear how the corresponding variations in all other off-shell $T$'s may be found with the help of the results of Sec. III. For example, in an obvious notation, Eq. (21) gives

$$\tilde{t}_l(p,k;k^2,\lambda)=t_l(p,k;k^2,\lambda)+(pk)^{-1}(k^2-p^2)e^{i\delta_l(k,\lambda)}\int_0^\infty dr\,u_l(pr)\int_0^\infty dr'\Lambda_l(r,r')w_l^{\mathrm{GL}}(k,r';\lambda)\,, \tag{36}$$

so that the response of $\tilde{t}_l(p,k;k^2)$ to the change in phase shift is

$$\tilde{t}_l(p,k;k^2,\lambda)-\tilde{t}_l(p,k;k^2)=t_l(p,k;k^2,\lambda)-t_l(p,k;k^2)$$
$$+(pk)^{-1}(k^2-p^2)e^{i\delta_l(k)}\int_0^\infty dr\,u_l(pr)\int_0^\infty dr'\Lambda_l(r,r')[e^{i\lambda\eta_l(k)}w_l^{\mathrm{GL}}(k,r',\lambda)-w_l^{\mathrm{GL}}(k,r')]\,, \tag{37}$$

where $w_l^{\mathrm{GL}}(k,r;\lambda)$ is given by (34). Fully off-shell $T$'s may be studied in like fashion by invoking the subtracted Low equation.

It is easy to reduce Eqs. (26)–(37) to analogous formulas valid to first order in $\lambda$. These could conceivably be of use in checking the stability of a given set of off-shell $T$'s against small changes in the input data. However, we found the resulting equations to be no more transparent than the general relations just given, so we do not reproduce them here. At any rate, if a computer program has been written to solve the Gel'fand-Levitan equation, (7), it should not require much additional effort to modify it to solve the similar equation (32), with which the stability of the off-shell $T$'s against phase-shift changes of any magnitude may be investigated.

## V. INCORPORATING LONG-RANGE POTENTIALS SUGGESTED BY MESON THEORY

The description of the two-nucleon force in terms of various meson-exchange processes indicates that the long-range part of this interaction is well represented by a superposition of local Yukawa potentials.[12] By a judicious choice of phase-shift parametrization, the G-L $T$'s can be made to conform with this theoretical constraint. Simple restrictions on the range of the unitary transforms then suffice to guarantee that all the phase-shift-equivalent off-shell $T$'s generated according to the rules of Sec. III will have the effects of this prescribed long-range behavior built into them.

In order to understand how the Yukawa form of the long-range potential manifests itself in the Gel'fand-Levitan construction, we establish a relation between the analytic properties of $f_l(k)$ in the complex $k$ plane and the large-$r$ behavior of the Gel'fand-Levitan potential $L(r)$. This may then be translated into a connection between the functional form of $\delta_l(k)$ and that of the long-range part of $L(r)$ by means of the relation

$$\delta_l(k)=\mathrm{Im}\ln f_l(k)\,, \tag{38}$$

which may be read from Eq. (4).

Suppose for the moment that we know $L(r)$ and that we can write it as the Yukawa superposition

$$L(r)=\int_\mu^\infty d\alpha\,C(\alpha)e^{-\alpha r}\,. \tag{39}$$

As elsewhere in this paper, we assume that $L(r)$ does not support any bound states. Then, as Martin has shown,[13] $f_l(k)$ is analytic everywhere in the $k$ plane except on a branch cut along the positive imaginary axis running from $k=i\mu/2$ to $k=i\infty$. There is, in fact, a one-to-one correspondence between the potential weight function $C(\alpha)$ and the discontinuity of $f_l(k)$ across the cut such that the long-range (small $\alpha$) part of the potential is completely determined by the discontinuity of $f_l(k)$ across the segment of the cut closest to the real axis. It is therefore an immediate consequence of Martin's analysis that specification of the discontinuity of $f_l(k)$ for $k=i\mu/2$ to $k=ib\mu/2$, where $b$ is some real constant, is equivalent to prescribing $C(\alpha)$ for $\alpha=\mu$ to $\alpha=b\mu$, and thus, to fixing the longest-ranged components of the potential. With the hindsight provided by Martin, we show here, in a decidedly nonrigorous way, how this correspondence emerges naturally within the present formalism.

We use the representation[14]

$$K_l(r,r')=\int_0^\infty d[\rho_l^{(0)}(E)-\rho_l(E)]\varphi_l(k,r)\varphi_l^{(0)}(k,r') \tag{40}$$

to investigate the link between the analytic properties of $f_l(k)$ in the $k$ plane and the behavior of $K_l(r,r)$ for large $r$. Differentiation of the latter function with respect to $r$ then gives us $L(r)$ for large $r$, according to Eq. (9).

Inserting the relations (5a) and (5b) in (40), we have

$$K_l(r,r)=\frac{2}{\pi}\int_0^\infty dk\,k^{2l+2}[1-|f_l(k)|^{-2}]\varphi_l(k,r)\varphi_l^{(0)}(k,r)\,. \tag{41}$$

The integrand on the right-hand side of (41) is an even function of $k$,[15] and for real $k$,[16]

$$f_l^*(k)=f_l(-k) \tag{42}$$

so that

$$K_l(r,r)=\frac{1}{\pi}\int_{-\infty}^{\infty}dk\,k^{2l+2}\left[\frac{f_l(k)f_l(-k)-1}{f_l(k)f_l(-k)}\right]\times\varphi_l(k,r)\varphi_l^{(0)}(k,r)\,. \tag{43}$$

Now, $\varphi_l(k,r)$ may be decomposed into[17]

$$\varphi_l(k,r)=\tfrac{1}{2}ik^{-l-1}[f_l(-k)f_l(k,r)-(-)^l f_l(k)f_l(-k,r)]\,, \tag{44}$$

where $f_l(k,r)$ is an irregular solution of Eq. (1a) which goes as[18]

$$f_l(k,r)\underset{r\to\infty}{\sim} i^l e^{-ikr}\,. \tag{44'}$$

Similarly, we write

$$\varphi_l^{(0)}(k,r)=\tfrac{1}{2}ik^{-l-1}[f_l^{(0)}(k,r)-(-)^l f_l^{(0)}(-k,r)]\,, \tag{45}$$

where

$$f_l^{(0)}(k,r)=y_l(kr)-iu_l(kr)\underset{r\to\infty}{\sim} i^l e^{-ikr}\,. \tag{45'}$$

Using (44) and (45) in Eq. (43), we find that

$$K_l(r,r)=(-1/4\pi)[K_l^{++}(r,r)+K_l^{--}(r,r)-(-)^l K_l^{+-}(r,r)]\,, \tag{46}$$

where

$$K_l^{++}(r,r)=\int_{-\infty}^{\infty}dk\left[\frac{f_l(k)f_l(-k)-1}{f_l(k)}\right]f_l(k,r)f_l^{(0)}(k,r)\,, \tag{46'}$$

$$K_l^{--}(r,r)=\int_{-\infty}^{\infty}dk\left[\frac{f_l(k)f_l(-k)-1}{f_l(-k)}\right]f_l(-k,r)f_l^{(0)}(-k,r), \tag{46''}$$

and

$$K_l^{+-}(r,r)=\int_{-\infty}^{\infty}dk\left[\frac{f_l(k)f_l(-k)-1}{f_l(k)f_l(-k)}\right]\times[f_l(-k)f_l(k,r)f_l^{(0)}(-k,r)+f_l(k)f_l(-k,r)f_l^{(0)}(k,r)]\,. \tag{46'''}$$

Now, according to (44′) and (45′),

$$K_l^{+-}(r,r)\underset{r\to\infty}{\sim}(-)^l\int_{-\infty}^{\infty}dk\left[\frac{f_l(k)f_l(-k)-1}{f_l(k)f_l(-k)}\right]\times[f_l(-k)+f_l(k)]\,, \tag{47}$$

i.e., $K_l^{+-}(r,r)$ becomes, for very large $r$, a constant independent of $r$. [Since we have a good deal of freedom in parametrizing $f_l(k)$ from the data, and since it is consistent with the properties of $f_l(k)$ for potentials satisfying the regularity criteria cited above Eq. (1b), we can certainly arrange things so that this constant is finite.] We see then, from Eqs. (9) and (46), that

$$L(r)\underset{r\to\infty}{\sim}-\frac{1}{2\pi}\frac{d}{dr}[K_l^{++}(r,r)+K_l^{--}(r,r)]\,. \tag{48}$$

In fact, since for real $k$ Eq. (41) holds, as well as[19]

$$f_l^*(-k,r)=(-)^l f_l(k,r) \tag{49a}$$

and

$$f_l^{(0)*}(-k,r)=(-)^l f_l^{(0)}(k,r)\,, \tag{49b}$$

Eq. (48) simplifies slightly to

$$L(r)\underset{r\to\infty}{\sim}-\frac{1}{\pi}\mathrm{Re}\frac{d}{dr}K_l^{++}(r,r)\,, \tag{50}$$

where "Re" denotes "real part of."

What is the form of $K_l^{++}(r,r)$ for large $r$? We have

$$K_l^{++}(r,r)\underset{r\to\infty}{\sim}(-)^l\int_{-\infty}^{\infty}dk\left[\frac{f_l(k)f_l(-k)-1}{f_l(k)}\right]e^{-2ikr}\,. \tag{51}$$

If we make specific assumptions about the analytic properties of $f_l(k)$, we may consider the right-hand side of (51) as part of a contour integral. With all due hindsight, we suppose that $f_l(k)$ is analytic for all $k$ except on a branch cut extending from $k=i\mu/2$ to $k=i\infty$ along the positive imaginary axis. Then the continuation of $f_l(-k)$ off the real axis is analytic everywhere except on a cut along the negative imaginary axis from $-i\mu/2$ to $-i\infty$.[3] Consider a contour $C$ consisting of the real $k$ axis from $-K$ to $K$ and a semicircle of radius $K$ in the lower half plane, indented to avoid the branch cut of $f_l(-k)$. Since no singularities of the integrand are enclosed by $C$,

$$I(r)=\int_C dk\left[\frac{f_l(k)f_l(-k)-1}{f_l(k)}\right]e^{-2ikr}=0\,.$$

In the limit $K\to\infty$, the contribution to $I(r)$ from the two quarter-circle arcs vanishes, leaving the nonzero contributions from the real axis and from the indentation around the cut to cancel one another:

$$\int_{-\infty}^{\infty}dk\left[\frac{f_l(k)f_l(-k)-1}{f_l(k)}\right]e^{-2ikr}=\int_{-i\mu/2}^{-i\infty}dk\,e^{-2ikr}\,\mathrm{disc}[f_l(-k)]\,, \tag{52}$$

where

$$\mathrm{disc}[f_l(-k)] = \lim_{\epsilon \to 0^+} [f_l(-k+\epsilon) - f_l(-k-\epsilon)],$$
$$\mathrm{Re}(k) = 0, \quad \mathrm{Im}(k) < -\mu/2. \tag{52'}$$

Because[20]

$$f_l^*(-k^*) = f_l(k) \tag{53}$$

for $k$ not on either branch cut, $\mathrm{disc}[f_l(-k)]$ is purely imaginary. If, on the branch cut from $-i\mu/2$ to $-i\infty$, we let

$$k = -i\chi, \quad \chi > \mu/2 \tag{54}$$

we have

$$\mathrm{disc}[f_l(-k)] = 2i \lim_{\epsilon \to 0^+} \mathrm{Im} f_l(i\chi + \epsilon), \tag{55}$$

and Eq. (51) becomes

$$K_l^{++}(r,r) \underset{r\to\infty}{\sim} (-)^l 2 \int_{\mu/2}^{\infty} d\chi\, e^{-2\chi r} [\lim_{\epsilon \to 0^+} \mathrm{Im} f_l(i\chi + \epsilon)]. \tag{56}$$

Then, defining

$$\alpha = 2\chi \tag{57a}$$

and

$$\rho_l(\alpha) = (-)^{l+1} \chi \lim_{\epsilon \to 0^+} \mathrm{Im} f_l(i\chi + \epsilon) \tag{57b}$$

we may write Eq. (50) as

$$L(r) \underset{r\to\infty}{\sim} -\frac{2}{\pi} \int_{\mu}^{\infty} d\alpha\, \rho_l(\alpha) e^{-\alpha r}, \tag{58}$$

which shows that the long-range potential is indeed a superposition of Yakawa components. Equation (57b) also shows that the weight function $\rho_l(\alpha)$ for small $\alpha$, which determines the Yakawa components of longest range, is given by the discontinuity of $f_l(-k)$ along the segment of the cut nearest the real axis.

The foregoing discussion leaves us with the problem of designing the phase-shift parametrization $\delta_l(k)$ so that upon insertion in Eq. (4), it automatically yields a Jost function having the desired analytic structure. It would appear to be simplest just to work backwards, as indicated as the beginning of this section: Choose an appropriate functional form for $f_l(k)$ [containing factors of the type $\int_\mu^{N\mu} d\alpha\, B(\alpha) \ln(1+2ik/\alpha)$, for example] and fix its free parameters by fitting the phase shifts, via Eq. (38), and the discontinuity across the nearby segment of the cut.

Finally, it is clear that the effects of the long-range potential (58) will be incorporated in the families of off-shell $T$'s generated by the unitary transforms (16) if the transform kernel $\Lambda_l(r,r')$ is only allowed to modify the Gel'fand-Levitan radial functions at sufficiently small distances. This restriction can be enforced by replacing Eq. (16b) by a condition of the form

$$\lim_{r\to\infty} |\Lambda_l(r,r')/L(r)| < |C|, \tag{16b'}$$

where $L(r)$ is given by Eq. (58) and $C$ is some constant whose magnitude will depend on the significance one attaches to the given long-range potential.

## VI. CONCLUDING REMARKS

We have shown how families of off-shell $T$'s may be constructed from a parametrization of the empirical phase shifts by solving one Volterra integral equation [Eq. (7)] and performing a series of integrations. The response of these off-shell $T$'s to alterations in the phase-shift parametrization may then be found by solving another Volterra equation [Eq. (32)], quite similar in form to the first, and carrying out another set of integrations. Since Benn and Scharf[4] have already demonstrated the numerical feasibility of solving integral equations of essentially the same type as Eqs. (7) and (32), we believe that the computations required to implement our formalism will be relatively straightforward. There is no doubt that a full-scale, quantitative study of the uncertainties in the two-nucleon $T$ matrix off the energy shell involves considerable effort in calculation no matter how it is done; our approach offers economies at least in the organization of such an undertaking.

The tools required to generalize our analysis to encompass both the existence of bound states and the coupling of orbital angular-momentum channels are readily available. As first applied to scattering theory by Jost and Kohn[3] and by Levinson,[3] the Gel'fand-Levitan theory already includes the possibility of bound states. The Gel'fand-Levitan radial functions are then no longer uniquely determined by the phase shift. However, one would hope to minimize the added ambiguity in the two-nucleon problem by using data to help delineate the deuteron wave function. The generalization of Gel'fand-Levitan theory to coupled partial waves was carried out by Newton and Jost.[21] Fuda[22] has recently extended the expressions (14a)–(14c) to coupled channels containing bound states, and unitary transforms have been written for such channels by Haftel and Tabakin.[2] The subtracted Low equation for this case was developed by Mongan.[6] In principle, then, the contents of this paper can be extended to cover all two-nucleon partial waves without difficulty, omitting Coulomb effects. Inclusion of the latter in a convincing way promises to be very difficult.[23]

It should be noted that the regularity conditions stated above Eq. (1b) specifically exclude hardcore and other truly singular potentials. Since most standard results of scattering theory must be amended for such potentials,[7] this limitation is not peculiar to our method.

An inverse scattering theory distinct from that of Gel'fand and Levitan, but actually rather similar in construction, has been devised by Marchenko.[24] It could be used equally well as the starting point of our development.

Although a parametrization of high-energy phase shifts is a necessary ingredient of our method, we recognize that the interpretation of data at center-of-mass momenta as low as 2 $\mathrm{fm}^{-1}$ in terms of a nonrelativistic, energy-independent potential is exceedingly suspect on general physical grounds. If such a model of the two-nucleon interaction is to be at all self-consistent, the hope must be that there exists some parametrization compatible with the data for which high-momentum $T$ matrix elements play a minor role in determining low-energy multinucleon observables. An important prospective application of our formalism is an orderly search for that ideal model.

*Note added.* After the present paper had been submitted for publication, we learned of related work of Karlsson.[25] Karlsson uses the Marchenko method to formulate a momentum-space procedure for going directly from the phase shifts to the off-shell $T$ associated with the local potential defined by the inversion scheme. We thank S. Coon for bringing the preprint to our attention, and B. Karlsson for a copy of the paper.

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