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Application of the Burnett-Kroll Soft-Proton Theorem to Nucleon-Nucleon Bremsstrahlung*

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We give an explicit analytic formula for the leading terms in the square of the matrix element for nucleon-nucleon bremsstrahlung in the relativistic, model-independent, soft-photon theory of Nyman. Such a formula should simplify direct and detailed comparisons of this soft-photon theory with experiment and with the predictions of other models. These comparisons have been difficult because Nyman in his original calculation of $NN \to NN\gamma$ was forced by the extremely complicated matrix element to use entirely numerical procedures. To obtain an analytic result we have used the Burnett-Kroll theorem, which makes it possible to bypass many of the complications of a straightforward algebraic calculation. We also emphasize the general utility of the Burnett-Kroll theorem as a calculational tool for obtaining in an almost trivial fashion the leading terms in the squares of very complicated radiative matrix elements.

I. INTRODUCTION

A number of years ago it was suggested[1] that an investigation of the nucleon-nucleon bremsstrahlung process $N+N \to N+N+\gamma$ would give information about the off-mass-shell components of the nucleon-nucleon scattering amplitude. More recently, however, it has become clear, as a result of various low-energy theorems[2–6] that the leading terms in an expansion of the $NN\gamma$ amplitude in powers of the photon momentum k depend only on the on-mass-shell parts of the NN amplitude. Thus one must work harder than was originally supposed to get information about off-mass-shell terms. However, experimental developments since the original suggestion have made it quite feasible to examine in detail regions where higher-order terms might be important, so nucleon-nucleon bremsstrahlung remains a very interesting process.

There have been a large number of calculations of nucleon-nucleon bremsstrahlung, including, for example, those of Refs. 7–11. Typical ones include a number of calculations based on primarily nonrelativistic potential models,[8] the relativistic one-boson exchange model of Baier, Kühnelt, and Urban[9] and the completely relativistic calculation of Nyman[10] which uses soft-photon techniques. This latter calculation is of particular interest because it is both relativistic and model-independent. By "model-independent" we mean that the radiative amplitude depends only on the on-mass-shell NN amplitude through the leading two orders, $O(k^{-1})$ and $O(k^0)$, in the photon momentum k. This result[2–6,12] follows directly from gauge invariance and certain smoothness and analyticity assumptions.[12] Within the definition of model-independent there exists the freedom to choose different continuations of the NN amplitude to the slightly unphysical point corresponding to the radiative process or to choose different sets of kinematic variables to describe the NN amplitude. One can show however, given the above assumptions, that this freedom alters the result only in order k.[3,12–14] Thus within the region of strict applicability of Nyman's calculation, i.e., when the $O(k)$ terms are truly negligible, all other calculations must agree with it (provided of course that they reproduce the same NN scattering amplitude and satisfy the necessary smoothness conditions[12]). Thus the soft-photon theory provides a benchmark against which to check more elaborate models.

From a practical point of view however, the region of strict applicability of the soft-photon theory is significantly smaller than the region which has been explored experimentally. The ex-

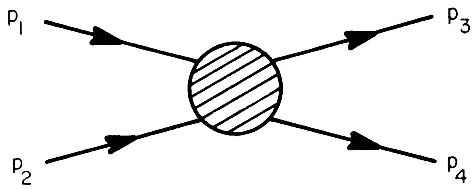

FIG. 1. Kinematics for the nonradiative process $NN \to NN$.

pansion parameter is really k/E,[2,7] where E is essentially an average kinetic energy of the nucleons. Thus for low energies k/E may be of order unity, the $O(k)$ terms may be nonnegligible, and the soft-photon theory as used by Nyman works rather poorly.[7] Nevertheless it has been customary, whether or not one expects the soft-photon theory to be strictly applicable, to compare Nyman's predictions with experiment or with the results of the particular model being considered.

For these reasons it would seem extremely useful to have available a simple analytic formula for the square of the radiative matrix element in the soft-photon theory which would simplify such direct comparisons with other models or with experiment. A nonrelativistic approximation to such a formula is known[7] but no such formula corresponding to Nyman's completely relativistic calculation seems to exist. In fact the amplitude Nyman obtained was so complicated that he found it impossible to obtain its square analytically either with straightforward trace techniques or using computer trace evaluation programs. Thus, he was forced to adopt a numerical representation for the γ matrices and other quantities in the amplitude and to evaluate the square of the amplitude numerically. The end results are then cross sections corresponding to particular geometries, incident energies, etc., and thus it is very difficult to make detailed comparisons with the predictions of various models or the results of particular experiments except for the specific points chosen by Nyman. In addition it is impossible, for example, to separate the contributions of each order in k, so as to learn something of the region of convergence of the soft-photon expansion, without a major programming effort comparable to his original calculation.

To remedy this situation we give in this paper an analytic formula for the square of the relativistic, model-independent $NN\gamma$ matrix element. That such a calculation is possible is due in part to the Burnett-Kroll (BK) theorem.[14,15] This theorem states essentially that the leading two terms in an expansion in k of the square of the matrix element for a radiative process summed over spins is given by a relatively simple operator operating on the unpolarized nonradiative cross section. Thus the theorem indicates, as does the Low theorem, that one must look at higher-order terms to learn information from a radiative process which is unobtainable from the corresponding nonradiative process. However, because of the simplicity of the final result and the relative simplicity of nonradiative matrix elements one can consider the BK theorem as a calculational tool useful for obtaining the leading terms in the squares of radiative matrix elements. Such an approach has proved useful previously in connection with a calculation of radiative K_{l3} decays.[16,17] In the present case it allows one to obtain the two leading terms in the square of the extremely complicated matrix element obtained by Nyman in a completely *trivial* fashion. Such an approach also allows one to see general features of the final result which are not immediately evident from a completely numerical calculation, as for example the fact that the nucleon magnetic moments do not contribute through the leading two orders in k.

Thus the purpose of this paper is twofold. First, as discussed above, we want to give an explicit analytic form for the square of the matrix element for the $NN\gamma$ process so that direct, detailed comparisons of Nyman's model-independent soft-photon calculation with other $NN\gamma$ calculations and with experiment can be easily made. Second, we

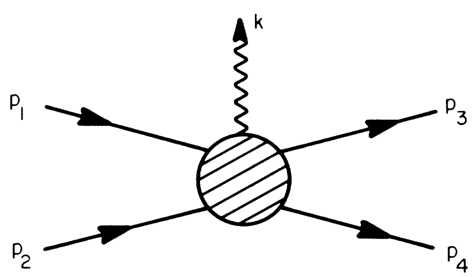

FIG. 2. Kinematics for the radiative process $NN \to NN\gamma$.

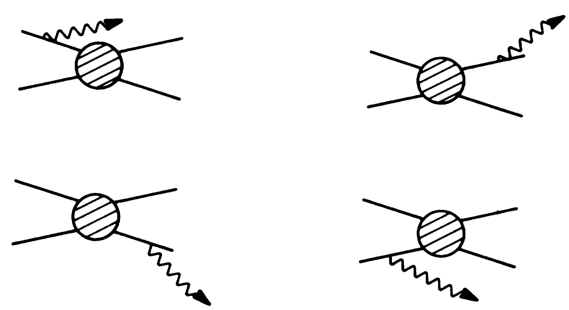

FIG. 3. Diagrams involving radiation from external lines which contribute to the Low result.

want to acquaint others working in the field of radiative processes with the utility of the BK theorem as a calculational tool for obtaining in a simple fashion the important terms in squares of very complicated radiative matrix elements. Such techniques should be particularly useful when one wants only an estimate for which the leading terms are sufficient or for processes, e.g., radiative decays, where kinematic constraints limit one to the soft-photon region.

In Sec. II we outline briefly our notation and the procedure used to obtain the matrix element. In Sec. III we discuss the difficulty of using straightforward trace techniques to evaluate the square of the matrix element, to understand why Nyman was forced to use a numerical representation. We then outline the BK theorem and show how it can be used to obtain an analytic result for the leading terms in the square of the $NN\gamma$ matrix element in a form suitable for numerical comparison with other models and with experiment.

II. RADIATIVE MATRIX ELEMENT

In this section we review briefly the calculation of Nyman leading to the matrix element for $NN \to NN\gamma$. We first define the nonradiative matrix element to establish our notation and then summarize the substance of the Low theorem and quote the general result. This result is then used to get the specific matrix element for $NN \to NN\gamma$.

Consider first the nonradiative, i.e. elastic, NN scattering process with kinematics as shown in Fig. 1. We define the unsymmetrized amplitude for this process as[18]

$$T(NN \to NN) = \sum_{\alpha=1}^{5} F_\alpha(\nu, \Delta)\, \bar{u}_3 t_\alpha u_1 \bar{u}_4 t^\alpha u_2 , \qquad (2.1)$$

where

$$t_1 = 1 , \quad t_2 = \frac{1}{\sqrt{2}}\sigma_{\mu\nu} = \frac{i}{2\sqrt{2}}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) ,$$

$$t_3 = i\gamma_5\gamma_\mu , \quad t_4 = \gamma_\mu , \quad t_5 = \gamma_5 .$$

Sums on implicit Lorentz indices are always implied, e.g.,

$$\bar{u}_3 t_4 u_1 \bar{u}_4 t^4 u_2 \to \sum_\mu \bar{u}_3 \gamma_\mu u_1 \bar{u}_4 \gamma^\mu u_2 .$$

The invariants $F_\alpha(\nu, \Delta)$ are related to the NN scattering parameters,[10, 19] e.g., to the phase shifts. They are functions of the variables $\nu = P_1 \cdot P_2 + P_3 \cdot P_4$ and $\Delta = P_1 \cdot P_3 + P_2 \cdot P_4$ which in the elastic scattering limit are essentially just the usual Mandelstam variables $s = (P_1 + P_2)^2$ and $t = (P_1 - P_3)^2$, i.e., $\nu \to s - 2m^2$ and $\Delta \to 2m^2 - t$, where m is the nucleon mass.

For the radiative process of Fig. 2 the T matrix can be expanded in powers of the photon energy k as

$$T(NN \to NN\gamma) = \frac{a}{k} + b + ck + \cdots . \qquad (2.2)$$

The coefficients a and b can be obtained, according to the Low theorem,[2-6] by adding a piece to the contributions from those diagrams involving radiation from external lines (Fig. 3) sufficient to make the result gauge invariant. It is a further consequence of the theorem that these coefficients involve only quantities which can be obtained from the on-mass-shell nonradiative process. Nyman gives a detailed discussion of application of Low's theorem to this specific process. Rather than repeat this discussion we will simply take the general Low result for a radiative matrix element from a recent review[16] and apply it directly to obtain the matrix element for $NN \to NN\gamma$ in a form appropriate to our later discussion.

Thus from Eq. (2.9) of Ref. 16 we have the general result for the matrix element for a radiative process $1 + \cdots \to 2 + \cdots + \gamma$, where 1 and 2 are fermions and where $\cdots$ stands for any number of bosons:

$$T(1+\cdots \to 2+\cdots+\gamma) = \sum_i \eta_i Q_i \frac{\epsilon \cdot P_i}{k \cdot P_i} \bar{u}_2 T_0 u_1 + \sum_i Q_i D^\mu(P_i) \bar{u}_2 \frac{\partial T_0}{\partial P_i^\mu} u_1 + \bar{u}_2 \frac{\epsilon \cdot \gamma k \cdot \gamma}{2k \cdot P_2}\left[Q_2 + \frac{\kappa_2}{2m_2}(\gamma \cdot P_2 + m_2)\right] T_0 u_1$$
$$+ \bar{u}_2 T_0 \left[Q_1 + \frac{\kappa_1}{2m_1}(\gamma \cdot P_1 + m_1)\right]\frac{k \cdot \gamma \epsilon \cdot \gamma}{2k \cdot P_1} u_1 + O(k) . \qquad (2.3)$$

Here P_i, Q_i, κ_i, m_i, and η_i are, respectively, the momentum, charge, anomalous magnetic moment, mass, and a phase (+1 for final particles, −1 for initial particles) corresponding to the ith particle. The photon momentum and polarization vector are, respectively, k^μ and ϵ^μ and $D^\mu(P_i) = (\epsilon \cdot P_i / k \cdot P_i) k^\mu - \epsilon^\mu$. T_0 is the nonradiative, on-mass-shell T matrix with the spinors factored out, i.e., $T(1+\cdots \to 2+\cdots) = \bar{u}_2 T_0 u_1$. It is however evaluated at values of the momenta P_i satisfying the four-momentum conservation equation appropriate to the radiative process. To obtain the result for $NN \to NN\gamma$ one must generalize the above equation to cases involving two fermion lines. Such a generalization is trivial, however, and leads to the following result for the specific case $NN\to NN\gamma$:

$$T(NN\to NN\gamma)=\left(\sum_{i=1}^{4}\eta_i Q_i\frac{\epsilon\cdot P_i}{k\cdot P_i}\right)\sum_{\alpha=1}^{5}F_\alpha(\nu,\Delta)\,\bar{u}_3 t_\alpha u_1\bar{u}_4 t^\alpha u_2+\sum_{\alpha=1}^{5}\left(\sum_{i=1}^{4}Q_i\,D^\mu(P_i)\frac{\partial}{\partial P_i^\mu}F_\alpha(\nu,\Delta)\right)\bar{u}_3 t_\alpha u_1\bar{u}_4 t^\alpha u_2$$

$$+\sum_{\alpha=1}^{5}F_\alpha(\nu,\Delta)\left\{\bar{u}_3\frac{\epsilon\cdot\gamma k\cdot\gamma}{2k\cdot P_3}\left[Q_3+\frac{\kappa_3}{2m}(\gamma\cdot P_3+m)\right]t_\alpha u_1\bar{u}_4 t^\alpha u_2+\bar{u}_3 t_\alpha\left[Q_1+\frac{\kappa_1}{2m}(\gamma\cdot P_1+m)\right]\right.$$

$$\left.\times\frac{k\cdot\gamma\epsilon\cdot\gamma}{2k\cdot P_1}u_1\bar{u}_4 t^\alpha u_2+(1\leftrightarrow 2,\ 3\leftrightarrow 4)\right\}+O(k)\,. \qquad (2.4)$$

With the appropriate choices of charges and anomalous magnetic moments the above equation gives the result for pp, np, or nn bremsstrahlung. With a little Dirac algebra one can show that it gives results identical to Nyman's Eqs. (24) and (25).

One further comment is in order. In the pp and nn cases one must make the full amplitude antisymmetric in the interchange of identical particles. This means that one should subtract from Eq. (2.4) a piece differing only by the interchange $3\leftrightarrow 4$. However, by using the Fierz[20] transformation in the form

$$(t_\alpha)_{\phi\sigma}(t^\alpha)_{\tau\nu}=\sum_{\beta=1}^{5}C_{\alpha\beta}(t_\beta)_{\phi\nu}(t^\beta)_{\tau\sigma}\,, \qquad (2.5)$$

with

$$C_{\alpha\beta}=\frac{1}{4}\begin{pmatrix}1&1&1&1&1\\6&-2&0&0&6\\4&0&-2&2&-4\\4&0&2&-2&-4\\1&1&-1&-1&1\end{pmatrix},$$

the added terms can be put in exactly the same form as the original terms except that $F_\alpha(\nu,\Delta)$ should be replaced by $\sum_\beta C_{\beta\alpha}F_\beta(\nu,\Delta')$, where $\Delta'=P_1\cdot P_4+P_2\cdot P_3$, which reduces in the elastic limit to $2m^2-u$ with $u=(P_1-P_4)^2$. Consequently antisymmetrization amounts to the replacement in Eq. (2.4) [or in Eq. (2.1)]

$$F_\alpha(\nu,\Delta)\to F_\alpha(\nu,\Delta)-\sum_\beta C_{\beta\alpha}F_\beta(\nu,\Delta')\,,$$

and so does not change the form of the result. Thus we can handle all three cases pp, pn, and nn, simultaneously by using Eq. (2.4) for the matrix element with the understanding that in the final result one must make the appropriate choices of charges, magnetic moments, and elastic scattering parameters F_α corresponding to the particular process under consideration.

III. BURNETT-KROLL THEOREM FOR THE SQUARE OF THE MATRIX ELEMENT

Given the matrix element of Eq. (2.4) it is *in principle* a straightforward procedure to square it, sum on spins, and using conventional trace techniques eventually obtain an analytic form for the square of the matrix element. Such an expression would in general take the form

$$\sum_{\text{spins}}|T(NN\to NN\gamma)|^2$$

$$\sim\text{Trace}\sum_{\text{spins}}\left(\frac{a^2}{k^2}+\frac{2\,\text{Re}\,ab^*}{k}+(b^2+2\,\text{Re}\,ac^*)+O(k)\right), \qquad (3.1)$$

where as discussed by Nyman the leading two terms, those of order $1/k^2$ and $1/k$, are model-independent and given from Eq. (2.4). The third term, the $O(k^0)$ term, involves both known parts, b^2, and an unknown structure-dependent part proportional to c. In this particular case, however, there are so many terms in the original matrix element of Eq. (2.4) that Nyman found such a straightforward approach impossible even when assisted by computer trace evaluation routines and was thus forced to adopt a completely numerical approach, i.e., to adopt a numerical representation for the γ matrices and evaluate the T matrix before squaring. In this section we want to show how the BK theorem allows one to bypass such involved numerical computations and obtain in a very simple fashion an analytic form for the two leading terms (the a^2 and ab terms) in the square of this matrix element.

First to get a feeling for the magnitude of the problem let us estimate the number of terms involved in the square of Eq. (2.4). If we consider the sums on i in brackets as single multiplying factors, but multiply out all other brackets we find that the T matrix contains $(8+8+3\times 8\times 4)$

= 112 terms. (Note that the sum on α involves 8 terms since the $t_2 t^2$ term when expanded in terms of γ matrices gives 4 terms.) Thus there are $(112)^2 \times 16 = 200\,704$ terms, each of which will be the product of 2 traces which in the most complicated cases involve the trace of 12 γ matrices times the trace of 6, or the trace of 8 times another trace of 8. By using a less straightforward but more intelligent approach one can reduce the initial number of terms which must be considered by a large factor, at least 25. For example using

$$F_1[1]\times[1] + F_2\left[\frac{1}{\sqrt{2}}\sigma_{\mu\nu}\right]\times\left[\frac{1}{\sqrt{2}}\sigma^{\mu\nu}\right] = (F_1 + 2F_2)[1]\times[1] - \tfrac{1}{2}F_2[\gamma_\mu\gamma_\nu]\times[\gamma^\mu\gamma^\nu], \tag{3.2}$$

and redefining $F_6 = (F_1 + 2F_2)$ one reduces the number of terms in the α sum back to five and thus the total number of terms by almost a factor of 3. Furthermore, three fourths of the terms are automatically zero because one or the other of the traces has an odd number of γ matrices. Even with these simplifications, however, we are left with several thousand traces to evaluate, some involving as many as 12 γ matrices, which is clearly impossible to do by hand.

The beauty of the BK theorem is that it allows one to bypass the evaluation of many of these traces in that it expresses the leading two terms of the square of the radiative matrix element as a very simple operator operating on the square of the nonradiative matrix element, in both cases summed on spins. Thus, one needs to do only those traces associated with the nonradiative process, which normally are comparatively simple.

To prove the BK theorem one starts with the general Low formula for the matrix element given in Eq. (2.3) and utilizes a number of Dirac algebra tricks, described in detail in Ref. 14 or 15 or in the review of Ref. 16. The result [Eq. (3.8) of Ref. 16], in the notation of Eq. (2.3) is

$$\sum_{\text{spins}} |T(1+\cdots \to 2+\cdots+\gamma)|^2 = \left[\hat{Q}^2 + \hat{Q}\sum_i Q_i D^\mu(P_i)\frac{\partial}{\partial P_i^\mu}\right] \times \sum_{\text{spins}} |T(1+\cdots\to 2+\cdots)|^2 + O(k^0), \tag{3.3}$$

where

$$\hat{Q} = \sum_i \eta_i Q_i \frac{\epsilon\cdot P_i}{k\cdot P_i}.$$

For our particular process we must generalize this formula to cases involving two fermion lines. Again the generalization is very easy and in this case leaves the formula unchanged.

The first step in applying this formula is to calculate the square of the nonradiative matrix element. Such a calculation is straightforward and in the present case easily carried out by hand.

The result is

$$\mathfrak{M}_{NN} \equiv m^4 \sum_{\text{spins}} |T(NN\to NN)|^2 = G_1(P_1\cdot P_2)(P_3\cdot P_4) + G_2(P_1\cdot P_3)(P_2\cdot P_4) + G_3(P_1\cdot P_4)(P_2\cdot P_3) + G_4 m^4 + G_5 m^2(P_1\cdot P_2 + P_3\cdot P_4) + G_6 m^2(P_1\cdot P_3 + P_2\cdot P_4) + G_7 m^2(P_1\cdot P_4 + P_2\cdot P_3), \tag{3.4}$$

where

$$G_1 = 4F_2^2 + 2F_3^2 + 2F_4^2 - 2F_1F_2 - 2F_2F_5 - 4F_3F_4,$$
$$G_2 = F_1^2 + F_5^2 - 2F_2^2,$$
$$G_3 = 4F_2^2 + 2F_3^2 + 2F_4^2 + 2F_1F_2 + 2F_2F_5 + 4F_3F_4,$$
$$G_4 = F_1^2 + F_5^2 + 6F_2^2 + 4F_3^2 + 4F_4^2,$$
$$G_5 = 2F_1F_4 + 6F_2F_3 - 6F_2F_4 - 2F_3F_5,$$
$$G_6 = F_1^2 - F_5^2 + 2F_3^2 - 2F_4^2,$$
$$G_7 = 2F_1F_4 + 6F_2F_3 + 6F_2F_4 + 2F_3F_5.$$

Consider now the square of the radiative matrix element. In order to be able to write the final formula in a compact form we define the quantities

$$A^\mu = \sum_{i=1}^4 \eta_i Q_i \frac{P_i^\mu}{k\cdot P_i}, \qquad W_{ij}^\mu = k\cdot P_i P_j^\mu - k\cdot P_j P_i^\mu, \tag{3.5}$$

and the permutation operator $\mathcal{P} = \sum_i \mathcal{P}_i$ with

$$\begin{aligned} \mathcal{P}_1 f(1,2,3,4) &= f(1,2,3,4),\\ \mathcal{P}_2 f(1,2,3,4) &= f(2,1,4,3),\\ \mathcal{P}_3 f(1,2,3,4) &= f(3,4,1,2),\\ \mathcal{P}_4 f(1,2,3,4) &= f(4,3,2,1), \end{aligned} \tag{3.6}$$

where $f(1,2,3,4)$ is some function of the four particle indices. Observe that A^μ, the scalars ν, Δ,

Δ', the square of the nonradiative matrix element $\mathfrak{M}_{NN}$, and the BK operator in Eq. (3.3) are all invariant under the permutations $\mathcal{P}_i$. Thus, using Eq. (3.3) and the polarization sum $\sum_{\text{pol}}\epsilon^\mu\epsilon^\nu=-g^{\mu\nu}$, we can write

$$\mathfrak{M}_{NN\gamma}\equiv m^4\sum_{\text{spins, pol}}|T(NN\to NN\gamma)|^2=-\mathcal{P}\left[\tfrac{1}{4}A^2+Q_1A\cdot\left(\frac{P_1}{k\cdot P_1}\,k\cdot\frac{\partial}{\partial P_1}-\frac{\partial}{\partial P_1}\right)\right]\mathfrak{M}_{NN}+O(k^0)\,. \tag{3.7}$$

Using the relation

$$\frac{\partial}{\partial P_1^\mu}=(P_2)_\mu\frac{\partial}{\partial\nu}+(P_3)_\mu\frac{\partial}{\partial\Delta}\,,$$

this becomes

$$\begin{aligned}\mathfrak{M}_{NN\gamma}=-A^2\mathfrak{M}_{NN}+\mathcal{P}\Bigg\{&\frac{Q_1}{k\cdot P_1}\left(A\cdot W_{12}\frac{\partial}{\partial\nu}+A\cdot W_{13}\frac{\partial}{\partial\Delta}\right)\mathfrak{M}_{NN}\\&+\frac{Q_1}{k\cdot P_1}[G_1A\cdot W_{12}(P_3\cdot P_4)+G_2A\cdot W_{13}(P_2\cdot P_4)+G_3A\cdot W_{14}(P_2\cdot P_3)\\&+m^2G_5A\cdot W_{12}+m^2G_6A\cdot W_{13}+m^2G_7A\cdot W_{14}]\Bigg\}+O(k^0)\,.\end{aligned} \tag{3.8}$$

We understand in this equation that the derivatives act only on the invariant functions $G_i(\nu,\Delta)$, i.e., on $F_\alpha(\nu,\Delta)$, as we have written explicitly in the last terms the contributions from the derivatives acting on the explicit momentum dependence of $\mathfrak{M}_{NN}$.

Equation (3.8) thus gives an analytic form for the leading terms, i.e., the $1/k^2$ part (contained in the A^2 term) and the $1/k$ part (contained in the remaining terms) of the square of the $NN\gamma$ matrix element. One can easily write it completely in terms of dot products of four-vectors using the definitions of Eq. (3.5) and then if desired express each dot product in terms of the angles, energies, and momenta appropriate to the particular coordinate system or experimental situation being considered. These leading terms constitute only a very small portion, roughly 14%, of the original 200 000 terms in the square of the matrix element. They are, however, at least in the kinematic region where the soft-photon approximation is valid, the most important terms. Thus the BK theorem has given a simple, quick method for choosing from the many terms in the square of a very complicated radiative matrix element, those which may be expected to be the most important, and expressing the result in analytic form.

If one wants to reproduce exactly the numerical results of Nyman, one must also include the b^2 term of Eq. (3.1) which comes from the square of the $O(k^0)$ terms in the amplitude, since it was included in his completely numerical calculation. The evaluation of this term is straightforward in principle, but in practice extremely complicated, since it contains the bulk of the original 200 000 terms. It proved possible however to find tricks, symmetries, and simplifications which reduced the number of traces sufficiently that the result could be obtained by computer using the symbolic trace evaluation program SCHOONSCHIP written by Veltman. The result still contains several hundred terms however and is too long to reproduce here. The difficulty of course is that, while these terms are necessary to reproduce Nyman's result and while they do give a model-independent contribution of $O(k^0)$, they do not give the complete $O(k^0)$ contribution. There are still additional contributions from the ac term of Eq. (3.1) which contain via the amplitude c completely unknown "structure-dependent" contributions and contributions which depend on the particular choice of the NN amplitude.[13,14] Thus one cannot calculate the square of the amplitude consistently to order k^0 in a model-independent way.

One further comment is in order. To use the BK theorem in the form given above one must sum on spins, which in some cases will be a disadvantage as it makes it impossible to get formulas for polarizations. (In contrast, with the numerical procedure used by Nyman results for particular spins can be obtained without much additional effort.) If one wants polarization information one can use a recent generalization[21] of the BK theorem which removes the requirement that all spins be summed and gives the leading terms in the square of the radiative amplitude for a particular spin configuration in terms of an operator acting on the square of the nonradiative amplitude for the same spin configuration. The operator is slightly more complicated than that of Eq. (3.3) and now involves magnetic moment terms, but should allow one to obtain analytic results for the leading terms in polarizations in a fashion, analogous to that used here, which should be much simpler than straightforward evaluation of the traces.

Finally we want to emphasize that these same techniques clearly can be applied to other radiative processes. They should be particularly useful for obtaining results in a quick and easy way when one is primarily interested in the soft photon region or when the accuracy given by keeping just the leading terms is sufficient. In particular they should be quite helpful when considering radiative decays, where phase-space considerations often limit one to the soft-photon region where the $O(k^0)$ term can really be expected to be small.

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