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Estimate of the Importance of Retardation in Nuclear Binding*

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A potential form alternate to the customary Yukawa form is deduced. This nonlocal potential incorporates retardation of the meson and nucleon normalization factors. A phenomenological potential patterned after the Reid soft-core potential is constructed. This new phenomenological potential is compared with the Reid potential off the energy shell. The off-shell changes lead to a decrease in the binding energy of nuclear matter of 1.5 MeV/particle in the $^3S_1$-$^3D_1$ state and an increase of 0.1 MeV/particle in the $^1S_0$ state.

1. INTRODUCTION

The best phenomenological nucleon-nucleon potentials fit the elastic scattering data using essentially sums of Yukawa forms.[1] This adoption of Yukawa forms in phenomenological potentials is based on one's conception of the underlying meson-exchange mechanism. However, a Yukawa potential is only an approximate nonrelativistic representation of the effect of mesonic degrees of freedom.

In this paper, an alternate potential form is deduced by direct examination of the one-meson-exchange mechanism including some relativistic effects. These relativistic effects arise from retardation in the meson propagation and from spinor normalization factors. The alternate potential form, which is nonlocal, is adopted here as our basic building block for a new phenomenological potential.

Starting from the same meson-exchange mechanism, the Yukawa and the alternate form are equivalent at low momenta. However, the two possible forms are not equivalent with respect to off-shell properties which play a significant role in nuclear calculations.

The focus here is to build phenomenological potentials using the generalized Yukawa forms and to study the induced off-shell changes. Of course, this is an *ad hoc* procedure to examine the possible significance of retardation. We find that it is significant and one should therefore turn to a more fundamental treatment of meson degrees of freedom.

Our procedure for estimating the importance of retardation consists of fitting the Reid phase shifts using the generalized Yukawa form and examining the associated off-shell effects both for the two-nucleon transition matrix and for nuclear matter. In Sec. 2 an alternate choice of potential form is defined to include retradation and spinor normalization effects. The "generalized Yukawa form" is nonlocal and therefore is given in momentum space for each partial wave (Sec. 3). The new potential form is compared to the corresponding Yukawa form and found to be different for both on- and off-diagonal matrix elements.

Observables are determined by the transition matrix. The T matrix was constructed using sums of generalized Yukawa forms by numerically solving the nonrelativistic Lippmann-Schwinger equation. The parameters of the new interaction were fitted to the Reid phase shifts, so that only off-energy shell changes result from adopting the new basic form (Sec. 4). Of course, the potential parameters needed to fit the same data differ from Reid's values. The off-shell changes in the T matrix are examined in Sec. 5 and lead to a net +1.4-MeV/A change in the energy per particle of nuclear matter.

2. GENERALIZED YUKAWA POTENTIAL

The Yukawa potential represents the interaction between two nucleons as produced by the exchange of a meson. For a Yukawa interaction, the meson propagator, which describes the transfer of momentum between the nucleons, is of the form

$$V_{NR}=N\frac{1}{(\vec{K}'-\vec{K})^2+m^2}. \tag{1}$$

The Fourier transform of Eq. (1) leads to a local potential because V depends only on $\vec{K}'-\vec{K}$. The factor N is chosen to phenomenologically represent the spin dependence ($N=\vec{\sigma}_1\cdot\vec{\sigma}_2$), the tensor property $[N=(\vec{\sigma}_1\cdot\vec{K})(\vec{\sigma}_2\cdot\vec{K}')-\frac{1}{3}\vec{\sigma}_1\cdot\sigma_2]$, and the spin-orbit nature $[N=(\vec{\sigma}_1+\vec{\sigma}_2)\cdot\vec{K}'\times\vec{K}]$ of the nucleon-nucleon interaction. The potential both on ($|K|=|K'|$) and off ($|K|\neq|K'|$) the energy shell is defined in Eq. (1).

Another important feature of the Yukawa interaction is that the mesons are assumed to transfer momentum instantaneously. The subscript NR in (1) signifies that the Yukawa interaction omits retardation, i.e., Eq. (1) neglects the meson transit time. In a more fundamental approach, the factor N is related to nucleon spinors, $u(K)$, and coupling constants, g, by terms of the general form $N'=g^2\bar{u}\,\Gamma u\bar{u}\Gamma u$, where Γ is the appropriate Dirac invariant. The spinor normalization factors,[2] $u\alpha[(E+M)/2E]^{1/2}$, in this general form of N introduce a more complicated dependence on $\vec{K}$ and $\vec{K}'$ than used in phenomenological representations of N. A more complicated dependence on $\vec{K}$ and $\vec{K}'$ also appears in the meson propagation when retardation effects are considered.

These spinor normalization and retardation effects can be included in a phenomenological scheme by defining a generalized Yukawa form to be

$$V_R=N\left(\frac{E+M}{2E}\right)\left(\frac{E'+M}{2E'}\right)\frac{1}{(\vec{K}'-\vec{K})^2-(E'-E)^2+m^2},\tag{2}$$

where N represents the standard spin dependence, tensor property, and spin-orbit terms. Here $E^2=K^2+M^2$ and $E'^2=K'^2+M^2$, but $E\neq E'$ for off-diagonal matrix elements ($|\vec{K}|\neq|\vec{K}'|$).[3] The potential V_R, both on and off the energy shell, is defined in Eq. (2). The quantity, $(E'-E)^2$, represents the retardation effect as discussed by Salpeter and Bethe.[4]

One can understand the $(E'-E)^2$ term as a simulation of retardation effects by comparing the time evolution operator of a field theory and a potential theory. The time evolution operators are inherently different because a potential acts instantaneously while a meson propagates during a finite time interval. It can be shown that the $(E'-E)^2$ factor in the potential minimizes to some extent the difference between the two time development operators for one-meson exchange. A complete definition of retardation requires examination of higher-order terms.[5] Both V_R and V_{NR} are based on the underlying meson-exchange mechanism. In fact V_R and V_{NR} are equivalent when both $|K|$ and $|K'|$ are much less than the nucleon mass. For higher momenta, however, they differ due to the spinor normalization and retardation effects. It is extremely difficult to decide from theory which phenomenological form is preferable for use in the Schrödinger equation. Several authors have discussed the question of how to properly incorporate meson effects in a potential.[6] Rather than investigate this important unresolved question, we have adopted Eq. (2) as an alternate form for the construction of a phenomenological potential. In this way, we hope to estimate the effect of spinor normalization and retardation on two-body scattering and on nuclear binding.

3. MATRIX ELEMENTS OF THE POTENTIALS

In order to understand the potential definition given in Eq. (2), the S-wave part of that expression is now extracted. The ${}^1S_0$ channel part of V_R is

$$U_R=N({}^1S_0)\frac{E+M}{2E}\,\frac{E'+M}{2E'}\,\frac{1}{2KK'}Q_0(Z_R),\tag{3}$$

where

$$Z_R=(2EE'-2M^2+m^2)/(2KK'),\tag{4}$$

and Q_0 is a Legendre function of the second kind. In this section, we compare Eq. (3) both analytically and numerically to the ${}^1S_0$ wave part of V_{NR} which is given by

$$U_{NR}=N({}^1S_0)\frac{1}{2KK'}Q_0(Z_{NR}),\tag{5}$$

where

$$Z_{NR}=(K^2+K'^2+m^2)/(2KK').\tag{6}$$

To compare U_R and U_{NR} analytically, we define the ratio

$$R(K',K)\equiv\frac{\langle K'|U_R|K\rangle}{\langle K'|U_{NR}|K\rangle}.\tag{7}$$

On the momentum diagonal $|\vec{K}|=|\vec{K}'|$ and, hence, $E=E'$. For $E=E'$ we have $Z_R=Z_{NR}$ and the Legendre functions give identical contributions. So that

$$R(K,K)=\left(\frac{E+M}{2E}\right)^2\leq 1.\tag{8}$$

Thus the diagonal matrix elements of the generalized Yukawa form are smaller than the diagonal elements of the standard Yukawa form because of the spinor normalization factor. The ratio of Eq. (8) depends on K; $R(K,K)$ is near 1 for E near M and becomes $\frac{1}{4}$ for large energies.

The K dependence of the normalization also reduces the ratio of Eq. (7) when $|\vec{K}|\neq|\vec{K}'|$. However, in this off-diagonal case $Z_R\neq Z_{NR}$. In fact, $Z_R\leq Z_{NR}$; see the Appendix. Since $Q_0(Z)$ is a monotonically decreasing function as Z goes from

TABLE I. S-wave phase shifts (in rad).

Lab energy (MeV)	Yukawa	Generalized Yukawa
24	−0.447	−0.446
48	−0.536	−0.532
96	−0.600	−0.590
144	−0.618	−0.604
208	−0.623	−0.601
304	−0.616	−0.583
352	−0.610	−0.572
$V_0 = 82$ MeV	$m = 1\ \text{fm}^{-1}$	

TABLE III. ${}^3S_1$-${}^3D_1$ parameters.

	Reid	Generalization
V_4	−10.463	−7.547
V_5	+210.93	+163.58
V_6	−12 751.2	−11 810.6
V_7	+59 545.8	+65 531.8
V_8	−10.463	−10.545
V_9	+10.463	+14.079
V_{10}	+1407.08	+728.2
V_{11}	−10 041.0	−9168.2
V_{12}	+2835.64	+3335.0
V_{13}	−16 278.6	−19 356.3

one to infinity, the Legendre functions contribute a factor greater than one to the ratio when $|\vec{K}| \neq |\vec{K}'|$. This fact shows that retardation tends to increase the off-diagonal matrix elements of U_R while the spinor normalization decreases the matrix elements for all K and K'.

For the limit $K' \to 0$,

$$\frac{\langle 0|U_R|K\rangle}{\langle 0|U_{NR}|K\rangle} = \left(\frac{E+M}{2E}\right)\frac{K^2+m^2}{2M(E-M)+m^2}, \tag{9}$$

where the asymptotic expression $Q_0(Z) \to 1/Z$ for large Z has been used. From Eqs. (8) and (9) it follows that

$$\frac{\langle 0|U_R|K\rangle}{\langle 0|U_{NR}|K\rangle} \geq \frac{\langle K|U_R|K\rangle}{\langle K|U_{NR}|K\rangle}. \tag{10}$$

The off-diagonal matrix elements for $K'=0$ are reduced less than the diagonal matrix elements when retardation and normalization are taken into account. From Eq. (9) we see that $R(0,K) > 1$ for $K^2 > m(m+2M)$. For a large enough energy, $\langle 0|U_R|K\rangle$ is actually greater than $\langle 0|U_{NR}|K\rangle$. In fact

$$\lim_{K' \to \infty} R(K', K) = \infty .$$

However, $\langle K'|U_R|K\rangle$ still goes to zero for large K', as can be seen from Eq. (3).

The net effect of normalization and retardation on the S-wave matrix elements is now clear. On and near the momentum diagonal, normalization is the greater effect and the matrix elements are decreased. Far off the momentum diagonal retardation is more important and the matrix elements are increased.

Since the phase shifts depend on potential matrix elements both on and off the momentum diagonal, the effect of these competing changes on the S-wave phase shift is not obvious. Numerical results for a standard Yukawa potential and the generalized Yukawa potential are shown in Table I. Both potentials are arbitrarily taken to have a strength, V_0, of +82 MeV and a range of 1 fm^{-1}.

The phase shifts of the generalized Yukawa potential are smaller than those of the standard Yukawa potential for each energy calculated. The change is greater at higher energies. In the energy range 24 to 352 MeV, the normalization and retardation effects weaken the potential. The effect at 352 MeV is about 7% of the phase shift. The calculation shows that both retardation and normalization effects contribute to this reduction.

The comparison of potentials given above does not enable us to conclude anything about the off-shell effects in the two-nucleon interaction because the on-shell T matrix has been changed. In the next section a phenomenological potential based on the generalized Yukawa form is constructed. After the new potential is fitted to the phase shifts, the T matrix of the generalized form is compared to the T matrix generated by the Reid potential.

TABLE II. ${}^1S_0$ parameters.

	Reid	Generalization
V_1	−10.463	−9.40
V_2	−6602.4	−6765.0
V_3	+45 389.4	+49 830.0

TABLE IV. Phase-shift comparison, ${}^1S_0$ (in rad).

Reid	Lab energy (MeV)	Generalization
0.861	24	0.862
0.684	48	0.684
0.440	96	0.437
0.263	144	0.258
0.080	208	0.076
−0.129	304	−0.129
−0.216	352	−0.212

4. A NEW PHENOMENOLOGICAL POTENTIAL

In this section, the Reid soft-core potential[1] is generalized in the $^1S_0$ and $^3S_1$-$^3D_1$ channels by replacing each standard Yukawa by its corresponding generalized Yukawa form. The Reid soft-core potential consists of the nonrelativistic one-pion exchange potential, short-range Yukawa potentials, and derivative or Thomas forms. Reid adjusted the strengths and ranges to fit the two-nucleon data. To fit the same $^1S_0$ and $^3S_1$-$^3D_1$ data with the generalized Yukawa form, it is necessary to adjust the potential strengths. (The ranges were kept unchanged.)

In the $^1S_0$ channel, the Reid soft-core potential can be written as

$$V(^1S_0) = V_1\frac{e^{-x}}{x} + V_2\frac{e^{-4x}}{4x} + V_3\frac{e^{-7x}}{7x}. \tag{11}$$

The V_i are potential strengths in MeV and $x=mr$ with $m=0.7\ \text{fm}^{-1}$. In momentum space this is

$$\langle K'|V(^1S_0)|K\rangle = \frac{1}{2KK'}\{V_1Q_0[Z(m)]/m + V_2Q_0[Z(4m)]/4m + V_3Q_0[Z(7m)]/7m\}, \tag{12}$$

where

$$Z(M) = (K^2+K'^2+M^2)/2KK'.$$

The generalized Yukawa form given by Eq. (3) is

$$\langle K'|V(^1S_0)|K\rangle = \frac{1}{2KK'}\left(\frac{E+M}{2E}\right)\left(\frac{E'+M}{2E'}\right)\left[\frac{V_1}{m}Q_0\left(\frac{2EE'-2M^2+m^2}{2KK'}\right) + \frac{V_2}{4m}Q_0\left(\frac{2EE'-2M^2+(4m)^2}{2KK'}\right) + \frac{V_3}{7m}Q_0\left(\frac{2EE'-2M^2+(7m)^2}{2KK'}\right)\right]. \tag{13}$$

In the $^3S_1$-$^3D_1$ channel, the Reid potential is given by

$$V(^3S_1\text{-}^3D_1) = V_c + V_T S_{12} + V_{LS}\vec{L}\cdot\vec{S}, \tag{14}$$

with

$$V_c = V_4\frac{e^{-x}}{x} + V_5\frac{e^{-2x}}{2x} + V_6\frac{e^{-4x}}{4x} + V_7\frac{e^{-6x}}{6x},$$

$$V_T = V_8\left(1+\frac{3}{x}+\frac{3}{x^2}\right)\frac{e^{-x}}{x} + V_9\left(\frac{12}{x}+\frac{3}{x^2}\right)\frac{e^{-4x}}{x} + V_{10}\frac{e^{-4x}}{4x} + V_{11}\frac{e^{-6x}}{6x},$$

and

$$V_{LS} = V_{12}\frac{e^{-4x}}{4x} + V_{13}\frac{e^{-6x}}{6x}.$$

The generalization of the central part, V_c, and the spin-orbit part V_{LS} parallels generalization of Eq. (12) to Eq. (13). For the tensor term in momentum space, the pertinent S-D and D-D matrix elements obtained from Eq. (14) are

$$\langle K2|V_T|K'2\rangle = V_8\left(\frac{1}{2mKK'}Q_2[Z(m)] + \frac{3}{10m^3}\{Q_1[Z(m)] - Q_3[Z(m)]\}\right) + \frac{3V_9}{10m^3}\{Q_1[Z(4m)] - Q_3[Z(4m)]\} + \frac{V_{10}}{8mKK'}Q_2[Z(4m)] + \frac{V_{11}}{12mKK'}Q_2[Z(6m)], \tag{15}$$

$$\langle K0|V_T|K'2\rangle = \frac{V_8}{m^3}\left\{\frac{K^2Q_2[Z(m)]+K'^2Q_0[Z(m)]}{2KK'} - Q_1[Z(m)]\right\} + \frac{V_9}{m^3}\left\{\frac{K^2Q_2[Z(4m)]+K'^2Q_0[Z(4m)]}{2KK'} - Q_1[Z(4m)]\right\} + \left(\frac{V_{10}}{4m} - \frac{64V_9}{m}\right)I_{02}^{(1)}(4m) + \frac{V_{11}}{6m}I_{02}^{(1)}(6m). \tag{16}$$

TABLE V. Phase-shift comparison $^3S_1$-$^3D_1$ (in rad).

Blatt-Biedenharn convention						
	Reid		Lab energy		Generalization	
$^3S_1$	$^3D_1$	ϵ	(MeV)	$^3S_1$	$^3D_1$	ϵ
1.426	−0.050	0.032	24	1.425	−0.052	0.033
1.106	−0.116	0.043	48	1.104	−0.119	0.046
0.751	−0.217	0.069	96	0.752	−0.216	0.069
0.526	−0.286	0.105	144	0.531	−0.283	0.096
0.314	−0.354	0.167	208	0.323	−0.350	0.139
0.092	−0.438	0.281	304	0.098	−0.423	0.217
0.007	−0.480	0.342	352	0.012	−0.452	0.261

The generalization chosen for Eqs. (15) and (16) was analogous to Eq. (13), i.e., Z_{NR} is replaced by Z_R and an over-all spinor normalization introduced. However, the difficult $I_{02}^{(1)}$ terms (see Haftel and Tabakin[7]) were left unchanged.

With these generalized forms, the potential strengths given in Tables II and III were determined by fitting the Reid phase shifts; the corresponding phase shifts are given in Tables IV and V. On shell, the T matrix of the generalized potential is closely equivalent to that of the Reid potential. The $^1S_0$ phase shift fit is very good. At low energies the fit in the $^3S_1$-$^3D_1$ channel is quite close. As a result we expect the deuteron properties to be changed very little. Above 150 MeV, the ϵ and $^3D_1$ phase shift fits fall somewhat below the Reid values.

It is of interest to examine the changes of the potential parameters needed to obtain the approximate on-shell equivalence. One important feature is that the strengths of the short-range repulsions,

TABLE VI. Binding energy of nuclear matter at saturation density $K_F = 1.36$ fm^{-1}.

	Reid	Generalized potential
M^*/M	0.640	0.640
U_0	79.00	79.00
Potential energy/A		
$^1S_0$	−15.57	−15.68
$^3S_1$	−15.16	−13.68
$^3D_1$	1.45	1.50
Sum	−29.28	−27.86
Kinetic energy/A	23.01	23.01
KE/A + PE/A	−6.27	−4.85
Other partial waves	−3.59	−3.59
E/A	−9.86	−8.44
Wound integrals		
$\kappa(^1S_0)$	0.0221	0.0197
$\kappa_{00}(^3S_1)$	0.0298	0.0327
$\kappa_{02}(^3S_1$-$^3D_1)$	0.0644	0.0763
$\kappa_{20}(^3S_1$-$^3D_1)$	0.0000	0.0000
$\kappa_{22}(^3D_1)$	0.0000	0.0000
Sum	0.1163	0.1287

V_3 and V_7, had to be increased to compensate for the weaker repulsive effect of V_R compared to V_{NR} (see Table I). The size of the shift in the one-pion coefficients, V_1 and V_4, is notable. However, Noble and Richards[8] have shown that changes of the unitarization scheme can produce one-pion exchange phase-shift modifications of the same order of magnitude. Therefore we consider these shifts reasonable.

Reid cancelled the $1/r^3$ singularity in the tensor force by setting $V_8 = -V_9$. For the generalized

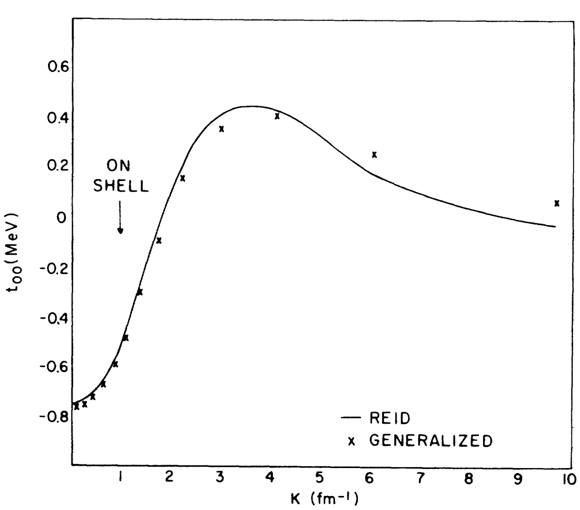

FIG. 1. The function $t_{00}(K, K')$ in the $^1S_0$ channel with $K' = 0.96$ fm^{-1}.

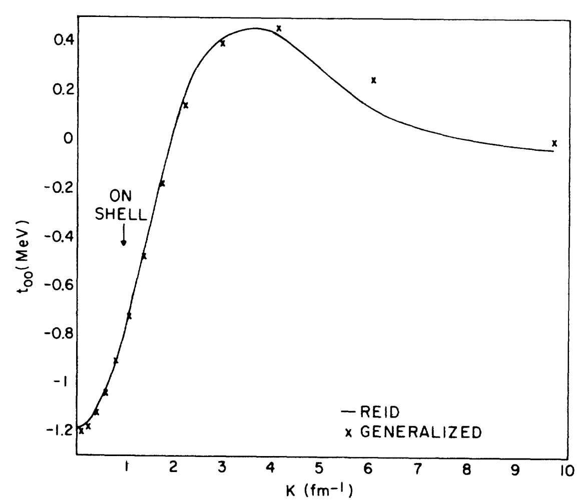

FIG. 2. The function $t_{00}(K, K')$ in the $^3S_1$-$^3D_1$ channel with $K' = 0.96$ fm^{-1}.

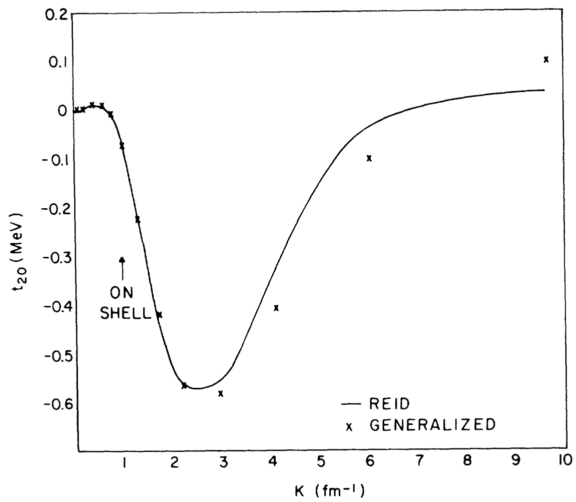

FIG. 3. The function $t_{20}(K, K')$ in the $^3S_1$-$^3D_1$ channel with $K' = 0.96$ fm^{-1}.

Yukawa case $V_8 \neq -V_9$ was used to fit the phase shifts, but no singularity is generated.

Having fixed the on-shell values, we attribute the off-shell modifications of the T matrix to the introduction of retardation and spinor normalization factors. The importance of these changes is examined in the next section.

5. OFF-SHELL T MATRIX AND NUCLEAR MATTER

Having used the generalized Yukawa interactions to fit the phase shifts, we next examine the full T matrix which is determined by the Lippmann-Schwinger equation, $T(\omega) = V + V(\omega - H_0)^{-1}T(\omega)$.

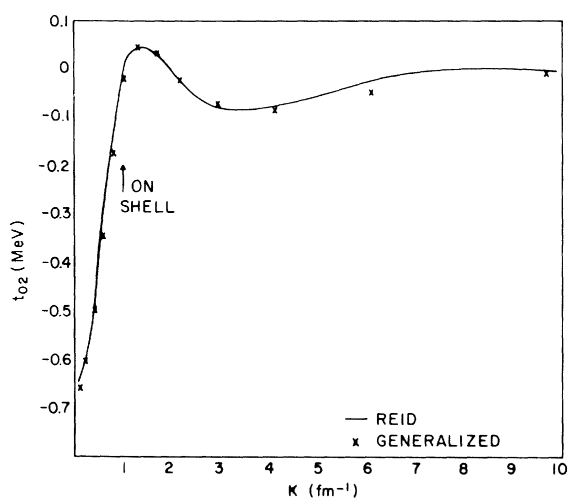

FIG. 4. The function $t_{02}(K, K')$ in the $^3S_1$-$^3D_1$ channel with $K' = 0.96$ fm^{-1}.

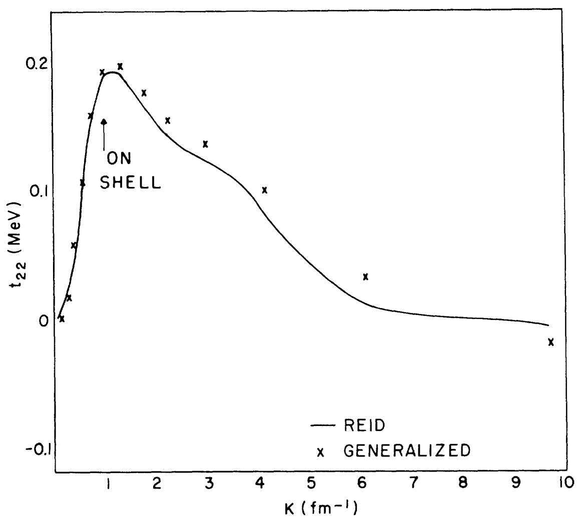

FIG. 5. The function $t_{22}(K, K')$ in the $^3S_1$-$^3D_1$ channel with $K' = 0.96$ fm^{-1}.

The numerical methods of Ref. 7 were used. It is of interest to see how the modifications in the potential matrix elements discussed in Sec. 3 effect the off-shell T matrix. The on-shell T matrix has already been fixed by using the new parameters (Tables II and III) to fit the phase shifts.

In Figs. 1–5 we present the $^1S_0$ and $^3S_1$-$^3D_1$ half off-shell T matrix elements $t^{\alpha}_{LL'}(K, K')$[9] as a function of K for the Reid soft-core potential and its generalization. For all states, the far off-shell T matrix elements of $U_R(K \gg K')$ were increased and approach zero more slowly than those of U_{NR}. Near the energy shell, the T matrix elements of U_R and U_{NR} are close to each other.

Since the T matrix elements of U_R and U_{NR} differ the most far off the energy shell, the question arises whether the differences influence nuclear

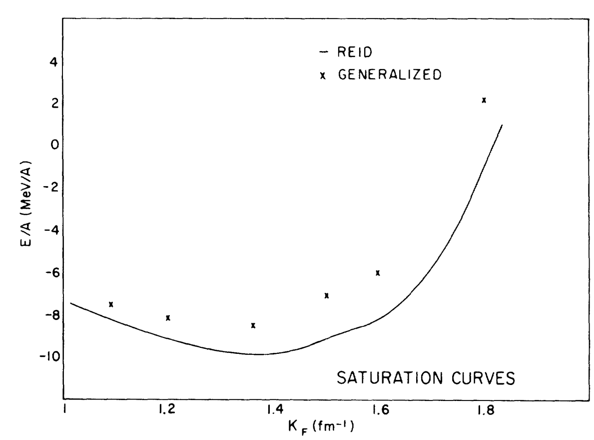

FIG. 6. Saturation curves for the Reid soft-core potential and the generalized potential.

observables. Nuclear matter calculations provide insight into this question due to their sensitivity to off-shell T matrix elements.[10]

In Fig. 6 we plot the saturation curves for the Reid potential and its generalization. The nuclear matter calculation also used the method of Ref. 7. The state by state contributions to the binding energy and to the wound intergral, κ,[11] for the $^1S_0$ and $^3S_1$-$^3D_1$ partial waves are shown in Table VI. The same $L>0$ partial waves are included in both cases. The Reid potential saturates at $K_F \simeq 1.36$ fm^{-1} with 9.86-MeV binding per particle. Different values are obtained by other authors,[12] because of their assumptions concerning higher partial waves. The generalized potential saturates at $K_F \simeq 1.35$ fm^{-1} with 8.44-MeV binding per particle. For our present study it is only the difference in binding $E_{NR} - E_R = 1.42$ MeV/A that is of interest.

By examining Table VI, wee see U_R gives less binding and larger wound integrals than U_{NR} for all states except the $^1S_0$ state. The reason for the increased repulsion and increased wound integral is clear from Ref. 7. Larger off-shell matrix elements are associated with larger wound integrals. Larger wound integrals imply increased repulsion in nuclear matter. In the $^3S_1$-$^3D_1$ channel the differences in the far off-shell elements of the T matrix dominate. Consequently U_R has a larger wound integral and more repulsion than U_{NR}.

In the $^1S_0$ state, the off-shell T matrix elements of U_{NR} are slightly larger up to $K' \approx 4$ fm^{-1}, at which point the slower falloff of U_R takes over. The net effect is a slight increase in attraction and decrease in the wound integral. The plots of T matrix elements show that we should expect this decrease in the wound integral only in the $^1S_0$ state.

Our change in nuclear matter binding is about twice the magnitude but of opposite sign of that found by Brown, Jackson, and Kuo.[13] Their prescription involves a different normalization factor $(MM'/EE')^{1/2}$, but no change in the argument, z of Q_0, i.e., no retardation effect. In their case a considerable decrease in far off-shell potential matrix elements occurs and a corresponding increase in binding of 0.5 MeV/A results. The difference between their result and ours clearly results from the different off-shell prescriptions. Including both the spinor normalization and retardation effect, our case leads to a decrease in binding energy of 1.4 MeV/A.

6. CONCLUSION

Generalized Yukawa interactions have been defined to incorporate retardation and spinor normalization effects. They have been used to construct phenomenological $^1S_0$ and $^3S_1$-$^3D_1$ potentials patterned after the Reid soft-core potential. It is found that spinor normalization decreases the potential matrix elements, whereas retardation increases the far off-diagonal matrix elements. When the on-shell T matrix is constrained to fit the phase shifts, the result of this competition is a net increase in far off-shell matrix elements. Consequently, the binding of nuclear matter is decreased 1.4 MeV/A. Using a qualitatively similar normalization, Brown, Jackson, and Kuo have found that a normalization change by itself causes a net increase in binding. We, therefore, conclude that our 1.4-MeV decrease in binding is primarly a result of retardation. While an improved $^3S_1$-$^3D_1$ fit may alter this result somewhat, no qualitative change is expected.

Based on this estimate, we feel an improved theory of the effects of retardation is needed. Perhaps the time development approach of Johnson and Baranger[14] can be of help. Certainly, a precise calculation of the effect of retardation can only be formulated with relativistic kinematics.

APPENDIX

Proof that $K^2 + K'^2 \geq 2EE' - 2M^2$:

$$(K^2 - K'^2)^2 \geq 0\,, \tag{A1}$$

$$\begin{aligned}(K^2 - K'^2)^2 &= K^4 + K'^4 - 2K^2K'^2 \\ &= (K^4 + K'^4 + 2K^2K'^2 + 4M^4 + 4M^2K^2 + 4M^2K'^2) - (4K^2K'^2 + 4M^2K^2 + 4M^2K'^2 + 4M^4) \\ &= (K^2 + K'^2 + 2M^2)^2 - 4(K^2 + M^2)(K'^2 + M^2) \\ &= \{K^2 + K'^2 + 2M^2 + 2[(K^2 + M^2)(K'^2 + M^2)]^{1/2}\}\{K^2 + K'^2 + 2M^2 - 2[(K^2 + M^2)(K'^2 + M^2)]^{1/2}\} \geq 0\,.\end{aligned}$$

Since the first factor is >0, the second factor must be too,

$$K^2 + K'^2 + 2M^2 - 2EE' \geq 0\,, \tag{A2}$$

$$K^2 + K'^2 \geq 2EE' - 2M^2\,. \tag{A3}$$

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[3]Here $p_1=(\vec{p},E)$, $p_2=(-\vec{p},E)$, $p_3=(\vec{p}',E')$, and $p_4=(-\vec{p}',E')$. All particles are on the mass shell. However, four-momentum is not conserved for finite collision times and we have $E\neq E'$ in our off-shell matrix elements. This definition of off-shell matrix elements parallels that of Coester and Scheirholz and should not be confused with a dispersion relation approach.

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[9]For uncoupled channels

$$t^{\alpha}_{LL}(K',K)\equiv e^{-i\,\delta_L(K')}T^{\alpha}_{LL}(K,K',K'^2+i\epsilon)\,.$$

For coupled channels

$$t^{\alpha}_{L'L}(K',K)\equiv\sum_l T^{\alpha}_{L'l}(K,K',K'^2+i\epsilon)[U^+e^{-i\Delta(K')}U]_l\,,$$

with

$$U=\begin{pmatrix}\cos\epsilon & \sin\epsilon\\ -\sin\epsilon & \cos\epsilon\end{pmatrix},$$

$$\Delta(K')=\begin{pmatrix}\delta_{J-1}(K') & 0\\ 0 & \delta_{J+1}(K')\end{pmatrix}.$$

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Neutron-Deuteron and Neutron-Neutron Total Cross Sections in the Range 25-60 MeV*

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Measurements of the n-d total cross section to a precision of 1% are reported for the energy range 25–60 MeV. Nearly monoenergetic neutron beams were used whose energy was determined to about ±100 keV. The experimental values of σ_{nd} for energies ≥36 MeV are several percent larger than previous measurements in this range. Assuming charge symmetry, σ_{nn} can be calculated from the $T=1$ phase shifts. The differences between experimental values of $(\sigma_{nd}-\sigma_{np})$ and σ_{nn} calculations are small and fairly well accounted for by correction terms based on the method of Glauber.

I. INTRODUCTION

In previous experiments at Harwell[1] and Harvard[2] in the ranges 15–120 MeV and 90–150 MeV, respectively, the difference between the n-d and the n-p total cross sections, $\sigma_{nd}-\sigma_{np}$, was measured directly using a (D_2O-H_2O) difference technique. The differences, $\sigma_{nd}-\sigma_{np}$, corrected for double scattering by the method of Glauber,[3] give inferred values of σ_{nn}. For $E_n\geq 40$ MeV these agreed to within experimental uncertainty with the values of σ_{nn} calculated from $T=1$ phase parameters derived from p-p measurements. This agreement is surprising, since the "size" of the wave packet of the incident nucleon is not small compared to the size of the deuteron and the impulse approximation is indeed approximate.

Experimental values of σ_{nd} were calculated at Harwell and Harvard by adding together the measured values of $\sigma_{nd}-\sigma_{np}$ and σ_{np}. The over-all precision of the Harwell values[1] of σ_{nd} depends on that of earlier Harwell measurements[4] of σ_{np} and is