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Plane-Wave Matrix Approach to Direct Reactions

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A new method based on plane-wave representations is developed and applied to the theory of direct reactions. The coefficients of the plane-wave expansions of propagators are determined either by the requirement that they fit the scattering matrix for an auxiliary potential both on and off the momentum shell or by the use of an effective unit operator. Such expansions lead to the reduction of typical direct reaction problems to simple matrix algebra, the basic matrices being either plane-wave matrices or auxiliary potential matrices. Particular attention is given here to finite-range distorted-wave matrix elements and coupled-channels theory, where the possible advantages of the present approach can be understood even at a formal level.

I. INTRODUCTION

The progress of direct reaction theory, from the early success with plane waves[1] in the 1950's to the remarkable successes of the distorted-wave approach[2] in the 1960's, has recently reached something of an impasse. The necessity of including finite-range effects, multistep processes,[3,4] and complicated structure details in order to properly understand a wide range of nuclear reactions is becoming clearer and clearer with each successive refinement of the theoretical calculations. The number of "anomalous" angular distributions[5] and spectroscopic factor problems[6] is no longer to be ignored and indeed is to be expected.

The impasse mentioned above is that if one extrapolates the present calculational form of the theory to include finite-range and multistep processes simultaneously, high-speed computers now available will not be adequate. One is also faced with a cost problem because recent advances in computing tend to significantly reduce only the time element rather than the over-all cost. The aim of the present paper is to suggest new methods for calculating direct interaction transition probabilities. These methods are expected to be considerably faster than those currently available for similar situations except in the limit of zero-range approximations.

The existing methods of dealing with distorted waves in the case of finite-range involve either: (i) exact[7] numerical methods in which coordinate transformations are required and carried through for the L^2 integrable (form factor) part of the matrix elements; or (ii) approximate methods involving either approximate expansions about the zero-range limit[8] or approximate forms of the distorted waves such as the WKB approach. The method proposed here involves replacing the distorted waves by a finite (discrete) sum over plane waves so that coordinate transformations become simple algebra as in the original plane-wave Born approximation. Unlike conventional momentum representations the coefficients in our series expansion are not chosen by inverse Fourier transformations acting on the distorted waves, but rather by the requirement that the coefficients yield a representation which exactly fits the elastic scattering

transition matrix elements both on and off the momentum shell.

The coupled-channel Born approximation as first discussed by Penny and Satchler[9] is a topic of current interest. This method allows one to investigate such effects as core excitation and charge exchange on direct reaction cross sections. Glendenning and Mackintosh[10] have shown that the "source term" method introduced by Ascuitto and Glendenning[3] is equivalent to the formulation of Penny and Satchler. Limited calculations[3,11] using this method have shown the importance of this approach; however, it is obvious that even in the cases investigated that the computational effort involved must be extremely large, both in time and computer capacity. This is mainly due to the use of standard techniques in order to obtain the solutions of the coupled-channel equations. The standard techniques of numerical integration require a relatively large amount of time and a large amount of storage if one wishes to retain the values of the wave functions. The method proposed here replaces the coupled-channel matrix elements by plane-wave or simple potential matrix elements. While it is not yet apparent that there is a large savings in computation time, there definitely will be a savings in the storage required.

In Sec. 2 the plane-wave expansions of the elastic scattering distorted waves and their Green's operators or propagators are derived. The discrete nature of the summation is justified by the use of an upper integration limit which implies the use of box eigenfunctions and follows the philosophy of "shell model in the continuum by discretization" proposed recently by Lane and Robson.[12] In its lowest form, wherein only one plane wave is used, the method is closely related to the well-known "on-shell" approximation for propagators. The particular case of a long-range distortion is discussed in Sec. 2 because of its importance in nuclear reactions. In Secs. 3–5 expressions are derived for three general types of situations involving "two-body" final states; the problem of extending the theory to the general N-body situation being preserved for future evaluation. The three situations discussed here are: (i) conventional distorted-wave Born approximation (DWBA); (ii) coupled channels with no mass transfer; and (iii) coupled-channels Born approximation. The last is simply a combination of the first two and is included because it represents the currently desired form of direct reaction theory. Possible extensions of the ideology presented here are briefly discussed in the final section.

2. PLANE-WAVE REPRESENTATIONS

In this section we introduce our fundamental hypotheses concerning plane-wave expansions. The first assumption is the existence of a relatively simple operator U which we term the distortion potential. Usually it can be chosen to fit the on-shell elastic scattering, but this is not a necessity, since it could be chosen as a soluble central potential which is used to improve the convergence of the associated perturbation series for the remaining interactions. The corresponding distorted wave is assumed to satisfy the Lippmann-Schwinger equations

$$\chi^{(\pm)}_{\gamma E} = \phi_\gamma + G_0^{\pm}(E)U^{\pm}\chi^{(\pm)}_{\gamma E} = \phi_\gamma + G_\gamma^{\pm}(E)U^{\pm}\phi_\gamma , \qquad (2.1)$$

in which $G_0^{\pm}$, $G_\gamma^{\pm}$ are the outgoing (+) or ingoing (−) propagators at energy E,

$$G_0^{\pm}(E) = \lim_{\epsilon\to 0^+}(E \pm i\epsilon - H_\gamma)^{-1} , \qquad (2.2)$$

$$G_\gamma^{\pm}(E) = \lim_{\epsilon\to 0^+}(E \pm i\epsilon - H_\gamma - U^{\pm})^{-1} , \qquad (2.3)$$

and ϕ_γ is the unperturbed (plane-wave×intrinsic state) eigenstate of the Hamiltonian operator H_γ representing the noninteracting system labeled γ. The distortions $U^{\pm}$ are given by $U^{+} = U$ and $U^{-} = U^{\dagger}$ which allows for U to be non-Hermitian.

The corresponding "elastic scattering" transition matrix elements are given by

$$t_{\gamma'\gamma}(E) = \langle\phi_{\gamma'}|U|\chi^{(+)}_{\gamma E}\rangle = \langle\chi^{(-)}_{\gamma' E}|U|\phi_\gamma\rangle$$
$$= \langle\phi_{\gamma'}|U + UG_\gamma^{+}(E)U|\phi_\gamma\rangle . \qquad (2.4)$$

In the present work the distorted waves $\chi_{\gamma E}$ will only appear in matrix elements where the interaction which gives rise to the transition is of a short-range character, thus, the expansion of the distorted waves in terms of plane waves

$$\langle\vec{r}_\gamma|\chi^{(+)}_{\gamma E}\rangle = \sum_{n=1}^{\infty}\langle\vec{r}_\gamma|\phi_n\rangle D^{(+)}_{n\gamma}(E) \qquad (2.5)$$

will be presumed over a range $0 \le r_\gamma \le R_i$. Since the Fourier-type series is expected to converge, the series is truncated at some value of $n = N$, the chosen value yielding a desired level of accuracy for the ensuing calculation.

This corresponds to the prescription proposed by Lane and Robson[13] for the introduction of an effective unit operator when working in a truncated space of nonorthogonal states. This is given by

$$1 = \sum_{n,m=1}^{\infty}|\phi_n\rangle N_{nm}\langle\phi_m| , \qquad (2.6)$$

where ϕ is a plane wave in the truncated coordinate space. N_{nm} is fixed by requiring that the right-hand side of Eq. (2.6) indeed does give unity within the subspace of terms $n = 1, 2, \ldots, N$.

The most general method we have found for determining the coefficients of an expansion in non-

orthogonal states is to use an auxiliary operator Y, i.e.,

$$\langle\phi_{n'}|\,Y\,|\chi^{+}_{\gamma E}\rangle = \sum_{n=1}^{N}\langle\phi_{n'}|\,Y\,|\phi_n\rangle D^{(+)}_{n\gamma}(E)\,, \tag{2.7}$$

which yields

$$D^{(+)}_{n\gamma}(E) = \sum_{n'=1}^{N}[\underline{Y}^{-1}]_{nn'}\langle\phi_{n'}|\,Y\,|\chi^{+}_{\gamma E}\rangle\,. \tag{2.8}$$

Two choices for the operator Y are appealing. The first is the unit operator in which case it should be remembered that $\langle\phi_n|\,\phi_{n'}\rangle^{R_i}_0$ is not diagonal due to the radial truncation we have imposed. Such a choice of expansion coefficients has been discussed elsewhere[10,12,13] for resonance reaction calculations. The second choice is $Y = U$ which relates the expansion coefficients to the potential matrix itself,

$$D^{(+)}_{n\gamma}(E) = \sum_{n'=1}^{N}[\underline{U}^{-1}]_{nn'}\,t_{n'\gamma}\,. \tag{2.9}$$

Equation (2.8) is the general result for the expansion coefficients $D^{(+)}_{n\gamma}(E)$. In distorted-wave theory one also needs the reverse order due to the presence of time-reversed states, i.e., we need the expansions

$$\langle\chi^{(-)}_{\gamma E}\,| = \sum_{n=1}^{N} D^{(-)}_{\gamma n}(E)\langle\phi_n|\,, \tag{2.10}$$

and we obtain by similar procedures the result,

$$D^{(-)}_{\gamma n}(E) = \sum_{n'=1}^{N}\langle\chi^{(-)}_{\gamma E}\,|\,X\,|\phi_{n'}\rangle[\underline{X}^{-1}]_{n'n}\,, \tag{2.11}$$

wherein X (like Y) is an auxiliary operator. Equations (2.8) and (2.11) represent the basic results of this section and are the basis for the results obtained in Secs. 3 and 5.

In some applications of scattering theory one is also interested in plane-wave expansions for Green's operators. The most general representation is via a double series

$$G^{\pm}_0(E) = \sum_{n,n'=1}^{N}|\phi_n\rangle g^{\pm}_{nn'}(E)\langle\phi_{n'}|\,, \tag{2.12}$$

or

$$G^{\pm}_\gamma(E) = \sum_{n,n'=1}^{N}|\phi_n\rangle G^{\pm}_{nn'}(E)\langle\phi_{n'}|\,, \tag{2.13}$$

in which a coordinate representation is valid only for $r \le R_i$ and $r' \le R_i$.

As in the case of distorted-wave expansions there are two appealing ways to choose the matrix coefficients $g^{\pm}_{nn'}$, $G^{\pm}_{nn'}$. One is to use the operator forms[12,13]

$$G^{+}_0(E) = (E - H_\gamma - L_\gamma)^{-1}, \tag{2.14}$$

$$G^{+}_\gamma(E) = (E - H_\gamma - U_\gamma - L_\gamma)^{-1}, \tag{2.15}$$

and the corresponding matrix representations[12]

$$g^{+}_{nn'}(E) = [(\underline{E} - \underline{H}_\gamma - \underline{L}_\gamma)^{-1}]_{nn'}\,, \tag{2.16}$$

$$G^{+}_{nn'}(E) = [(\underline{E} - \underline{H}_\gamma - \underline{L}_\gamma - \underline{U}_\gamma)^{-1}]_{nn'}\,, \tag{2.17}$$

where L_γ is the Bloch operator. The second is to require that $g_{nn'}$ or $G_{nn'}$ fit the exact potential scattering matrix; i.e., we solve

$$\underline{t} = \underline{U} + \underline{U}\,\underline{g}^{+}\,\underline{t} = \underline{U} + \underline{U}\,\underline{G}^{+}_\gamma\,\underline{U}\,, \tag{2.18}$$

which gives

$$\underline{g}^{+} = \underline{U}^{-1} - \underline{t}^{-1}\,, \tag{2.19}$$

$$\underline{G}^{+}_\gamma = \underline{U}^{-1}\,\underline{t}\,\underline{U}^{-1} - \underline{U}^{-1} = \underline{g}^{+}(\underline{t}\,\underline{U}^{-1})\,, \tag{2.20}$$

as well as the auxiliary equations

$$(\underline{1} - \underline{g}^{+}\underline{U})^{-1} = \underline{1} + \underline{g}^{+}\,\underline{t} = \underline{U}^{-1}\,\underline{t}\,, \tag{2.21}$$

$$(\underline{1} + \underline{G}^{+}_\gamma\underline{U}) = \underline{U}^{-1}\,\underline{t}\,, \tag{2.22}$$

and their reversed forms

$$(\underline{1} - \underline{U}\,\underline{g}^{+})^{-1} = \underline{1} + \underline{t}\,\underline{g}^{+} = \underline{t}\,\underline{U}^{-1}\,, \tag{2.23}$$

$$(\underline{1} + \underline{U}\,\underline{G}^{+}_\gamma) = \underline{t}\,\underline{U}^{-1}\,. \tag{2.24}$$

In this method of determining $\underline{g}^{+}$ or $\underline{G}^{+}_\gamma$ one needs to know $\underline{t}$ and $\underline{U}^{-1}$ by alternative procedures, a result which is clear from Eqs. (2.19) and (2.20) above. Clearly Eqs. (2.9) and (2.22) are consistent, as expected, for if we insert Eq. (2.1) into Eq.(2.5) and make use of Eq. (2.22) we obtain Eq. (2.9). Similarly Eqs. (2.16) and (2.17) can be shown to be consistent with Eq. (2.8) when $Y = 1$ owing to the use of the same unit operator as given in Eq. (2.6) in each case.

The theory of Sec. 4 as well as some of the results in Sec. 5 are based on the result for $\underline{g}^{+}$ given by Eq. (2.19) above. Before turning to specific applications it is appropriate to discuss two general features of the present ideology.

First there is the problem of an upper limit R_i (which we assume is the range of integration of the direct reaction matrix element) in the case that U includes a long-range Coulomb interaction. In this case U is not negligible for $r > R_i$ and our interpretation of the matrix elements $U_{nn'}$ and $t_{n\gamma}$ needs to be specified more precisely. The simplest method of ensuring that the foregoing holds is to require that the series coefficients are chosen to represent the t matrix only over the region of interest, i.e., we separate the t-matrix integrals into two parts,

$$t_{\gamma'\gamma} = \langle\phi_{\gamma'}|\,U\,|\chi^{(+)}_{\gamma E}\rangle^{R_i}_0 + \langle\phi_{\gamma'}|\,U\,|\chi^{(+)}_{\gamma E}\rangle^{\infty}_{R_i}$$

$$= t^{R_i}_{\gamma'\gamma} + t^{\infty}_{\gamma'\gamma}\,,$$

and choose $g_{nn'}$ to fit only $t^{R_i}_{\gamma'\gamma}$. The results ob-

tained are then valid provided $t^{R_i}_{\gamma'\gamma}$ (and similarly $U^{R_i}_{\gamma'\gamma}$) are used in place of $t_{\gamma'\gamma}$ (and $U_{\gamma'\gamma}$). Note that $\chi^{(+)}_{\gamma E}$ is still the solution of U over all space and the introduction of the limit R_i is used only as an artifice to allow faster convergence of the series for the problems it is to be used to solve.

Second, there is the question of representing the distorted waves $\chi^{(+)}_{\gamma E}$ which appear in the $D^{(+)}$ coefficients. One has to choose between partial-wave series and angle representations. In the latter case the eigenstates ϕ_n have a definite wave number k_n and a definite direction $\hat{k}_n$ (or perhaps more conveniently a sum over definite directions chosen to yield at least partial diagonalization of the matrix Y). The simplest and probably the best choice in that case is the set of eigenstates of a cube of side $2R_i$ because for a central operator $U(r)$ or the effective unit operator one can block diagonalize according to reflection symmetries about the three Cartesian axes.

In the case of partial-wave methods the representations involve only a sum over angular momentum $(J\alpha)$ states with corresponding radial coefficients $\chi^{(+)}_{\gamma E, J\alpha}(r)$ (where α represents other quantum numbers necessary to obtain a complete set of angular momenta). Each partial-wave state $\chi^{(+)}_{\gamma E, J\alpha}$ is to be expanded into noninteracting (spherical Bessel) radial states $j_l(k_n r)$ so that the sum over n involves only a sum over the values of k_n. As an example we consider a simple potential $U(r)$ where the only relevant quantum number is the orbital quantum number l. The distorted partial wave is then simply expanded according to the foregoing as

$$\chi^{(+)}_{\gamma E, l}(k_\gamma, r) = \sum_n j_l(k_n r) D^{(+)}_{nl,\gamma l}(E) , \tag{2.25}$$

where

$$D^{(+)}_{nl,\gamma l}(E) = \sum_{n'} [(\underline{U}^{R_i}_l)^{-1}]_{nn'} t^{R_i}_{n'l,\gamma l}(E) , \tag{2.26}$$

with

$$t^{R_i}_{n'l,\gamma l}(E) = \int_0^{R_i} dr\, r^2 j_l(k_n r) U(r) \chi^{(+)}_{\gamma E, l}(k_\gamma, r) , \tag{2.27}$$

and

$$[\underline{U}^{R_i}_l]_{nn'} = \int_0^{R_i} dr\, r^2 j_l(k_n r) U(r) j_l(k_n r) . \tag{2.28}$$

In practice it will usually be necessary to convert the above expansion back to angle representation by using

$$i^l j_l(k_n r) Y_{lm}(\hat{r}) = (4\pi)^{-1} \int d\hat{k} Y_{lm}(\hat{k}) e^{i\vec{k}_n \cdot \vec{r}} , \tag{2.29}$$

with $\vec{k}_n = k_n \hat{k}$, so that the full expansion becomes

$$i^l \chi^{(+)}_{\gamma E, l}(k_\gamma, r) Y_{lm}(\hat{r}) = \sum_n \int d\hat{k} e^{i\vec{k}_n \cdot \vec{r}} D^{(+)}_{nl,\gamma l}(E, \hat{k}) , \tag{2.30}$$

with the coefficients being given by

$$D^{(+)}_{nl,\gamma l}(E, \hat{k}) = (4\pi)^{-1} Y_{lm}(\hat{k}) D^{(+)}_{nl,\gamma l}(E) . \tag{2.31}$$

The disadvantage here is the occurrence of a continuous sum over $\hat{k}$ rather than a finite sum over $\hat{k}_n$, but this is at least partially compensated for by the simplification of the matrix inversion for the D matrix which is diagonal in the angular momentum. Which method will prove most effective depends primarily on the energy of the distorted waves and remains to be investigated in numerical calculations.

3. DISTORTED-WAVE MATRIX ELEMENTS

From this point on we shall simplify the notation by adopting an obvious matrix notation. The conventional distorted-wave matrix elements have the form

$$T_{\beta\alpha} = \langle \chi^{(-)}_{\beta E} | \hat{V} | \chi^{(+)}_{\alpha E} \rangle , \tag{3.1}$$

where $\chi^{(+)}_{\alpha E}$, $\chi^{(-)}_{\beta E}$ are the solutions of the potentials U_α, $U^\dagger_\beta$, respectively. Substituting the expansions (2.5) and (2.10) given in Sec. 2 for $\chi^{(+)}_{\alpha E}$, $\chi^{(-)}_{\beta E}$ we immediately obtain the result

$$\underline{T} = \underline{D}^{(-)}_{[\beta]} \underline{\hat{V}}_{[\beta][\alpha]} \underline{D}^{(+)}_{[\alpha]} , \tag{3.2}$$

in which $[\beta]$ and $[\alpha]$ signify the source of the representation. In practical calculations only the physically interesting matrix elements of $\underline{T}$ are needed, e.g.

$$T_{\beta\alpha} = \sum_{\beta'} \sum_{\alpha'} D^{(-)}_{\beta\beta'}(E = E_\beta) \hat{V}_{\beta'\alpha'} D^{(+)}_{\alpha'\alpha}(E = E_\alpha) , \tag{3.3}$$

which indicates the usual on-energy-shell relationship. The conventional plane-wave theory is obtained by replacing the distortion matrices $\underline{D}$ by unit matrices which is a good approximation only when (see Eq. 2.9)

$$\underline{t} \simeq \underline{U} \quad \text{or} \quad \chi_{\gamma E} \simeq \phi_\gamma ,$$

i.e., when the plane-wave Born approximation is valid for potential scattering. For most applications such a relation is not valid and a proper solution in terms of the above is necessary.

Equation (3.3) can also be written in terms of the t matrices by making use of Eqs. (2.9) and (2.11). We obtain

$$T_{\beta\alpha} = \sum_{\beta'\alpha'} \sum_{n'n} t_{\beta n'} [\underline{U}_\beta{}^{-1}]_{n'\beta'} \hat{V}_{\beta'\alpha'} [\underline{U}_\alpha{}^{-1}]_{\alpha' n} t_{n\alpha} . \tag{3.4}$$

Using Eq. (2.4) this can then be written as

$$T_{\beta\alpha}=\sum_{\beta'\alpha'}\sum_{n'n}\langle\chi^{(-)}_{\beta E}|U_\beta|\phi_{n'}\rangle[\underline{U}_\beta^{-1}]_{n'\beta'}\times\hat{V}_{\beta'\alpha'}[\underline{U}_\alpha^{-1}]_{\alpha'n}\langle\phi_n|U_\alpha|\chi^{(+)}_{\alpha E}\rangle\,. \tag{3.5}$$

Of course in this equation direct evaluation of the matrix elements of the distorting potentials is needed in terms of the distorted waves.

In the angle representation $T_{\beta\alpha}$ becomes simply a double sum over plane-wave matrix elements $\hat{V}_{\beta'\alpha'}$, each of which we assume will be much easier to evaluate than the original distorted-wave integrals. It is this factor which hopefully yields the element of speed for the present method. In the case of partial-wave representations the simplest example is again afforded by central potentials where the expansions

$$\chi^{(+)}_{\gamma E}=4\pi\sum_{lm}i^l\chi^{(+)}_{\gamma E,l}(k_\gamma,r_\gamma)Y_{lm}(\hat{r}_\gamma)Y^*_{lm}(\hat{k}_\gamma)\xi_\gamma \tag{3.6}$$

are appropriate. The state ξ_γ is representative of the product of intrinsic states for the colliding pair in channel γ. The DWBA integral then becomes

$$T_{\beta\alpha}=(4\pi)^2\sum_{l_\beta l_\alpha}\sum_{m_\beta m_\alpha}Y_{l_\beta m_\beta}(\hat{k}_\beta)Y^*_{l_\alpha m_\alpha}(\hat{k}_\alpha)\times\iint d\vec{r}_\beta d\vec{r}_\alpha[i^{-l_\beta}\chi^{(+)}_{\beta E,l_\beta}(k_\beta,r_\beta)Y^*_{l_\beta m_\beta}(\hat{r}_\beta)]\times\langle\xi_\beta|\hat{V}|\xi_\alpha\rangle[i^{l_\alpha}\chi^{(+)}_{\alpha E,l_\alpha}(k_\alpha,r_\alpha)Y_{l_\alpha m_\alpha}(\hat{r}_\alpha)]\,. \tag{3.7}$$

The expansions given by Eqs. (2.24) and (2.25) are now used to give

$$T_{\beta\alpha}=\sum_{l_\beta l_\alpha}\sum_{n_\beta n_\alpha}(2l_\beta+1)(2l_\alpha+1)D^{(+)}_{n_\beta l_\beta,\beta l_\beta}(E)D^{(+)}_{n_\alpha l_\alpha,\alpha l_\alpha}(E)\times\int d\hat{k}'\int d\hat{k}P_{l_\beta}(\hat{k}_\beta\cdot\hat{k}')\hat{V}_{n_\beta n_\alpha}(\hat{k}',\hat{k})P_{l_\alpha}(\hat{k}\cdot\hat{k}_\alpha)\,, \tag{3.8}$$

with

$$\hat{V}_{n_\beta n_\alpha}(\hat{k}',\hat{k})=\int e^{i\vec{k}_{n_\beta}\cdot\vec{r}_\beta}\xi^*_\beta\hat{V}\xi_\alpha e^{i\vec{k}_{n_\alpha}\cdot\vec{r}_\alpha}d\tau\,, \tag{3.9}$$

being plane-wave Born matrix elements with initial- and final-wave numbers $\vec{k}_{n_\alpha}=k_{n_\alpha}\hat{k}$ and $\vec{k}_{n_\beta}=k_{n_\beta}\hat{k}'$. As expected $T_{\beta\alpha}$ is represented by wave number sums and angular integrations over plane-wave matrix elements. In this case the usefulness of the expansion depends on the ease with which the double Legendre transformations can be calculated as well as on the ease of performing plane-wave matrix element evaluations. The distortion factors $D^{(+)}_{n_\beta l_\beta,\beta l_\beta}(E)$, $D^{(+)}_{n_\alpha l_\alpha,\alpha l_\alpha}(E)$ are given by Eq. (2.26) in Sec. 2 and are relatively easy to calculate even by conventional techniques.

It is useful to gain some insight into the present method by considering just one term in the sum over each of n_β, n_α, namely, the "on-shell" choice $k_{n_\beta}=k_\beta$. In this case the distorted-wave partial series is evaluated by replacing each distorted radial function by the product of a distortion factor and a plane radial function

$$\chi^{(+)}_{\gamma E,l}(k,r)\to D^{(+)}_{kl,kl}\,j_l(kr)\,, \tag{3.10}$$

where

$$D^{(+)}_{kl,kl}=\frac{\int_0^{R_i}dr\,r^2j_l(kr)U(r)\chi^{(+)}_l(k,r)}{\int_0^{R_i}dr\,r^2j_l^2(kr)U(r)} \tag{3.11}$$

is the distortion factor. In this lowest form we see that the difference between a distorted wave and a plane wave is "measured" by the projected space $j_l(kr)U(r)$. The actual "measure" in the DWBA integral is in terms of $\hat{V}$ multiplied by radial functions appropriate to the other channel involved. Only if these two "measures" are similar can we expect the "on-shell" form of the theory to work successfully. In general this is not the case, but explicit calculations[14] show that for transfer reactions and high partial waves (where only Coulomb distortions matter) the on-shell approximation is reasonably accurate. The on-shell character of Coulomb scattering is a direct consequence of its long range.

Although the approximation given by Eq. (3.10) is usually a decided improvement over simple plane-wave Born approximation one finds it is a poor approximation for a few special partial waves – namely, those for which the denominator in Eq. (3.11) for $D^{(+)}_{kl,kl}$ almost vanishes. This occurs when the Coulomb potential almost cancels the nuclear potential due to the positive definite nature of $j_l^2(kr)$ everywhere. Such a cancellation does not in general occur in the matrix elements of $\hat{V}$, at least not for the same value of l, so that the projected space $j_l(kr)U(r)$ is a spurious measure and such partial waves are poorly represented. We are therefore, in general, forced to use the full matrix form of the theory.

The choice $Y=1$ instead of $Y=U(r)$ would apparently not meet with such catastrophes, since we obtain

$$D^{(+)}_{kl,kl}=\frac{\int_0^{R_i}dr\,r^2j_l(kr)\chi^{(+)}_l(k,r)}{\int_0^{R_i}dr\,r^2j_l^2(kr)}\,, \tag{3.12}$$

and the denominator has a positive definite integrand. Numerical calculations using Eq. (3.12) show that the use of a single plane wave is still not a sufficiently accurate representation of the distorted wave for the low partial waves.

The alternative choice of $X=1=Y$ instead of

$X = U_\beta$, $Y = U_\alpha$ yields the simpler form for $T_{\beta\alpha}$,

$$T_{\beta\alpha} = \sum_{\beta'\alpha'} \sum_{nn'} \langle \chi_\beta^{(-)} | \phi_n \rangle [\underline{1}^{-1}]_{n'\beta'} \hat{V}_{\beta'\alpha'} [\underline{1}^{-1}]_{\alpha' n} \langle \phi_n | \chi_\alpha^{(+)} \rangle . \tag{3.13}$$

Calculations[15] have been made using both Eqs. (3.5) and (3.13). While both methods converged reasonably fast the case with $X = 1 = Y$ was found to be superior due to the easier computation and the better rate of convergence of the series used. More general statements concerning the relative merits of Eqs. (3.5) and (3.13) must await more extensive testing via numerical calculations.

4. COUPLED-CHANNELS THEORY

In this section we are concerned with solving the coupled-channel problem by the use of the expansions introduced in Sec. 2. This method differs markedly from alternative methods based on numerically integrating the coupled Schrödinger differential equations.[16] We restrict our treatment to the situation wherein no mass transfer is concerned, so that the Hamiltonian describing the unperturbed channels in initial and final states is the same (i.e., we allow for direct inelastic scattering and in some situations charge-exchange scattering). From the integral equations for a scattering system where all rearrangement channels are elimintated the transition matrix relating all the retained channels is

$$\mathcal{T}_{\gamma'\gamma} = \langle \phi_{\gamma'} | V_{\gamma'} | \Psi_\gamma^{(+)} \rangle , \tag{4.1}$$

with

$$\Psi_\gamma^{(\pm)} = \phi_\gamma + G_\gamma^\pm V_\gamma \phi_\gamma = \phi_\gamma + G_0^\pm V_\gamma^\pm \Psi^{(\pm)} \tag{4.2}$$

and

$$G_\gamma^{(\pm)} = \lim_{\epsilon \to 0^+} (E \pm i\epsilon - H_\gamma - V_\gamma^\pm)^{-1} . \tag{4.3}$$

As in the case of a simple potential U we also allow for non-Hermitian interactions $V^+ = V$, $V^- = V^\dagger$ which may arise due to the elimination of rearrangement channels and truncation of the γ channels to some finite number.

Using the Lippmann-Schwinger equation for $\Psi_\gamma^{(+)}$ in the defining equation for $\mathcal{T}_{\gamma'\gamma}$ with the expansion for G_0^+ given by Eq. (2.12) we immediately obtain the matrix relation

$$\underline{\mathcal{T}} = (\underline{1} - \underline{V}\,\underline{g}^+)^{-1} \underline{V} , \tag{4.4}$$

where

$$\underline{g}^+ = \underline{U}^{-1} - \underline{t}^{-1} \quad \text{or} \quad (\underline{E} - \underline{H}_\gamma - \underline{L}_\gamma)^{-1} ,$$

corresponding to Eqs. (2.19) and (2.16), respectively. The expansion over n in the representation of G_0^+ now includes all excited states of the colliding pairs which are to be included in the coupled-channels calculation as well as a sum over all the momentum eigenstates attached to each excited state.

A more obvious form of the theory results with $\underline{g}^+ = \underline{U}^{-1} - \underline{t}^{-1}$ by writing $\underline{V} = (\underline{V} - \underline{U}) + \underline{U}$ and the use of simple matrix manipulation which yields

$$\begin{aligned} \underline{\mathcal{T}} &= \underline{t} + \underline{D}^{(-)} [\underline{1} - (\underline{V} - \underline{U}) \underline{g}^+ \underline{D}^{(-)}]^{-1} (\underline{V} - \underline{U}) \underline{D}^{(+)} \\ &= \underline{t} + \underline{D}^{(-)} [\underline{1} - (\underline{V} - \underline{U}) \underline{G}_\gamma^+]^{-1} (\underline{V} - \underline{U}) \underline{D}^{(+)} , \end{aligned} \tag{4.5}$$

where $\underline{D}^{(+)}$ and $\underline{D}^{(-)}$ are defined in Sec. 2. An obvious choice of $\underline{U}$ is the diagonal part of V (in channel space) so that the first term $t_{\gamma'\gamma}$ is diagonal in γ and the second term to lowest order in $(\underline{V} - \underline{U})$ is off-diagonal and given by DWBA as represented by $\underline{V} - U \to \hat{V}$ in Sec. 3.

The nice feature of the problem is that all the matrix elements are either plane-wave type or simple potential matrix elements. The only difficulty which could arise is the size of the matrices which instead of being $N \times N$ (single channel) become $(nN) \times (nN)$, where n is the number of channels. The speed of the problem depends critically then on the number N which in turn depends on the particular operator V being solved. Numerical testing is necessary for typical situations in order to assess the practical merit of this technique relative to conventional methods.

In the case of collective models being used for $H_\gamma + V_\gamma$ such as rotational or vibrational models the operators $V_\gamma - U_\gamma$ and U_γ can be chosen as the nonspherical and spherical components (rotations) or the one-phonon creation and zero-phonon component (vibrations), respectively. In both cases the operator $V_\gamma - U_\gamma$ has a radial form which is typically a differential of the radial part of U_γ. This similarity in form should lead to a rapid convergence for the N series in each channel because U_γ provides a good "measure" of the residual interaction $V_\gamma - U_\gamma$ in this situation.

5. COUPLED-CHANNELS BORN APPROXIMATION

The usual form of DWBA still applies in this case if we substitute V for U in the distorted waves and regard V as the operator used in the immediately preceding section. The matrix elements to be evaluated are now

$$T_{\beta\alpha} = \langle \Psi_\beta^{(-)} | \hat{V} | \Psi_\alpha^{(+)} \rangle , \tag{5.1}$$

where $\Psi^{(-)}$ and $\Psi^{(+)}$ are given by Eq. (4.2) and which on using similar arguments to the DWBA result [Eq. (3.2)] leads to a formally equivalent

answer; i.e.,

$$\underline{T} = \underline{D}^{(-)}_{[\beta]} \hat{\underline{V}}_{[\beta][\alpha]} \underline{D}^{(+)}_{[\alpha]}$$

or (5.2)

$$\underline{T} = \underline{\mathcal{T}}_{[\beta]} \underline{V}^{-1}_{[\beta]} \hat{\underline{V}}_{[\beta][\alpha]} \underline{V}^{-1}_{[\alpha]} \underline{\mathcal{T}}_{[\alpha]}$$

if we put $X = V_\beta$, $Y = V_\alpha$, except that the matrices in channel classes $[\alpha]$ and $[\beta]$ are now $(n_\gamma N_\gamma \times n_\gamma N_\gamma)$ with $\gamma = \alpha, \beta$ and contain plane waves relative to each excited state for the coupled channels appropriate to either the initial or final states. The matrices $\mathcal{T}_{[\beta]}$, $\mathcal{T}_{[\alpha]}$ have matrix elements $\mathcal{T}_{\beta\beta'}$, $\mathcal{T}_{\alpha\alpha'}$ and are each given by Eq. (4.4) for $\mathcal{T}_{\gamma'\gamma}$ in terms of their appropriate distortion operator U_γ. Again the problem is reduced to matrix algebra with matrix elements which are plane wave or potential matrix elements of our simple U operators. This is most clearly seen by substituting for $\mathcal{T}$ from Eq. (4.4) into Eq. (5.2) which yields

$$\underline{T} = (\underline{1} - \underline{V}_{[\beta]} \underline{g}^{+}_{[\beta]})^{-1} \hat{\underline{V}}_{[\beta][\alpha]} (\underline{1} - \underline{g}^{+}_{[\alpha]} \underline{V}_{[\alpha]})^{-1} . \quad (5.3)$$

Equations (5.2) and (5.3) for $\underline{T}$ represent the future desired level of direct reaction theory wherein multistep processes are allowed for and finite range for $\hat{V}$ is facilitated by the plane-wave character of the equation.

In the situation where channel coupling $V - U$ is weak the approximation of keeping $V - U$ only to first order will considerably alleviate the matrix inversions, i.e.,

$$\begin{aligned} \underline{T} \simeq\ & \underline{D}^{(-)}_{[\beta]} \hat{\underline{V}}_{[\beta][\alpha]} \underline{D}^{(+)}_{[\alpha]} \\ & + \underline{D}^{(-)}_{[\beta]} \hat{\underline{V}}_{[\beta][\alpha]} \underline{D}^{(+)}_{[\alpha]} \underline{g}^{+}_{[\alpha]} (\underline{V}_{[\alpha]} - \underline{U}_{[\alpha]}) D^{(+)}_{[\alpha]} \\ & + \underline{D}^{(-)}_{[\beta]} (\underline{V}_{[\beta]} - \underline{U}_{[\beta]}) \underline{g}^{+}_{[\beta]} \underline{D}^{(-)}_{[\beta]} \hat{\underline{V}}_{[\beta][\alpha]} \underline{D}^{(+)}_{[\alpha]} , \end{aligned} \quad (5.4)$$

or more compactly

$$\underline{T} = \underline{T}^{DW}_{[\beta][\alpha]} + \underline{T}^{DW}_{[\beta][\alpha]} \underline{f}_{[\alpha]} \underline{T}^{DW}_{[\alpha]} + \underline{T}^{DW}_{[\beta]} \underline{f}_{[\beta]} \underline{T}^{DW}_{[\beta][\alpha]} , \quad (5.5)$$

where the "distorted-wave elements" T^{DW} are given by

$$\underline{T}^{DW} = \underline{D}^{(-)} \underline{T}^{PW} \underline{D}^{(+)} , \quad (5.6)$$

and

$$\underline{f}_{[\gamma]} = \underline{t}^{-1}_{[\gamma]} (\underline{1} - \underline{U}_{[\gamma]} \underline{t}^{-1}_{[\gamma]}) = \underline{U}_\gamma \underline{t}_\gamma^{-1} \underline{G}^{+}_\gamma \underline{t}_\gamma^{-1} \underline{U}_\gamma , \quad (5.7)$$

and $\underline{T}^{PW}$ is the plane-wave form of $\underline{V} - \underline{U}$ or $\hat{\underline{V}}$ wherever appropriate. The last two terms of Eq. (5.5) represent the two-step processes which involve DWBA for each step of the process, and are connected by the appropriate matrices $\underline{f}$. The only inversion is for $\underline{t}$ which is block diagonal in channel space if U_γ is chosen to be diagonal. Equation (5.5) with "on-shell approximations" for $T^{DW}_{[\gamma]}$ and a simple number (proportional to the Cauchy value of $-i\pi$) for $\underline{f}_{[\gamma]}$ have recently been used by Bindal and Koshel[14] with some success for transfer reactions. Hopefully the generality of the present approach will prove useful in assessing the merits of such approximations. The question of whether to use angle or partial-wave representations also remains to be assessed in these more complicated situations.

6. CONCLUSIONS

The present approach represents an attempt to return to plane waves by the use of plane-wave expansions for wave functions or propagators involved. The use of momentum eigenstates is not in itself very novel. The power of the present method (when $Y = U$) lies in the use of integral equations for the potential scattering operator which allows us to fix the coefficients in the propagator expansions. In the case of DWBA evaluation this has the merit of exactly evaluating the operator UG_0U and approximately evaluating the operator UG_0V within the $N \times N$ subspace. The similarity of these operators in some applications leads us to hope that this method of choosing the expansion coefficients will lead to the optimum convergence rate while at the same time it allows us to retain the DWBA philosophy of fitting elastic scattering amplitudes. Since for $N \to \infty$ the method is exact, all we need is good convergence for the method to be useful for small N. Similar arguments hold for coupled-channels theory with $UG_0^+\hat{V}$ being replaced by $UG_0^+(V - U)$ and $(V - U)G_0^+(V - U)$. Only computer codes using the above methods will ascertain whether the plane-wave approach is superior to current methods. Such codes for DWBA and coupled channels are being developed.

The extension of the present method to other problems appears to be interesting in various cases. In particular reactions involving "three-body channels" can perhaps be treated in the present formalism by use of the method of Vincent and Fortune[17] if the reaction is sequential[18] and the initial stage is direct. The only change is that the state ξ_β will be "resonant" rather than bound. The method appears likely to be quite useful for almost any reaction at relativistic energies since the plane-wave theory itself becomes relatively accurate. The repeated use of Eq. (2.12) in Faddeev's equations[19] also leads to a matrix representation of the three-body problem although the size of the matrices involved will undoubtedly become more formidable than here. The numerical evaluation of the approach and its application to the phenomena indicated herein will be reported

on in separate communications. Since the methodology presented here has been in part justified by a calculation, the extension of the formalism to more complicated problems is of importance. The applications presented in this paper are aimed at nuclear physics, but there seems to be no obvious reason why the theory should not prove equally useful in atomic or molecular collisions.

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K-Electron-Capture-to-Positron-Emission Ratio in the Decays of ^{15}O and ^{19}Ne

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The K/β^+ ratio in the decays of ^{19}Ne and ^{15}O have been measured as $(9.6 \pm 0.3) \times 10^{-4}$ and $(10.7 \pm 0.6) \times 10^{-4}$, respectively. A gas-flow proportional counter, operating in anticoincidence with the surrounding plastic scintillator, was used. Theoretical K/β^+ ratios for ^{19}Ne and ^{15}O were computed, using exchange-overlap corrections calculated by Vatai and, separately, exchange corrections extrapolated from the results of Bahcall for $14 \leq Z \leq 37$. The experimental results were found to be in better agreement with Vatai's calculations.

I. INTRODUCTION

Electron exchange and overlap corrections, Bk, for K capture to positron emission ratios have been calculated by Bahcall[1] for nuclides in the range $14 \leq Z \leq 37$ and by Vatai[2] for nuclides in the range $13 \leq Z \leq 37$. The effect of the exchange-overlap correction is to reduce the theoretical value for K/β^+ ratios by a factor of $(1 - Bk)$. The calculations of Bahcall and Vatai differ by about 7% in the value of Bk at $Z = 14$ and the discrepancy increases on extrapolation into the region $Z \leq 14$, as shown in Table I for $Z = 8$ and $Z = 10$.

A recent measurement[3] of the K/β^+ ratio in the decay of ^{30}P is in agreement with Vatai's calculations. Ledingham *et al.*[3] also include a summary of other experimental measurements of K/β^+ ratios for $Z \leq 15$, including the results given in this paper. We describe below measurements of K/β^+ ratios in the decays of ^{19}Ne and ^{15}O.