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δ-Function Model for the Two-Nucleon Interaction

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An interaction of the form $V_{LJST}(r)=\sum(v_i/r_i)\delta(r-r_i)$ is used to make a fit to the Livermore X phase shifts (0–460 MeV). Only states with $L\leq 2$ are considered; coupling between states is at present neglected but effective central $^3S_1$ and $^3D_1$ interactions are given. No more than four terms (per state) are required. In states which have a repulsion the position of the innermost δ function can be chosen arbitrarily within wide limits; e.g., for the $^1S_0$ state $0.1542\leq r_1\leq 0.95$ F. Families of hard-core interactions in which r_c and r_1 are arbitrary are also determined. We find $0.1542\leq r_c\leq 0.7$ F for the $^1S_0$ state. All $^1S_0$ interactions give essentially the same D-state phase shifts within the same energy range (these differ from $^1D_2$ experimental phase shifts). The main virtues of this type of potential are the ease with which the parameters may be determined and its relative simplicity in calculations for $A\geq 3$ and nuclear matter. It may also be written in the separable form $\langle rLJST|V|r'LJST\rangle=\sum(v_i/r_i)\delta(r-r_i)\delta(r'-r_i)$.

I. INTRODUCTION

By far, the most important sources of our knowledge of the two-nucleon interaction which is used in many-body investigations are the properties of the deuteron and the elastic scattering data for two nucleons for laboratory energies below ≈400 MeV. An interaction which is compatible with these data is by no means unique; restrictions have to be imposed and a variety of assumptions can be made. There are two very fundamental and well-founded restrictions: (1) The interaction possesses the strong-interaction symmetries and (2) the one-pion-exchange (OPE) interaction dominates at relative distance greater than about 2 F. Equivalent restrictions on the two-nucleon scattering matrix have proven indispensable in obtaining a unique reduction of the scattering data to phase parameters.

As a matter of principle, the phase shifts, even if known at all energies, can give no indication of the degree or type of nonlocality which is present in the interaction. As a consequence a variety of forms for the interaction have been constructed;

some are based on particle-theoretical considerations; all are in a large part phenomenological. The usefulness of a particular interaction is determined by the extent to which it is compatible with as many nuclear properties as can be calculated accurately. Surveys of these general matters have been given by Moravcsik,[1] Signell,[2] and Moszkowski.[3]

There are additional uncertainties in the form of the two-nucleon interaction which arise as a consequence of pion production at high energies. One cannot use the scattering data above ≈ 450 MeV for the purpose of constructing a nonrelativistic energy-conserving Hamiltonian. The resulting indeterminancy in the potential will be discussed under the assumption that nonrelativistic quantum mechanics applies and that the phase shifts for energies above a maximum (400–500 MeV) are in principle unobtainable. We shall restrict ourselves solely to the case where the assumed potential is local.

The first uncertainty applies to the innermost region of the potential. Briefly, a limited energy range implies limited penetration into the innermost region; hence little detail can be obtained; as the angular momentum is increased the situation becomes worse.

The second uncertainty applies to the detailed spatial variation of the potential. The amount of detail which can be obtained for a radial region of dimension Δr depends on the range of wave numbers employed in the scattering. If only very small wave numbers are used then only an average range and average strength are determined (effective-range theory). If all wave numbers could be used then the potential would be determined everywhere (Gelfand-Levitan theory; the potential is not unique if there is a bound state). If the wave numbers are restricted to an interval of size Δk then one expects that only gross information is determined in Δr, where

$$\Delta r \Delta k \approx \text{const.} \tag{1}$$

If the region in question is repulsive, the local wave number is smaller than the incident wave number; if the region is attractive, the local wave number is greater. Thus less information is expected for a repulsive region, especially when it is also an inner region, than for an attractive region. Since there is only one weakly bound state in the two-nucleon system, one expects, at the maximum usable energy, that the average local wave number in an attractive region has the same order of magnitude as the incident wave number. For $E_{\text{lab}} = 500$ MeV, $k_{\max} \approx 2.5\ \text{F}^{-1} = \Delta k$. The uncertain principle suggests that the constant in (1) is of order unity; we have then

$$\Delta r \approx 0.4\ \text{F}. \tag{2}$$

These crude estimates indicate that one can divide the potential into several regions in each of which only gross properties of the interaction need be specified. One of the simplest ways of doing this is to assume that the interaction is concentrated at one radial value within each region; we are thereby led to a δ-function interaction.

In this paper we consider, then, an interaction, which for the central case can be written

$$\frac{M}{\hbar^2} V(r) = \sum_{i=1}^{m} \frac{v_i}{r_i} \delta(r - r_i). \tag{3}$$

M is the nucleon mass, the v_i are dimensionless strength parameters, and the r_i are the concentration radii. These latter will henceforth be referred to as interaction points since we shall work primarily in the space defined by the radial coordinate r. The main virtue of this interaction is, of course, its simplicity. The two-body problem, even with tensor forces, is solved by finite linear algebra. The few-body and many-body problems are inherently difficult; one expects however, that if a reasonably accurate solution can be obtained, it would be done more easily with a δ function than with continuous local interactions. Recent calculational progress in the solution of the three-body problem has been made possible, in part, by the use of nonlocal interactions which are the sum of separable terms. More generally the δ-function interaction can be written as follows:

$$\frac{M}{\hbar^2} \langle r | V | r' \rangle = \frac{M}{\hbar^2} V(r) \delta(r - r') \tag{4a}$$

$$= \sum_{i=1}^{m} \frac{v_i}{r_i} \delta(r - r_i) \delta(r_1 - r'). \tag{4b}$$

Equation (4b) shows that the δ-function interaction also possesses the separable form. Nonlocal generalizations of (4b) are briefly discussed in Sec. VI.

The main objections to a δ-function interaction will probably stem from its discontinuous nature. Even though this form of the interaction can be made to reproduce the two-body data quite well, we do not suggest it as a candidate for the "true" two-nucleon interaction but rather as a useful model. The argument given above and the results of this paper do suggest that the detailed spatial structure of an assumed local potential are to a certain extent inconsequential as long as we are dealing with nuclear systems in which nucleon energies are less than those considered here. We believe it is a matter of determining how many δ

functions are needed, rather than whether they can be used at all. Further discussion on this matter will be found in Sec. VII. It is important to remember that the above discussion assumes that a local potential is adequate; this of course need not be the case.

A brief sketch of our results will now be given. Since previous determinations of a local two-nucleon interaction have all required a strong state dependence, we have considered each state separately. In Sec. V we give evidence that state dependence is necessary.

A list of two-nucleon states for $J \leq 4$ is given in Table I. We do not anticipate a need to consider states for which $J \geq 4$ since the interaction in these states is presumably dominated by the OPE tail; in applications a perturbation treatment is presumably adequate. In the present paper we shall only consider $J \leq 2$. The coupling between the $^3S_1$ and $^3D_1$ states is neglected; the $^3P_2$-$^3F_2$ state is not considered at all. Work on these important omissions is in progress. The derived $^3S_1$ effective central interaction, along with that for the $^1S_0$ state ("S-state interaction") should prove useful in those investigations in which it is desirable to avoid the full complexities of the two-nucleon interaction. We have used the phase parameters of MacGregor, Arndt, and Wright[4] which will be referred to as MAW-X.

We have sought the minimum number of interaction points for which a good fit to the data could be obtained. The interaction divides naturally into a strong inner region and a relatively weak tail region. The division occurs between 2 and 3 F. We find that the inner region requires a minimum of two interaction points in all states except the $^1D_2$ state, which requires only one. In those states which require a repulsion in the inner region (because of negative phase shifts at high energies) we found that the position (r_1) of the repulsion could be chosen within wide limits; the remaining parameters were then uniquely determined. For the $^1S_0$ state, for example, $0.1542 \leq r_1 \leq 0.95$ F. At the lower extreme v_1 is infinite.

TABLE I. Two-nucleon states for total angular momentum $J \leq 4$, listed according to charge state and spin, parity (S, π) designation.

	$(np-pn)$		$(np+pn), nn, pp$	
J	$0,-$	$1,+$	$0,+$	$1,-$
0			$^1S_0$	$^3P_0$
1	$^1P_1$	$^3S_1, ^3D_1$		$^3P_1$
2		$^3D_2$	$^1D_2$	$^3P_2, ^3F_2$
3	$^1F_3$	$^3D_3, ^3G_3$		$^3F_3$
4		$^3G_4$	$^1G_4$	$^3F_4, ^3H_4$

For the $^3D_2$ and $^1D_2$ states, which do not require a repulsion, we find that the interaction parameters in the minimal determination are essentially unique. This difference is in accord with the crude analysis given below.

The tail region requires a minimum of two interaction points in all states except the $^3S_1$, which requires only one. The parameters for this region are essentially unique for each state. Since the tail region has indefinite extent, a representation by a finite number of δ functions must fail in some respect. We find this to occur only when $L > 0$ and at the lowest energies; for this situation the very outermost part of the tail cannot be neglected. We doubt that a refined description of this outermost region is worthwhile; in applications it would enter as a very small perturbation which would be overwhelmed by the S-state interaction at low energies.

Since the innermost region is rather obscure we also consider interactions which contain a hard core of arbitrary radius r_0; we find that the minimum number of interaction points and the parameters of the tail region do not change. In those states which require a repulsion both r_0 and r_1 can be chosen arbitrarily within limits. For example, in the $^1S_0$ state $0.1542 < r_0 < 0.7$ F.

A detailed presentation and discussion of the results are given in Sec. IV. In Sec. V we report certain observations which we feel are interesting for the general problem of nuclear forces. In Secs. II and III we present the necessary mathematical apparatus which is needed for determining phase shifts and interaction parameters. In Sec. VI we consider formal matters such as the construction of the off-shell T matrix. Finally in Sec. VII we suggest how the δ function interaction model might be used in nuclear problems.

δ-functions interactions (with nonzero ranges) were employed in 1957 by Raphael[5] and in 1959 by Bosterli.[6] Raphael applied the Gauss-Jacobi mth-order quadrature method to the scattering integral equation; this became formally identical to the m-δ-function interaction problem. The positions of the δ functions, except for scale, were given by the zeros of the orthogonal polynomials associated with the quadrature method. The strengths of the δ functions could be determined if the (continuous) interaction was known. In Raphael's method they were determined by fitting the experimental data; he considered only the $^1S_0$ data current in 1957 and maximum $m = 3$. In our approach, the positions of the δ functions are chosen to give a good fit to the data. Bosterli considers only the $^1S_0$ and $^3S_1$-$^3D_1$ states; the interaction is assumed to take place at one point, but consists of central and tensor parts. The low-

energy ($E < 30$ MeV) and bound-state properties were used to determine the parameters.

II. BASIC EQUATIONS

Strong interactions are invariant with respect to spatial rotations, spatial inversion, time reversal, nucleon permutation, and rotations in isotopic spin space. A complete set of good quantum numbers which specify a state of the two nucleon systems are J, M_J (total angular momentum), S (total spin), T, M_T (total isotopic spin); these will be denoted collectively by α. Parity is given by $(-1)^{S+T+1}$. We shall use the set of basic states

$$|\alpha r L\rangle , \tag{5}$$

which are eigenstates of the magnitude of relative position $|\vec{r}_1 - \vec{r}_2|$ and magnitude of orbital angular momentum $\vec{L}$, neither of which are conserved.

The specific nuclear interaction will be assumed to take the form

$$\frac{M}{\hbar^2}\langle \alpha r L | V | \alpha r' L' \rangle = \delta(r - r') \sum_{i=1}^{m} \frac{v_i^{\alpha L L'}}{r_i} \delta(r - r_i) = \sum_{i=1}^{m} \frac{v_i^{\alpha L L'}}{r_i} \delta(r - r_i)\delta(r' - r_i) . \tag{6}$$

Invariance with respect to spatial rotations and rotations in the isotopic spin space (charge independence) imply that these matrix elements are independent of M_J and M_T, respectively. The weaker assumption of charge symmetry, i.e., only n-n and p-p alike, implies a dependence on M_T^2. The dimensionless strength parameters $v_i^{\alpha L L'}$ are real and symmetric in L, L'; they vanish unless $L = L' = J$ or L, $L' = J \pm 1$, $J \mp 1$. The latter case occurs only when a tensor force is present.

The state of relative motion satisfies the equation:

$$\left(p^2 + \frac{M}{\hbar^2} V\right) |\Psi_k\rangle = k^2 |\Psi_k\rangle , \tag{7}$$

where $\vec{p}$ is the momentum conjugate to $\vec{r}_1 - \vec{r}_2$, M is the nucleon mass, $k^2 = (M/2\hbar^2)E$, E is the laboratory energy and is twice the energy of relative motion. We also have

$$\frac{\hbar^2}{M} = 41.47 \text{ MeV F}^2 ,$$

$$k(\text{in F}^{-1}) = 0.1098[E(\text{in MeV})]^{1/2} . \tag{8}$$

The symbol E will always indicate the laboratory energy. We now write Eq. (7) in the $\alpha r L$ representation: As usual we introduce

$$u_L(k, r)/r = \langle \alpha r L | \Psi_k \rangle \tag{9}$$

and suppress the index α throughout. We have

$$\left(\frac{d}{dr^2} - \frac{L(L+1)}{r^2} + k^2\right) u_L(k, r) - \sum_{i, L'} \frac{v_i^{LL'}}{r_i} \delta(r - r_i) u_{L'}(k, r) = 0 . \tag{10}$$

Although we shall now only treat the uncoupled case, the formalism for the coupled case can be obtained by a straightforward generalization.

We proceed to replace the Schrödinger equation (10) by boundary conditions at r_i. To this end it is only necessary to consider the neighborhood of the point r_i, for which Eq. (10) can be written

$$[-t + k^2] u(k, r) - \frac{v_i}{r_i} \delta(r - r_i) u(k, r) = 0 . \tag{11}$$

For convenience the angular momentum labels are suppressed. For $r_i < r < r_{i+1}$, $u(k, r) = u_i(k, r)$, where the latter is a solution of the free-particle equation

$$(-t + k^2) u_i(k, r) = 0 , \tag{12}$$

and is continuous everywhere. We now assume that everywhere in the neighborhood of r_i, $u(k, r)$ takes the form

$$u(k, r) = [1 - \theta(r - r_i)] u_{i-1}(k, r) + \theta(r - r_i) u_i(k, r) . \tag{13}$$

The unit step function $\theta(r - r_i)$ is defined along with its derivatives in the sense of distribution theory.

Upon substituting (13) into Eq. (11), making use of (12) and the linear independence of $\delta(r - r_i)$ and $\delta'(r - r_i)$, the following boundary conditions are found to hold at r_i:

$$u_i(k, r_i) = u_{i-1}(k, r_i) \equiv u(k, r_i) , \tag{14a}$$

$$\left[\frac{d}{dr} u_i(k, r) - \frac{d}{dr} u_{i-1}(k, r)\right]_{r = r_i} = \frac{v_i}{r_i} u(k, r_i) . \tag{14b}$$

We now define

$$\varphi_i(k, r) = F_L(kr) + \tan\delta_i(k) G_L(kr) , \tag{15}$$

where

$$F_L(x) = x j_L(x) , \tag{16a}$$

$$G_L(x) = -x n_L(x) , \tag{16b}$$

are the standard regular and irregular solutions of the free-particle equation. We may now write

$$u_i(k, r) = A_i \varphi_i(k, r) . \tag{17}$$

Since this form contains two arbitrary constants, it is a general solution of the free-particle equation. The condition of regularity at $r=0$ implies

$$u_0(r) = A_0 F_L(kr)\,; \tag{18}$$

thus

$$\tan\delta_0(k) = 0\,. \tag{19}$$

We shall also consider the case where there is a hard core with radius $r_0 < r_i$. We have then

$$u_0(r) = \begin{cases} 0\,, & r \le r_0 \\ A_0[F_L(kr) + \tan\delta_0 G_L(kr)]\,, & r_0 \le r \le r_1\,, \end{cases} \tag{20}$$

where now

$$\tan\delta_0 = -F_L(kr_0)/G_L(kr_0)\,. \tag{21}$$

The non-hard-core case is simply obtained from Eq. (21) by taking the limit $r_0 \to 0$.

Beyond the final interaction point we employ the standing-wave asymptotic form

$$u_f(k,r) = F_L(kr) + \tan\delta_f(k) G_L(kr)\,. \tag{22}$$

Thus

$$A_f = 1\,. \tag{23}$$

The desired phase shift is $\delta_f(k)$. The outgoing-wave case may be treated similarly.

We now derive the equation relating $\tan\delta_{i-1}(k)$ and $\tan\delta_i(k)$. From Eqs. (14a) and (17) we have

$$u(k,r_i) = A_{i-1}\varphi_{i-1}(k,r_i) = A_i\varphi_i(k,r_i)\,. \tag{24}$$

We now multiply Eq. (14b) by $u(k,r_i)$. With appropriate use of Eqs. (24) and the derivative of (17), the factors $A_{i-1}A_i$ appear on both sides of the equation and may be canceled. We have than

$$\varphi_{i-1}(k,r_i)\left[\frac{d}{dr}\varphi_i(k,r)\right]_{r=r_i} - \varphi_i(k,r_i)\left[\frac{d}{dr}\varphi_{i-1}(k,r)\right]_{r=r_i} = \frac{v_i}{r_i}\varphi_{i-1}(k,r_i)\varphi_i(k,r_i)\,. \tag{25}$$

In this we substitute the definition (15) and then employ the Wronskian relation

$$G(kr)\frac{d}{dr}F(kr) - F(kr)\frac{d}{dr}G(kr) = k\,. \tag{26}$$

Quite straightforwardly we arrive at

$$\tan\delta_i(k) - \tan\delta_{i-1}(k) = -\frac{v_i}{kr_i}\varphi_{i-1}(k,r_i)\varphi_i(k,r_i)\,. \tag{27}$$

This is the single most useful equation in this paper. We can easily solve for $\tan\delta_i(k)$:

$$\tan\delta_i(k) = \frac{\tan\delta_{i-1}(k) - (v_i/kr_i)F_L(kr_i)\varphi_{i-1}(k,r_i)}{1 + (v_i/kr_i)G_L(kr_i)\varphi_{i-1}(k,r_i)}\,, \tag{28}$$

$$\varphi_{i-1}(k,r_i) = F_L(kr_i) + \tan\delta_{i-1}(k)G_L(kr_i)\,. \tag{28'}$$

If we know $v_i, \ldots, v_f$ and $r_1, \ldots, r_f$ we can find $\tan\delta_f$ by repeated use of this equation. We start with $\tan\delta_0$, which either vanishes or is given by Eq. (21) when a hard core is present. The quantity $\delta_i(k)$ is clearly the phase shift which would result if the interaction beyond r_i were not present; we shall refer to it as the variable phase.

The constants A_i are easily found from the continuity equation (24),

$$A_{i-1} = A_i\varphi_i(k,r_i)\varphi_{i-1}^{-1}(k,r_i)\,, \tag{29}$$

and the asymptotic condition $A_f = 1$. We shall refer to the above as the sequential form of the solution. Other forms will be discussed in Sec. VI.

In the next section we shall relate the parameters v_i to the final phase shifts $\delta_f(k)$. We need to express $\tan\delta_i(k)$ in terms of $\tan\delta_{i+1}(k)$, which in turn will be expressed in terms of $\tan\delta_{i+2}$, etc. From Eq. (27) we obtain, after a change of indices and notation,

$$\tan\bar{\delta}_i(k) = \frac{\tan\bar{\delta}_{i+1}(k) + (v_{i+1}/kr_{i+1})F_L(kr_{i+1})\bar{\varphi}_{i+1}(k,r_{i+1})}{1 - (v_{i+1}/kr_{i+1})G_L(kr_{i+1})\bar{\varphi}_{i+1}(k,r_{i+1})}\,, \tag{30}$$

where

$$\bar{\varphi}_{i+1}(k,r) = F_L(kr) + \tan\bar{\delta}_{i+1}(k)G_L(k,r)\,. \tag{31}$$

If we know $\tan\delta_f$, v_f, $v_{f-1}, \ldots, v_{i+1}$, r_f, $r_{f-1}, \ldots, r_{i+1}$, then $\tan\bar{\delta}_i(k)$ can be found. Numerically, $\tan\bar{\delta}_i(k) = \tan\delta_i(k)$; however, we shall have need to distinguish between the two methods of determination.

It should be clear from the above treatment that if we include a Coulomb potential in addition to the δ functions, that the results are modified only by replacing $F_L(kr)$ and $G_L(kr)$ by $F_L(\eta, kr)$ and $G_L(\eta, kr)$, which are the standard regular and irregular Coulomb functions. The phase shift obtained is then relative to these functions.

In the above development we have made special provisions for the presence of a hard-core interaction. We now inquire to what extent a δ-function interaction with very large strength can be interpreted as a hard core. The phase shift resulting from a single interaction point is given by

$$\tan\delta_1(k) = \frac{-(v_1/kr_1)F_L^2(kr_1)}{1 + (v_1/kr_1)G_L(kr_1)F_L(kr_1)}\,. \tag{32}$$

The amplitude of the wave function for $r \leq r_1$ is easily found to be

$$A_0 = \frac{1}{1 + (v_1/kr_1)G_L(kr_1)F_L(kr_1)} . \tag{33}$$

For sufficiently large values of v_1 the hard-core values are obtained except when $G_L(kr_1)$ or $F_L(kr_1)$ have zeros. For very large values of v_1, deviations from hard-core values occur when $|k - k_0| r_1 \lesssim C/v_1$, where $k_0 r_1$ is the zero and C is of order unity. Under these conditions the deviation of $\delta_1(k)$ is quite small since $\tan\delta_1(k)$ assumes extreme values ($\sim \infty$ or 0) at these zeros. On the other hand, the amplitude goes from zero to unity as a zero is approached. A long as we are dealing with phase shifts we need not distinguish between a very large-strength δ function and a hard core. The hard-core formalism is conveniently obtained by simply taking the limit $v_1 \to \infty$.

Continuous Variable-Phase Equation

Let us consider the limit in which the interaction becomes continuous. To this end we rewrite Eq. (3):

$$\frac{M}{\hbar^2} V(r) = \frac{M}{\hbar^2} \sum_i \overline{V}(r_i) \Delta r_i \delta(r - r_i) , \tag{34}$$

where

$$\Delta r_i = r_i - r_{i-1} . \tag{35}$$

Upon comparing Eq. (34) with Eq. (3) we find

$$\frac{M}{\hbar^2} \overline{V}(r_i) \Delta r_i = \frac{v_i}{r_i} . \tag{36}$$

We now write $\tan\delta_i(k) = \tan\delta(k, r_i)$ and $\tan\delta_{i-1}(k) = \tan\delta(k, r_{i-1})$ and define

$$\Delta \tan\delta(k, r_i) = \tan\delta(k, r_i) - \tan\delta(k, r_{i-1}) . \tag{37}$$

(This is the change that occurs at the interaction point r_i.) Equation (27) can now be put in the following form:

$$\frac{\Delta \tan\delta(k, r_i)}{\Delta r_i} = -\frac{M}{\hbar^2 k} \overline{V}(r_i) \varphi_{i-1}(k, r_i) \varphi_i(k, r_i) . \tag{38}$$

The number of interaction points is now increased in a fashion such that Δr_i becomes infinitesimal; the right-hand side of Eq. (34) becomes an integral which yields $V(r) = \overline{V}(r)$. Since v_i/r_i is now infinitesimal, so is $\Delta\tan\delta(k, r_i)$. In the limit we have

$$\frac{d}{dr} \tan\delta(k, r) = -\frac{M}{\hbar^2 k} V(r) \varphi^2(k, r) , \tag{39}$$

where

$$\varphi(k, r) = F_L(kr) + \tan\delta(k, r) G_L(kr) . \tag{40}$$

The variable-phase equation (39) is usually derived by other means.[7] It is a first-order nonlinear differential equation of the Ricatti type. It requires one boundary condition, which for a well-behaved potential is $\tan\delta(k, 0) = 0$; for a hard core we use

$$\tan\delta(k, r_0) = -F_L(k, r_0)/G_L(kr_0) .$$

The phase shift is obtained from $\tan\delta(k, r)$ for a value of r sufficiently beyond the range of the interaction.

Effective-Range Theory

The effective-range formula

$$k \cot\delta(k) = -\frac{1}{a} + \tfrac{1}{2} r_{\text{eff}} k^2 \tag{41}$$

adequately describes the phase-shift behavior of $S(n, p)$ states for energies up to $\approx$40 MeV. According to Noyes,[8]

$$^1S_0(n, p): \quad a = -23.679 \pm 0.028 \text{ F}, \quad r_{\text{eff}} = 2.51 \pm 0.11 \text{ F}; \tag{42}$$

$$^3S_1: \quad a = 5.397 \pm 0.011 \text{ F}, \quad r_{\text{eff}} = 1.727 \pm 0.014 \text{ F}.$$

Equation (41) implies

$$\frac{1}{a} = -[k \cot\delta(k)]_{k=0} , \tag{43}$$

$$r_{\text{eff}} = 2\left[\frac{d}{dk^2} k \cot\delta(k)\right]_{k=0} . \tag{44}$$

From Eq. (28) we obtain, for S states,

$$k \cot\delta_i(k) = \frac{k \cot\delta_{i-1}(k) + (v_i/r_i) \cos kr_i \psi_{i-1}(k, r_i)}{1 - (v_i/r_i)(\sin kr_i/k)\psi_{i-1}(k, r_i)} , \tag{45}$$

where

$$\psi_{i-1}(k, r_i) = k \cot\delta_{i-1}(k) \frac{\sin kr_i}{k} + \cos kr_i . \tag{46}$$

We define

$$\frac{1}{a_i} = -[k \cot\delta_i(k)]_{k=0} , \tag{47}$$

$$\rho_i = \left[2 \frac{d}{dk^2} k \cot\delta_i(k)\right]_{k=0} . \tag{48}$$

Thus $a = a_f$ and $r_{\text{eff}} = \rho_f$. Straightforwardly we obtain

$$\frac{1}{a_i} = \frac{1}{D_i}\left[\frac{1}{a_{i-1}} - \frac{v_i}{r_i}\left(1 - \frac{r_i}{a_{i-1}}\right)\right] , \tag{49}$$

$$\rho_i = \frac{1}{D_i}\left\{\rho_{i-1}\left[1 + v_i\left(1 - \frac{r_i}{a_i}\right)\right] - v_i r_i\left[\left(1 - \frac{r_i}{a_{i-1}}\right)\left(1 - \frac{1}{3}\frac{r_i}{a_i}\right) + \left(1 - \frac{r_i}{a_i}\right)\left(1 - \frac{1}{3}\frac{r_i}{a_{i-1}}\right)\right]\right\}, \quad (50)$$

where

$$D_i = 1 - v_i\left(1 - \frac{r_i}{a_{i-1}}\right). \quad (51)$$

For a hard core we have $a_0 = r_0$, $\rho_0 = \frac{2}{3}r_0$.

At sufficiently low energies an interaction with a single δ function can reproduce the phase shifts if we choose

$$1/v_1 = -(1 - 3r_{\text{eff}}/2a)^{1/2}, \quad (52a)$$

$$r_1 = a(1 + 1/v_1). \quad (52b)$$

However, the single-δ-function phase shifts begin to deviate from the two-nucleon experimental values at energies less than 10 MeV.

III. DETERMINATION OF THE INTERACTION PARAMETERS

The procedure we used for determining the interaction parameters is relatively simple. Its success is attributed to the simplicity of the δ function and to the fact that only three or four interaction points are required for each state. The determination of the r_i depends to a certain extent on the state being considered and will be discussed under the proper heading in the next section. The v_i are determined by a general procedure which will now be presented.

As it will appear in Eq. (113), $\tan\delta_f(k)$ is a rational fraction in the m strengths v_i. By choosing m distinct wave numbers $k^{(i)}$, one obtains m nonlinear algebraic equation from which the v_i can be found. The method by which we solve these equations is given below; however, certain general consideration will first be given. In principle, all but one of the v_i can be eliminated by Sylvester's method. The resulting polynominal equation in the one remaining v_i has degree 1, 2, 8 for $m = 1$, 2, 3, respectively; for higher m the degree increases rapidly. We have handled the multiplicity of solutions by simply disregarding those which did not possess characteristics commonly attributed to the interaction; e.g., the $^1S_0$ interaction consists of a repulsion followed by an attraction which goes to zero monotonically as the distance increases.

The process of elimination is excessively tedious even for $m = 3$. For large m it is more practical to consider an iterative method of solution based on Eq. (27). For m wave numbers $k^{(i)}$, we can write the m equations

$$v_i = -k^{(i)} r_i \frac{\tan\bar{\delta}_i(k^{(i)}) - \tan\delta_{i-1}(k^{(i)})}{\varphi_{i-1}(k^{(i)}, r_i)\bar{\varphi}_i(k^{(i)}, r_i)}, \quad (53)$$

in which $\tan\delta_{i-1}(k)$ is given by Eq. (28) (with $i \to i-1$); it is a function of $v_1, \ldots, v_{i-1}$ only. In addition, $\tan\bar{\delta}_i(k)$ is given by Eq. (30); it is thus a function of $v_{i+1}, \ldots, v_f$ and the final phase shift $\delta_f(k)$.

We have used the Gauss-Seidel iteration scheme for which the nth approximation is

$$\begin{aligned} v_1^{(n)} &= f_1(v_2^{(n-1)}, v_3^{(n-1)}, \ldots; k^{(1)}), \\ v_2^{(n)} &= f_2(v_1^{(n)}, v_3^{(n-1)}, \ldots; k^{(2)}), \\ v_3^{(n)} &= f_3(v_1^{(n)}, v_2^{(n)}, \ldots; k^{(3)}), \end{aligned} \quad (54)$$

etc. Convergence to four decimal places is obtained after about 10 iterations and is not particularly sensitive to the initial guesses $v_2^{(0)}, v_3^{(0)}, \ldots$.

We have also used another means of solution which starts by solving for v_1 and v_2 in terms of $v_3, v_4, \ldots$. This requires only the solution of a quadratic equation. The equations to be solved by iteration are then

$$\begin{aligned} v_3 &= f_3(v_1, v_2, v_4, \ldots; k^{(3)}) = g_3(v_3, v_4, \ldots; k^{(3)}), \\ v_4 &= f_4(v_1, v_2, v_3, \ldots; k^{(4)}) = g_4(v_3, v_4, \ldots; k^{(4)}), \end{aligned} \quad (55)$$

etc. The sign of the radical in the solution of the quadratic equation has to be chosen to give the desirable solution. This method converges more quickly, but involves rather cumbersome expressions.

IV. NUMERICAL RESULTS

In constructing the δ-function interaction using the MAW-X[4] phase shifts, we rather arbitrarily chose to fit the energy-dependent values rather than the energy-independent values. We have fitted the $^1S_0(n, p)$ phase shifts and have neglected the $^1S_0(p, p)$ values. MAW-X gives two analyses of the n-p data. The "unconstrained" solution gives a slightly negative value for the $^3S_1$-$^3D_1$ mixing parameter, ϵ_1, below 80 MeV; since this is contrary to expectations based on OPE considerations, "pseudodata" were introduced to constrain ϵ_1 to be positive. We consider here only the "constrained" solution. We have not taken the coupling into account in the present work. In MAW-X,[4] Table VII contains the relevant data for $^1S_0(n, p)$, $^3S_1$, $^1P_1$, $^3D_2$, and $^3D_1$ states; Table IV contains the $^3P_0$, $^3P_1$, and $^1D_2$ data.

The data for the latter group of states ($T = 1$) a are obtained from p-p scattering. The phase

shifts of MAW-X are the nuclear bar phase shifts defined and discussed by Stapp *et al.*[9]. They are approximately the phase shifts that would be obtained if the Coulomb and other weaker electromagnetic potentials were not present. This is more truly so when $L>0$, for then the Coulomb potential is quite small relative to the centrifugal barrier within the range of the nuclear force [ratio $=0.0347r/L(L+1)$]. Since the phase shifts derived from the available n-p data are consistent with the more accurate p-p (nuclear bar) phase shifts for $L>0$, we shall use the latter as if they were the "pure" nuclear $T=1$ phase shifts. We believe that the uncertainties introduced by this procedure are much less than those which occur in the application of the derived interaction to nuclear systems with more than two nucleons. With regard to the charge splitting in the lower-energy $^1S_0$ phase shifts, it should be remembered that the near zero energy resonance in this state magnifies the effect of small differences in the interaction. We have chosen to derive a $^1S_0$ interaction only from the n-p phase shifts because the above manner of accounting for Coulomb effects does not apply quite as well to the S state at low energies. A more detailed analysis of Coulomb effects will be given in the future.

S States

For the $^1S_0$ state a minimum of four interaction points are required to give a good fit to the data. The interaction parameters were determined as follows: r_1 was set at 0.5 F; v_1, v_2, v_3 were obtained as discussed in Sec. III with v_4 given and no hard core; the experimental data at $E_1=360$, $E_2=100$, and $E_3=3$ MeV were used. The value of r_2 was adjusted (and v_1, v_2, v_3 redetermined) until the scattering length, $a=-23.68$, was reproduced from Eq. (49) with $i=4$. The values of r_3, r_4, and v_4 were adjusted (and other parameters redetermined) such that a good over-all fit was achieved.

A family of $^1S_0$ interactions was obtained by varying r_1. The tail parameters, r_3, r_4, and v_4 were fixed at the values found above and v_1, v_2, v_3, and r_2 were determined as described above. The smallest possible value of r_1 is associated with $v_1=\infty$. For values of r_1 somewhat beyond 0.95 F, the scattering length, and hence the low-energy behavior, could not be fitted. The set of interactions obtained in this manner will be called the minimal repulsive-core (MRC) family. The values of the interaction parameters are given in Table II. It should be noticed that v_3 does not vary much; thus each member of the family possesses essentially the same tail.

When we introduce a hard core of arbitrary radius $r_0>0.1542$ F we find that four attractive interaction points are required to give a good fit to the data; however, r_1 is arbitrary within limits. The parameters of the hard-core families are determined as follows: Fix r_0 and r_1; use the MRC values for the tail parameters r_3, r_4, v_4; proceed to determine v_1, v_2, v_3, r_2 as in the MRC case.

For a given value of r_0, the value of r_1 can range from r_0 to a value for which the fit to the low-energy data begins to fail. As r_0 is increased the range of r_1 becomes more restricted; no solution can be obtained somewhat beyond $r_0=0.7$ F. The limit $r_1\to r_0$ is associated with $v_1\to-\infty$; this limiting interaction is qualitatively different from the other members and most likely implies a boundary condition in which the wave function just outside the core does not vanish.

In Table III we compare the phase shifts of one member of the MRC family with experiment. For other sets of parameters (MRC and hard-core families) the phase shifts differ from each other at high energies but still remain within the experimental error.

The MRC and hard-core families represent extreme cases. For a given core radius, $0.1542\leq r_c\leq 0.7$ F, one can give the repulsion any value between the MRC and hard-core values; by the procedure given above, the other parameters can be determined such that a good fit to the data is achieved.

TABLE II. $^1S_0(n,p)$ range (r_i in F) and dimensionless strength parameters v_i. All members have $r_3=2.60$, $r_4=3.80$, $v_4=-0.058$.

r_1	r_2	v_1	v_2	v_3
		MRC		
0.1542	1.226	∞	−0.971	−0.165
0.155	1.226	197.5	−0.971	−0.165
0.300	1.223	1.166	−0.979	−0.165
0.500	1.208	0.669	−1.028	−0.165
0.700	1.172	0.765	−1.182	−0.166
0.950	1.054	4.019	−3.439	−0.168
		$r_0=0.423$ (Reid hard core)		
0.450	1.249	−15.497	−0.897	−0.164
0.500	1.259	−5.389	−0.866	−0.164
0.900	1.610	−1.564	−0.253	−0.155
		$r_0=0.5$ (≈HJ and Yale hard cores)		
0.550	1.286	−10.086	−0.782	−0.163
0.600	1.307	−5.190	−0.722	−0.162
0.800	1.497	−2.260	−0.363	−0.158
		$r_0=0.7$		
0.750	1.669	−14.736	−0.233	−0.152

TABLE III. Comparison of $S(n,p)$ experimental phase shifts (degrees) with MRC phase shifts at various energies (in MeV). The scattering lengths and phase shifts at 3, 100, and 360 MeV are in agreement. The first interaction point, r_1, is in F.

r_1 \ E	1	10	60	160	260	460
$^1S_0$ exp.	62.41 ± 0.01	60.85 ± 0.26	40.14 ± 1.10	20.04 ± 2.02	5.89 ± 2.05	−14.95 ± 3.70
0.1542	62.44	60.86	39.88	20.02	5.99	−14.82
$^3S_1$ exp.	147.84 ± 0.01	102.12 ± 0.16	55.63 ± 0.46	25.46 ± 0.50	5.76 ± 0.80	−24.77 ± 2.48
0.3000	147.83	102.24	55.88	25.49	5.88	−24.47

In comparing $\delta_1(k)$ for the MRC family we find for all energies (≤ 460 MeV) that greater r_1 implies more repulsion. The second interaction point must then be correspondingly more attractive since the tail is essentially the same for all members. The least repulsion (in the present context) is thus given by a hard core with $r_c = 0.1542$ F.

For the $^3S_1$ state we have neglected the coupling to the $^3D_1$ state and have derived families of effective central interaction which yield the $^3S_1$ experimental phase shifts. The procedure for determining parameters and families is the same as in the $^1S_0$ case except that now only a single interaction point in the tail is required. The results are found in Tables IV and III. The smallest hard core is at $r_c = 0.3$ F.

P States

For the $^3P_0$, $^3P_1$, and $^1P_1$ states, a minimum of four interaction points are required to give a good fit to the data. The interaction parameters were determined as follows: r_1 was set at 1 F; v_1, v_2, v_3, v_4 were found as discussed in Sec. III (no hard core). The data $E_1 = 400$, $E_2 = 200$, $E_3 = 100$, $E_4 = 60$ MeV were used. The values of r_2, r_3, and r_4 were adjusted (and the v_i redetermined) such that a good over-all fit to the data was achieved. The MRC family was obtained by varying r_1; the parameters r_3, r_4, v_4 were held fixed at the values obtained for $r_1 = 1$ F; v_1, v_2, v_3 were then found by using the data at E_1, E_2, E_3; r_2 was adjusted (and the v_i redetermined) until v_3 acquired the same value as in the $r_1 = 1$ F case. The parameters are given in Tables V, VI, and VII. The smallest possible r_1 is associated with $v_1 = \infty$. The upper limit to r_1 is reached when it becomes impossible to reproduce v_3 for any value of r_2.

All $L > 0$ phase shifts are given in Table VIII. For all P states (and D states as well) the magnitudes of the 20-MeV phase shifts are too small.

TABLE IV. $^3S_1$ range (r_i in F) and dimensionless strength parameters v_i. All members have $r_3 = 2.54$ F.

r_1	r_2	v_1	v_2	v_3
		MRC		
0.3	1.245	∞	−1.550	−0.264
0.31	1.245	30.080	−1.550	−0.264
0.500	1.236	1.974	−1.598	−0.263
0.700	1.203	1.735	−1.808	−0.264
0.900	1.123	3.870	−2.958	−0.267
		$r_0 = 0.5$ (≈HJ and Yale hard cores)		
0.550	1.256	−9.141	−1.489	−0.262
0.800	1.299	−1.707	−1.221	−0.261
1.000	1.440	−1.769	−0.584	−0.258
		$r_0 = 0.548$ (Reid hard core)		
0.600	1.265	−10.073	−1.435	−0.261
0.800	1.313	−2.342	−1.143	−0.261
1.000	1.509	−2.114	−0.443	−0.255
		$r_0 = 0.7$		
0.750	1.334	−14.331	−1.037	−0.260
0.900	1.490	−4.360	−0.507	−0.256

TABLE V. $^1P_1$ range (in F) and dimensionless strength parameters. All members have $r_3 = 3.50$, $v_3 = 0.039$, $r_4 = 4.55$, $v_4 = 0.032$.

r_1	r_2	v_1	v_2
		MRC	
0.646	1.688	∞	0.540
0.650	1.688	155.09	0.540
0.8	1.706	4.173	0.500
1.0	1.783	2.121	0.348
1.2	2.105	1.780	0.119
		$r_0 = 0.428$ (Reid hard core)	
0.7	1.689	10.477	0.537
1.0	1.741	1.423	0.423
1.2	1.875	1.187	0.234
		$r_0 = 0.5$ (≈HJ and Yale hard cores)	
0.7	1.689	9.553	0.538
1.0	1.725	1.081	0.455
1.4	2.200	0.982	0.067
		$r_0 = 0.7$	
1.0	1.674	−0.624	0.579
1.4	1.591	−0.810	1.167

This could be remedied by adding another δ function at about 5 to 6 F with a strength smaller than v_4; however, at still lower energies the same situation remains. The difficulty lies in the finiteness of the tail. Variable-phase calculations show that the outer region of the tail becomes increasingly influential as the energy is lowered. We believe that the correction of the interaction which remedies this behavior (e.g., an OPE tail beginning at 5 F) would be a quite negligible perturbation in any application in which these interactions are likely to be used.

The minimal hard cores for $^3P_0$, $^3P_1$, and $^1P_1$ states occur at 0.6736, 0.723, and 0.646 F, respectively. One should note that these hard cores have more than twice the size of the S-state minimal hard core cores and are also larger than the Reid[10] ($r_0 = 0.4283$-F) or Hamada-Johnston[11] (HJ) and Yale[12] ($r_0 \approx 0.5$-F) values.

We do not believe that an exhaustive survey of hard-core families is as useful for P states as for S states. We have considered the Reid and Hamada-Johnston and Yale hard cores for the sake of comparison. We have also given an example of one hard-core family for a value of r_0 greater than the minimal hard core. For the $^3P_1$ and $^1P_1$ states, which have only negative phase shifts, one should notice that an attractive δ function is required when the hard core is greater than the minimal value. This is a general feature; hard cores greater than the minimal require a compensating attraction.

TABLE VI. $^3P_1$ range (in F) and dimensionless strength parameters. All members have $r_3 = 3.32$, $v_3 = 0.083$, $r_4 = 4.53$, $v_4 = 0.033$.

r_1	r_2	v_1	v_2
	MRC		
0.723	2.052	∞	0.261
0.728	2.052	144.06	0.261
0.900	2.100	3.992	0.230
1.100	2.321	2.053	0.142
	$r_0 = 0.428$ (Reid hard core)		
0.8	2.057	8.348	0.257
1.2	2.350	1.319	0.131
	$r_0 = 0.5$ (≈HJ and Yale hard cores)		
0.8	2.056	7.827	0.258
1.2	2.265	1.103	0.152
	$r_0 = 0.8$		
0.9	2.044	−3.986	0.267
1.2	1.995	−0.627	0.317
1.6	1.822	−0.912	0.938

D States

The coupling of the $^3D_1$ state to the $^3S_1$ state is neglected. An effective $^3D_1$ central interaction requires four interaction points. The phase shifts are all negative. The procedure for determining parameters is the same as for P states. We have only determined the MRC family. The parameters are found in Table IX. Note that the minimal hard core ($r_0 = 0.924$ F) is significantly larger than for the S and P states.

The $^3D_2$ state, unlike any state considered thus far, has phase shifts which are all positive; thus no repulsion is required. A minimum of four interaction points are required to obtain a good fit. Parameters are found in Table X, $r_0 = 0$. There is very little arbitrariness in the choice of r_1. A good fit was found for $r_1 = 1.65$ F. For r_1 somewhat greater than 1.7 F, v_3 could not be reproduced. As r_1 was decreased from 1.5 F the quality of the fit steadily deteriorated. We conclude that a four-interaction-point fit to the $^3D_2$ state is essentially unique for a fixed tail. The difference between the $^3D_2$ state and states considered previously is further discussed in Sec. V

We have *assumed* that by adding a hard core to the interaction, still only a small variation in r_1 is allowed. The hard-core interactions were thus determined as follows: r_0 is chosen arbitrarily, r_2, r_3, r_4, and v_4 are held fixed at the $r_0 = 0$ values; r_1 was adjusted until the MRC value of v_3 was reproduced.

TABLE VII. $^3P_0$ (in F) and dimensionless strength parameters. All members have $r_3 = 2.62$, $v_3 = -0.290$, $r_4 = 4.10$, $v_4 = -0.075$.

r_1	r_2	v_1	v_2
	MRC		
0.6736	1.601	∞	−0.569
0.680	1.601	103.83	−0.569
0.800	1.586	5.545	−0.608
1.000	1.511	2.966	−0.885
1.200	1.370	5.811	−2.873
	$r_0 = 0.483$ (Reid hard core)		
0.7	1.601	23.959	−0.570
1.0	1.547	1.954	−0.746
1.3	1.365	8.909	−6.020
	$r_0 = 0.5$ (≈HJ and Yale hard cores)		
0.7	1.601	23.024	−0.570
1.0	1.560	1.530	−0.699
1.2	1.485	1.728	−1.189
	$r_0 = 0.8$		
0.9	1.615	−5.281	−0.529
1.2	1.735	−1.239	−0.274

TABLE VIII. Comparison of $L > 0$ experimental phase shifts (degrees) with MRC values for various energies (in MeV). They are in agreement at 100, 200, and 400 MeV. The first interaction point, r_1, is in F.

r_1 \ E	20	40	160	260	360	460
$^1P_1$ exp.	−4.03 ± 0.06	−6.16 ± 0.17	−19.59 ± 0.82	−28.07 ± 1.26	−33.43 ± 2.19	−36.58 ± 3.35
0.646	−3.3	−5.93	−19.66	−27.94	−33.35	−35.98
$^3P_1$ exp.	−4.25 ± 0.03	−7.14 ± 0.05	−18.28 ± 0.08	−25.85 ± 0.17	−32.47 ± 0.27	−38.27 ± 0.37
0.723	−3.79	−7.04	−18.36	−25.74	−32.40	−38.64
$^3P_0$ exp.	7.02 ± 0.09	10.52 ± 0.16	4.69 ± 0.23	5.47 ± 0.33	−14.22 ± 0.54	−21.16 ± 0.76
0.6736	6.30	10.45	4.85	5.65	14.06	−22.29
$^3D_1$ exp.	−1.74 ± 0.06	−4.52 ± 0.15	−16.14 ± 0.38	−19.99 ± 0.56	−21.17 ± 0.85	−20.92 ± 1.66
0.924	−1.30	−4.23	−16.12	−19.66	−20.79	−22.20
$^3D_2$ exp.	2.97 ± 0.03	7.43 ± 0.11	22.98 ± 0.41	25.42 ± 0.63	23.83 ± 1.09	20.43 ± 1.67
1.65	2.22	7.11	23.05	25.60	23.65	21.75
$^1D_1$ exp.	0.53 ± 0.01	1.31 ± 0.01	5.90 ± 0.08	8.54 ± 0.13	10.08 ± 0.22	10.81 ± 0.37
1.34	0.27	1.04	5.81	8.67	10.02	11.29

The $^1D_2$ state also has only positive phase shifts. In this state however only three attractive interaction points are required to give a good fit to the phase shifts. The discussion for the $^3D_2$ state applies here also. Parameters are found in Table XI.

V. NUCLEAR FORCES

In this section we summarize the results of our investigation and attempt to relate it to the general problem of determining a local two-nucleon interaction phenomenologically. We shall also report our investigation into the state dependence of the interaction.

In having fitted the phase shifts ($E \leq 460$ MeV) for each state with the minimum number of interaction points, we have determined the (state-dependent) "skeleton" of the interaction. This skeleton can be "fleshed out" in a bewildering variety of ways owing to the inherent energy limitation in using the two-nucleon experimental data. In the Introduction we estimated the size of the region in which details of interactions were indeterminable to be $\Delta r \approx 0.4$ F; from our results we find, for each state, that within a region of size $(\Delta r)_L \approx 1$ F one need only specify an average strength and average range.

We have found, in those states which require a repulsion, that a MRC family of skeletons exists even though the number of interaction points has been kept to a minimum. In the $^3D_2$ and $^2D_1$ states, which do not require a repulsion, the skeleton is essentially unique. This difference is in accord with the qualitative arguments given in the Introduction by which we concluded that a repulsive region, especially when it is also an inner region, is less determinable that an attractive region. A striking illustration of this difference is obtained by comparing the $^3D_2$ MRC interaction ($r_0 = 0$ F, Table X) with the $^3D_1$ MRC interaction with $r_1 = 1.7$ F (Table IX). Except for the sign of the strength, the structures are quite similar. The attractive $^3D_2$ MRC family is very small compared to the repulsive $^3D_1$ family.

For those states which require a repulsion, the hard-core families give various ways in which the inner region can be fleshed out. For the $^3D_2$ and

TABLE IX. $^3D_1$ range (in F) and dimensionless strength parameters for the MRC family. All members have $r_3 = 3.60$, $v_3 = 1.96$, $r_4 = 4.80$, $v_4 = 0.064$.

r_1	r_2	v_1	v_2
0.924	2.434	∞	0.751
0.950	2.434	34.107	0.751
1.2	2.449	2.576	0.718
1.5	2.504	1.238	0.606
1.7	2.588	1.101	0.465
1.8	2.666	1.123	0.366

TABLE X. $^3D_2$ range (in F) and dimensionless strength parameters for the hard-core series. All members have $r_2 = 2.60$, $r_3 = 3.8$, $v_3 = -0.218$, $r_4 = 5.15$, $v_4 = -0.069$. The Reid hard core is at $r_0 = 0.428$; the HJ and Yale at $r_0 \approx 0.5$.

r_0	r_1	v_1	v_2
0.0	1.650	−1.092	−0.529
0.2	1.650	−1.092	−0.529
0.428	1.635	−1.122	−0.532
0.5	1.622	−1.154	−0.534
0.8	1.497	−1.661	−0.550
1.2	1.271	−17.234	−0.557

$^2D_1$ states the hard core is an additional, unessential feature which is allowed by the large uncertainty of the inner region for $L>0$.

At the other extreme we have the tail region, which did not have to be changed when the inner region was varied. Even though some variation in the tail region is possible, it appears to be a consequence of the uncertainty of the data rather than the actual existence of phase equivalent alternatives.

If we now consider the current collection of continuous models of the two-nucleon interaction, we find the same general features as above: Most have a common tail, which is taken from OPE theory, and a variety of inner regions (hard cores, soft cores, supersoft cores). The uncertainty in the details of the interaction over distances $(\Delta r)_L$ is eliminated by using only smoothly varying general forms with relatively few adjustable parameters. In contrast, we have eliminated this uncertainty by considering only the minimum number of interaction points required to give a good fit. More specific knowledge of the two-nucleon problem must come from sources other than the two-nucleon data; in Sec. VII we take up this matter.

Another aspect of the two-nucleon interaction which we shall consider is its state dependence. We shall not consider the general situation, which is quite complex; rather, we consider only the singlet-even states: $^1S_0$, $^1D_2$. For these states, state dependence simply means that we must use a different central potential in each state. All current local models contain such a dependence. In view of the large amount of arbitrariness in the interaction, one might inquire whether a single local central potential can be found which gives the phase shifts for both states. The work of Yoder and Signell[13] give a strong indication that this can not be done. Our investigation gives added evidence in support of their conclusion.

Let us suppose such a potential exists. One would expect that its skeleton could be found among the great variety of MRC and hard-core potentials which have been constructed for the $^1S_0$ state. What we find, however, is that all these $^1S_0$ potentials give essentially the same set of D-state phase shifts (which differs from the experimental $^1D_2$ phase shifts). This is demonstrated in Table XII, which contains all the extreme cases. An entirely similar type of phase equivalence was found when the $^3S_1$ interaction was used in calculating D-state phase shifts.

In view of these findings we are led to postulate the following theorem: If the potentials $V_1(r)$ and $V_2(r)$ are phase equivalent in the angular momentum state $L_{\min}$ for $0 \leq E \leq E_{\max}$, they will be equivalent for $L > L_{\min}$ in the same energy range. Owing to uncertainties inside the centrifugal barrier, one does not expect, in general, that phase equivalence in a particular angular momentum state implies phase equivalence in lower state; this has been confirmed by specific examples. If the theorem could be proved, the general problem of determining state dependence or independence would be greatly simplified when a local interaction is assumed.

Even though the singlet-even interaction is apparently state-dependent, OPE consideration require the tail ($r \gtrsim 3$ F) to be the same. The tails of the $^1S_0$ and $^1D_2$ have very similar δ-function representations. If one placed an additional interaction point between 4 and 5 F, the tails could easily be made identical without decreasing the quality of the fits. We now call attention to the hard-core $^1D_2$ interaction with $r_0 = 0.59$ F (Table XI); note that r_1 and v_1 are essentially the same as the $^1S_0$ minimum hard-core values (Table II). This example shows that the difference between

TABLE XI. $^1D_2$ range (in F) and dimensionless strength parameters for the hard-core series. All members have $r_2 = 2.40$, $r_3 = 3.80$, $v_3 = -0.045$. The Reid hard core is at $r_0 = 0.428$; the HJ and Yale at $r_0 \approx 0.5$.

r_0	r_1	v_1	v_2
0.000	1.340	−0.597	−0.171
0.428	1.305	−0.677	−0.172
0.500	1.276	−0.763	−0.172
0.590	1.226	−0.968	−0.173
0.900	1.020	−7.381	−0.176

TABLE XII. Demonstration of essential phase equivalence in the D state of the various $^1S_0$ interactions. The difference from the MRC ($r_1 = 0.155$) member is shown for both S and D states. Phase shifts below $E = 100$ MeV do not differ by more than 0.01 degrees in the S state and 0.03 degrees in the D state.

r_1 \ E	160		260		460	
	S	D	S	D	S	D
			MRC			
0.155	20.02	6.44	5.99	10.06	−14.92	18.33
0.5	−0.01	0.01	−0.04	0.07	−0.21	0.44
0.95	−0.06	0.09	−0.16	0.40	−0.92	2.06
			$r_0 = 0.5$			
0.6	0.04	0.03	0.09	0.02	−0.33	−0.17
0.8	0.11	0.10	0.25	0.12	1.02	0.16
			$r_0 = 0.7$			
0.75	0.15	0.24	0.38	0.46	1.48	1.33

the $^1S_0$ and $^1D_2$ interactions may be attributed solely to a difference in the repulsion; it is, however, by no means necessary.

Finally, we list the minimal hard-core radii for those states which require a repulsion: $^1S_0$ (0.1542 F), $^3S_1$ (0.3 F), $^3P_0$ (0.6736 F), $^3P_1$ (0.723 F), $^1P_1$ (0.646 F), $^3D_1$ (0.924 F). Thus an increase in orbital angular momentum is accompanied by a significantly enhanced inner repulsion; on the other hand, the remainder of the interaction does not undergo any such gross changes.

VI. GENERAL FORMALISM

There are several purposes to this section. First, we shall present a formalism capable of handling the general nonlocal δ-function interaction. Second, we shall present several practical methods by which the two-nucleon transition operator may be calculated when the δ-function interaction is local. Third, we obtain a closed form for the phase-shift function. Fourth, we treat the bound state.

We shall briefly summarize the abstract formalsim associated with the relative motion of two particles.[14] In r space we have

$$\int_0^\infty dr\,|r\rangle\langle r| = 1\,, \tag{56a}$$

$$\langle r|r'\rangle = \delta(r-r')\,. \tag{56b}$$

The free-particle wave function is

$$\langle r|k\rangle = F(kr) = \langle k|r\rangle\,. \tag{57}$$

In k space

$$\frac{2}{\pi}\int_0^\infty dk\,|k\rangle\langle k| = 1\,, \tag{58a}$$

$$\langle k|k'\rangle = \tfrac{1}{2}\pi\delta(k-k')\,. \tag{58b}$$

The state of relative motion is designated by $|u^{(\gamma)}(k)\rangle$. The standing-wave solution is indicated by $\gamma = s$, and the outgoing wave solution by $\gamma = +$. For the free-particle Green's function $g^{(\gamma)}(q)$ we have

$$\langle r|g^{(\gamma)}(q)|r'\rangle \equiv g^{(\gamma)}(r,r';q) = -\frac{1}{q}\begin{cases} F(qr)w^{(\gamma)}(qr')\,, & r \le r' \\ w^{(\gamma)}(qr)F(qr')\,, & r \ge r' \end{cases} \tag{59}$$

in which $w^{(\gamma)}(qr)$ is an irregular free-particle function. We have

$$w^{(s)}(qr) = G(qr)\,, \tag{60a}$$

$$w^{(+)}(qr) = G(qr) + iF(qr)\,, \tag{60b}$$

in which q may be any complex number. The wave operator equation is

$$\Omega^{(\gamma)}(q) = 1 + g^{(\gamma)}(q)U\Omega^{(\gamma)}(q)\,, \tag{61}$$

where $U = (M/\hbar^2)V$ and V is the interaction operator. The wave operator has the basic property that

$$|u^{(\gamma)}(k)\rangle = \Omega^{(\gamma)}(k)\,|k\rangle\,. \tag{62}$$

The interacting Green's function $\Gamma^{(\gamma)}(q)$ may be defined by

$$\Gamma^{(\gamma)}(q) = \Omega^{(\gamma)}(q)g^{(\gamma)}(q) \tag{63}$$

and satisfies the following equation:

$$\Gamma^{(\gamma)}(q) = g^{(\gamma)}(q) + g^{(\gamma)}(q)U\Gamma^{(\gamma)}(q)\,. \tag{64}$$

The transition operator is defined as

$$T^{(\gamma)}(q) = U\Omega^{(\gamma)}(q) \tag{65}$$

and satisfies the equation

$$T^{(\gamma)}(q) = U + Ug^{(\gamma)}(q)T^{(\gamma)}(q)\,. \tag{66}$$

$T^{(+)}(q)$ is usually referred to as the T matrix and $T^{(s)}(q)$ as the K matrix. We also have

$$T^{(\gamma)}(q) = U + U\Gamma^{(\gamma)}(q)U\,, \tag{67}$$

$$\Omega^{(\gamma)}(q) = 1 + g^{(\gamma)}(q)T^{(\gamma)}(q)\,. \tag{68}$$

For a well-behaved interaction, $u^{(\gamma)}(k,r)$ vanishes at the origin and hence is regular. Beyond the range of the interaction we have

$$u^{(\gamma)}(k,r) \to F(kr) - \frac{1}{k}\langle k|T^{(\gamma)}(k)|k\rangle w^{(\gamma)}(kr)\,. \tag{69}$$

The phase shift is obtained from either

$$-\frac{1}{k}\langle k|T^{(+)}(k)|k\rangle = e^{i\delta(k)}\sin\delta(k) \tag{70a}$$

or

$$-\frac{1}{k}\langle k|T^{(s)}|k\rangle = \tan\delta(k)\,. \tag{70b}$$

For the sake of notational clarity the index γ will be understood in what follows.

We now consider a general nonlocal δ-function interaction defined as follows:

$$\langle r|U|r'\rangle = \sum_{i,i'}^{m} U_{ii'}\delta(r-r_i)\delta(r'-r_{i'})\,. \tag{71}$$

The m interaction points are labeled such that

$$r_1 < r_2 < \cdots < r_m\,. \tag{72}$$

Equation (71) defines a finite matrix $\underline{U}$ which must be symmetric and real if U is Hermitian. The continuous limit is obtained as in Sec. III, by letting $m \to \infty$ and

$$U_{ii'} \to \langle r_i|U|r_{i'}\rangle dr_i\,dr_{i'}\,. \tag{73}$$

The local δ-function interaction considered pre-

viously is obtained if $\underline{U}$ is diagonal with elements

$$U_{ii'}=\frac{v_i}{r_i}\delta_{ii'}. \tag{74}$$

All quantities of interest can be expressed in terms of finite matrices which are completely analogous to the operators defined above. The formalism for these matrices requires $\underline{U}$, whose elements, e.g., are given by (74), and $\underline{g}(q)$, whose elements are given by

$$g_{ii'}(q)=\langle r_i|g(q)|r_{i'}\rangle, \quad i,i'=1,\ldots,m. \tag{75}$$

It follows that

$$\langle r|T(q)|r'\rangle=\sum_{i,i'=1}^{m}T_{ii'}(q)\delta(r-r_i)\delta(r'-r_{i'}), \tag{76}$$

$$\langle k|T(q)|k'\rangle=\sum_{i,i'=1}T_{ii'}(q)F(kr_i)F(k'r_{i'}), \tag{77}$$

$$u(k,r)=F(kr)+\sum_{i,i'=1}^{m}g(r,r_i;k)T_{ii'}(k)F(kr_{i'}). \tag{78}$$

From the analog of Eq. (66) we obtain

$$\underline{T}(q)=\underline{U}(1-g(q)\underline{U})^{-1}. \tag{79}$$

The main computational effort centers around the inversion of the m-dimensional matrix $1-\underline{g}(q)\underline{U}$. If an analytic expression is desired the construction of determinants is necessary. The method given above is of course just Fredholm's method adapted to the general nonlocal δ-function interaction.

When nonlocality is assumed to be present in the two-nucleon interaction it is usually ascribed to the inner region ($r<2$ F). For the local case we have seen that at most two interactions points are needed in the minimal description of this region. The situation in the nonlocal case should be similar except that now we have an additional parameter in the off-diagonal element $U_{12}=U_{21}$; this implies an additional arbitrariness. For two interaction points we readily find

$$\underline{\Omega}(q)=\frac{1}{D(q)}[\underline{1}-\mathrm{Tr}(\underline{g}\underline{U})+\underline{g}\underline{U}], \tag{80}$$

$$\underline{T}(q)=\frac{1}{D(q)}\left[\underline{U}-\begin{pmatrix}g_{22}&-g_{12}\\-g_{21}&g_{11}\end{pmatrix}\det\underline{U}\right], \tag{81}$$

$$-\frac{1}{k}\langle k|T(k)|k\rangle=\frac{-1}{kD(k)}\left(\sum_{i,i'=1}^{2}U_{ii'}F(kr_i)F(kr_{i'})+\det\underline{U}\bar{g}_{21}(k)F(kr_2)F(kr_1)\right), \tag{82}$$

where

$$D(q)=1+\frac{1}{k}\left(\sum_{i,i'=1}^{2}U_{ii'}w(kr_i)F(kr_{i'})+\det\underline{U}\bar{g}_{21}(k)w(kr_2)F(kr_1)\right), \tag{83}$$

$$\bar{g}_{21}(k)=-\frac{1}{k}[w(kr_2)F(kr_1)-F(kr_2)w(kr_1)]. \tag{84}$$

The phase shift obtained from (82) can be used as input into the outer region and the final phase shift can be obtained from the sequential form (28).

In the case where the interaction is purely local the Fredholm determinantal method is more complicated than is necessary. We shall present two methods for obtaining quantities of interest which take advantage of the local character of the interaction.

The first method is a generalization of the sequential method (see Sec. III). Let us define

$$\underline{U}^{(i)}=\sum_{j=1}^{i}\underline{U}_j, \tag{85}$$

where

$$[\underline{U}_j]_{ii'}=\frac{v_j}{r_j}\delta_{ji}\delta_{ji'}; \tag{86}$$

i.e., this is just the interaction at the point r_j. We have

$$\underline{U}^{(i)}-\underline{U}^{(i-1)}=\underline{U}_i. \tag{87}$$

The finite wave matrix associated with $\underline{U}^{(i)}$ is $\underline{\Omega}^{(i)}$. We have

$$\frac{1}{\underline{\Omega}^{(i)}}=\underline{1}-\underline{g}\underline{U}^{(i)}, \tag{88}$$

$$\frac{1}{\underline{\Omega}^{(i-1)}}=\underline{1}-\underline{g}\underline{U}^{(i-1)}. \tag{89}$$

After subtracting and rearranging we have

$$\underline{\Omega}^{(i)}=\underline{\Omega}^{(i-1)}+\underline{\Omega}^{i-1}\underline{g}\underline{U}_i\underline{\Omega}^{(i)}. \tag{90}$$

It is probably most convenient to find the interacting Green's functions:

$$\underline{g}^{(i)}=\underline{\Omega}^{(i)}\underline{g}. \tag{91}$$

This satisfies the equation

$$\underline{g}^{(i)}=\underline{g}^{(i-1)}+\underline{g}^{(i-1)}\underline{U}_i\underline{g}^{(i)}. \tag{92}$$

Owing to the simplicity of $\underline{U}_i$ we readily find

$$g^{(i)}_{jj'}=g^{(i-1)}_{jj'}+\frac{v_i}{r_i}\frac{g^{(i-1)}_{ji}g^{(i-1)}_{ij'}}{1-(v_i/r_i)g_{ii}}. \tag{93}$$

The sequential evaluation begins with $i=1$ and $\underline{g}^{(0)}=\underline{g}$. Owing to the fact that $g_{jj'}=g_{j'j}$ and the form of Eq. (93), one need only evaluate $g^{(i)}_{jj'}$ for $j\leq j'\leq i+1$ in order to preceed to the next interaction

point. After $\underline{\Gamma}(q)=\underline{g}^{(m)}(q)$ has been found, $\underline{T}(q)$ is found from Eq. (67).

The case where the interaction includes a hard core at r_0 is handled as follows: We consider $m+1$ interaction points for which a finite interaction at r_0 is the last to be added (the order of adding interaction points is immaterial). The limit of $\underline{T}^c(q)\equiv\underline{T}^{m+1}(q)$ is then found for $v_0\to\infty$. The results are readily found from Eq. (93) with $i=m+1$ and the equation for $T_{jj'}$:

$$T^c_{jj'}=\delta_{jj'}\frac{v_j}{r_j}\left[g^{(m)}_{jj'}-\frac{g^{(m)}_{j0}g^{(m)}_{0j'}}{g^{(m)}_{00}}\right]\frac{v_{j'}}{r_{j'}}\,,\quad j,j'\neq 0\,, \tag{94a}$$

$$T^c_{j0}=T^c_{0j}=\frac{v_j}{r_j}\frac{g^{(m)}_{j0}}{g^{(m)}_{00}}\,,\quad j\neq 0\,, \tag{94b}$$

$$T^c_{00}=\frac{-1}{g^{(m)}_{00}}\,. \tag{94c}$$

The sums in Eqs. (76), (77), and (79) must now include $i,\ i'=0$.

The next method we shall describe is quite general; however, when the interaction is local, quantities of interest can be obtained iteratively. The method is especially useful when a closed-form solution is desired; also it is readily extended to the bound-state problem.

Let us introduce the Green's function $\bar{g}(q)$ for which

$$\langle r|\bar{g}(q)|r'\rangle\equiv-\frac{1}{q}\begin{cases}0\,, & r\leq r'\\ w(qr)F(qr')-F(qr)w(qr')\,, & r\geq r'\end{cases}$$
$$\equiv\bar{g}(r,r';q)\,. \tag{95}$$

We shall refer to the formalism defined by $\bar{g}(q)$ and the interaction U as the "barred" formalism. The definitions of $\bar{\Omega}(q)$, $\bar{T}(q)$, etc. are completely analagous to the "unbarred" quantities and will be understood; the same also holds for the finite matrices $\underline{\bar{g}}(q)$, $\underline{\bar{\Omega}}(q)$, $\underline{\bar{T}}(q)$, etc.

In order to appreciate the advantages of the barred formalism let us consider the integral equation for $\bar{u}(k,r)$ when the interaction is local:

$$\bar{u}(k,r)=F(kr)+\int_0^r dr'\,\bar{g}(r,r';k)U(r')\bar{u}(k,r')\,. \tag{96}$$

This equation is well studied[15] and is fundamantal to the investigation of the general dependence of scattering quantities on the strength of the interaction. For a well-behaved $U(r)$ the iterative solution of Eq. (96) converges and is an entire function of the strength parameter. As we shall see,

$$u(k,r)=\bar{u}(k,r)/\bar{D}(k)\,, \tag{97}$$

where $\bar{D}(k)$ may be obtained once we know $\bar{u}(k,r)$ and is also an entire function of the interaction-strength parameter.

For an m-term local δ-function interaction the solution to Eq. (96) is obtained after just $m-1$ iterations. Herein lies the main advantage of the barred formalism for our study.

We shall first obtain the equations relating the barred and unbarred quantities in the general case. We introduce

$$\Delta(q)\equiv g(q)-\bar{g}(q) \tag{98}$$

for which, according to Eqs. (59) and (95),

$$\langle r|\Delta(q)|r'\rangle=-\frac{1}{q}F(qr)w(qr')\,. \tag{99}$$

From the defining equation (61) for $\Omega(q)$ and $\bar{\Omega}(q)$ we may obtain

$$\Omega(q)=\bar{\Omega}(q)+\bar{\Omega}(q)\Delta(q)T(q)\,, \tag{100}$$

$$T(q)=\bar{T}(q)+\bar{T}(q)\Delta(q)T(q)\,. \tag{101}$$

We find from Eqs. (99) and (56′) that

$$\langle r|\Delta(q)T(q)|r'\rangle=-\frac{1}{q}F(qr)\int_0^\infty dr''w(qr'')\langle r''|T(q)|r'\rangle\,. \tag{102}$$

A simple equation for the integral in (102) is readily obtained with the aid of Eq. (101). With this solution we may write

$$\Delta(q)T(q)=\frac{1}{\bar{D}(q)}\Delta(q)\bar{T}(q)\,, \tag{103}$$

where

$$\bar{D}(q)=1+\frac{1}{q}\int_0^\infty\int_0^\infty dr\,dr''w(qr)\langle r|\bar{T}|r''\rangle F(qr'')$$
$$=1-\mathrm{Tr}(\Delta\bar{T})\,. \tag{104}$$

It is important to note that Eq. (99) implies that $\Delta(q)$ has no inverse and may not be canceled in Eq. (103). The wave matrix may be put into the forms

$$\Omega(q)=\bar{\Omega}(q)+\frac{1}{\bar{D}(q)}\bar{\Omega}(q)\Delta(q)\bar{T}(q)$$
$$=\frac{\bar{\Omega}(q)}{\bar{D}(q)}\{1+[\Delta(q)\bar{T}(q)-\mathrm{Tr}\Delta(q)\bar{T}(q)]\}\,. \tag{105}$$

It can be shown that

$$\Omega(k)|k\rangle=\bar{\Omega}(k)|k\rangle/\bar{D}(k)\,. \tag{106}$$

Thus the so called "half-on-shell" matrix elements in k space have a particularly simple relationship. Equation (97) follows from this.

Having now established the relationship between the barred and unbarred quantities we shall turn our attention to the δ-function interaction. It suffices to consider only the finite matrices. Quite

directly we obtain

$$T_{ii'}(q) = \bar{T}_{ii'}(q) - \frac{1}{q\bar{D}(q)} \sum_{l=1}^{m} \bar{T}_{il}(q) F(qr_l) \times \sum_{l'=1}^{m} w(qr_{l'}) \bar{T}_{l'i'}(q) , \quad (107)$$

with

$$\bar{D}(q) = 1 + \frac{1}{q} \sum_{i,i'=1}^{m} w(qr_i) \bar{T}_{ii'} F(qr_i) . \quad (108)$$

We shall consider the purely local interaction given by (74). It is convenient to introduce $\underline{\bar{K}} = \underline{\bar{g}}(q)\underline{U}$, whose matrix elements are

$$\bar{K}_{ii'} = \bar{g}_{ii'}(q) \frac{v_i}{r_i} . \quad (109)$$

According to Eq. (95), $\bar{K}_{ii'}$ vanishes if $i \leq i'$; this implies that the mth power of $\underline{\bar{K}}$ vanishes. Upon iterating the basic equation for $\underline{\bar{\Omega}}(g)$ we then arrive at

$$\underline{\bar{\Omega}}(q) = 1 + \sum_{\nu=1}^{m-1} \underline{\bar{K}}^{\nu} . \quad (110)$$

The elements of $\underline{T}$ are simply obtained from

$$\bar{T}_{ii'}(q) = [U\Omega]_{ii'} = \frac{v_i}{r_i} \bar{\Omega}_{ii'} . \quad (111)$$

All the pertinent relations are now at hand to construct the full transition operator by means of Eq. (76) or (77). Phase shifts are conveniently obtained with the aid of Eq. (106) and the barred analog of Eq. (77); specifically

$$-\frac{1}{k}\langle k|\, T\,|k\rangle = \frac{-1}{k\bar{D}(k)} \sum_{i \geq i'} F(kr_i) T_{ii'}(k) F(kr_{i'}) . \quad (112)$$

For arbitrary m we have explicitly

$$-\frac{1}{k}\langle k|\, T(k)\,|k\rangle = -\frac{1}{k\bar{D}(k)} \left(\sum_{i=1}^{m} F(kr_i) \frac{v_i}{r_i} F(kr_i) + \cdots + \sum_{i_1 > \cdots > i_\nu} F(kr_{i_1}) \frac{v_{i_1}}{r_{i_1}} \bar{g}_{i_1 i_2} \frac{v_{i_2}}{r_{i_2}} \cdots \bar{g}_{i_{\nu-1} i_\nu} \frac{v_{i_\nu}}{r_{i_\nu}} F(kr_{i_\nu}) + \cdots \right.$$
$$\left. + F(kr_m) \frac{v_m}{r_m} \bar{g}_{m m-1} \frac{v_{m-1}}{r_{m-1}} \cdots \bar{g}_{21} \frac{v_1}{r_1} F(kr_1) \right) . \quad (113)$$

The explicit form of $\bar{D}(k)$ is obtained simply by comparing (108) with (112). These explicit forms give a global view of the structure of the phase-shift function.

The last topic we shall treat is the bound state for a local interaction. This occurs when $\langle q|\, T^{(+)}(q)\,|q\rangle$ is singular; since $\langle q|\, \bar{T}^{(+)}(q)\,|q\rangle$ is never singular for a local interaction, we must have $\bar{D}^{(+)}(q) = 0$. Owing to the presences of $\bar{D}^{(+)}(q)$ in the denominator of unbarred quantities, the unbarred formalism breaks down for bound states. This is not so for the barred formalism: Since we may write

$$\bar{u}^{(+)}(q, r) = \bar{D}^{(+)}(q) u^{(+)}(q, r) , \quad (114)$$

we have asymptotically

$$\bar{u}^{(+)}(q, r) \to \bar{D}^{(+)}(q) F(qr) - \frac{1}{q}\langle q|\, \bar{T}^{(+)}(q)\,|q\rangle w^{(+)}(qr) . \quad (115)$$

This is a decaying expotential when

$$\bar{D}^{(+)}(i\kappa) = 0 , \quad \kappa > 0 . \quad (116)$$

Since $\bar{u}^{(+)}(i\kappa, r)/r$ is regular at the origin, the bound-state eigenfunction may be written

$$u_B(r) = A\bar{u}^{(+)}(i\kappa, r) \quad (117)$$

in which A is chosen by the normalization condition:

$$\int_0^\infty u_B^2(r) dr = 1 . \quad (118)$$

The bound-state energy is

$$E_B = -\frac{\hbar^2}{M} \kappa^2 . \quad (119)$$

Explicitly we have

$$\bar{u}^{(+)}(i\kappa r) = F(i\kappa r) + \sum_{i \geq i'} \bar{g}^{(+)}(r, r_i; i\kappa) \times \frac{v_i}{r_i} \bar{\Omega}^{(+)}_{ii'}(i\kappa) F(i\kappa r_{i'}) . \quad (120)$$

Since locality has been assumed, the iterative solution for $\bar{\Omega}^{(+)}(i\kappa)$ may be used.

VII. DISCUSSION OF APPLICATIONS

Although a fundamental understanding of nuclear structure is advancing at a vigorous pace, there is still the nagging uncertainty with regard to the "true" form of the two-nucleon interaction. If there were certainty in the interaction, comparison of the results of a many-body approximate calculation with experimental data would be signifi-

cant. As it stands now, agreement or disagreement with experiment may be caused by the use of an incorrect interaction. On the other hand, if a calculational method is known to be accurate (and practical) for a particular nuclear system, then incorrect models of the interaction could be eliminated. Next to the two-nucleon system the three-nucleon system and nuclear matter are the simplest and are the natural testing ground for an interaction model. Let us assume that accurate calculations on these systems can be made with the δ-function model. One could then proceed to find which of the various δ-function models fits the data best. Because of the state dependence and uncertainty of the inner region of the interaction, the number of possible models is enormous. Hopefully, trends would be discovered which would narrow down the possibilities; e.g., the MRC families might be more reasonable than the hard-core families. Next, one should test how fleshing out the interaction affects the many-body results. This would be done by replacing a single interaction point by two or three others located in the same general region. The replacement will involve an additional number of parameters. With respect to the two-nucleon phase shifts, all but one or two (depending on the state and region) are arbitrary. If the parameters to be determined are strengths then at most a quadratic equation need be solved (see Sec. III). Although any degree of fleshing out can be accomplished in this manner, the number of parameters proliferates.

It is our expectation that a smooth fleshing out of the tail region will be inconsequential and that a smooth fleshing out of the intermediate region will not produce major effects. The hard-core families are extreme examples of nonsmooth fleshing out of the inner region with respect to the MRC skeletons. Nuclear-matter calculations (see below) show that major changes occur for sufficiently large hard cores.

If it occurs that a high degree of fleshing out is required in fitting a δ-function interaction to many-body data, then the model is essentially continuous and has lost much of its usefulness; however, one still has (in principle) a systematic means for constructing a "correct" local two-body interaction.

The program just described is somewhat ideal in its entirety. Haftel[16] and Mueller[17] have proceeded more practically. They perform nuclear-matter calculations with an S-state interaction using the $^1S_0$ and effective $^3S_1$ interaction obtained here. They compare the MRC families with the hard-core ($r_0 = 0.5$ F) families. Haftel employs the Brueckner method. For the MRC family he obtains binding energies of 27 to 31 MeV and for the hard-core case 15 to 19 MeV. Mueller employs the Jastrow method. In considering only two-body clusters he obtains results similar to Haftel's. When three-body clusters are included he finds that the binding energy decreases by about 5 MeV.

The δ-function model can also be extended in the direction of nonlocality, as discussed in Sec. VI. Extension of the model in the direction of momentum dependence causes the appearance of derivatives of δ functions. First derivatives appear in the boundary condition model of Lomon and Feshbach[18] and in the work of Schenter and Ford[19] as a replacement for a hard core. A cursory investigation on our part indicates that higher derivates of the δ function are too singular to be used in an interaction model.

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Parametrization of the Three-Body D Function. II*

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A new parametrization of the three-body D function based in part on numerical investigation is presented. It is correct in both weak coupling and the Efimov limit, and agrees remarkably well with "exact" numerical calculation in a boson model. It has the correct singularity position but not the correct strength at the three-body threshold. In the neutron-deuteron quartet system it agrees very well with exact calculation and casts light on the relative model independence of these calculations. We also show that the Efimov states become virtual states with increasing coupling.

I. INTRODUCTION

One of the most remarkable developments to emerge from recent interest in the three-body problem is the Efimov effect.[1] This is the fact that the number of three-body s-wave bound states for three identical bosons interacting via short-range nonsingular two-body forces grows to infinity as the two-body force increases to just bind two particles and then decreases for stronger binding. A detailed proof of the occurrence of this surprising phenomena has been given, as well as a discussion of how the number of bound states can be nonmonotonic in the potential strength.[2-4] As is shown in AN1 (Ref. 2) and AN2 (Ref. 4), the Efimov effect can be considered an infrared divergence of the Faddeev kernel in momentum space, or alternately as Efimov has shown[3] as a long-range effect in configuration space. In both treatments it is clear that the occurrence of the effect and the properties of the bound states are very weakly dependent on the details of the two-body forces. Rather they depend essentially only on the logarithm of the two-body scattering length and the logarithm of the range of the forces. This weak dependence has encouraged us to study the three-body problem near the Efimov limit in hopes of being able to write down some general characteristics and some of the general functions associated with the problem in a way that exploits the weak dependence on details.[5]

In this paper we study the momentum-space Faddeev equation.[6] In particular we examine the eigenvalues of the kernel of that equation and the Fredholm D function or determinant associated with it. We see how the Efimov divergence manifests itself in terms of these quantities in general and also by studying a particular simple model. The model we take is three identical bosons interacting via two-body separable potentials of the Yamaguchi sort. This model is the simplest soluble model we know.[7] Since the Efimov effect is presumed independent of the details of the model, our choice of convenient model is not restrictive. We find from numerical studies that for a wide range of coupling strengths and energies around but not only at the Efimov point, the eigenvalues are remarkably regular and can easily be represented in terms of a single analytic function of the trace of the Faddeev kernel and the trace of its square. It is then possible to construct an analytic expression for the three-body Fredholm denominator in terms of these eigenvalues. The function fits the directly calculated D function over a remarkably wide range of couplings and energies including essentially all negative energies for all couplings below the critical coupling that gives a zero-energy two-body state and for a wide range about that coupling. For weak couplings, we are able to explain a part of this agreement in terms of the fact that our expansion for the D function coincides with the first few terms of the Fredholm