Inverse-scattering theory at a fixed energy for the Klein-Gordon equation

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(Received 23 February 1998)

The inverse-scattering theory at a fixed energy for the scattering of a particle by a potential in the Schrödinger equation formulated by Alam and Malik, which is based on the earlier work of Hooshyar and Razavy, is extended, in this paper, to the scattering of spinless particles at relativistic energies governed by the Klein-Gordon equation. The differential equation is replaced by a set of difference equations. This reduces the inverse-scattering problem to solving a continued fraction equation. The solution provides the values of the potential at a number of points which are equal to (one plus the number of partial waves). The theory is tested for three widely different complex potentials, one of which is relevant to pion-nucleus scattering. The points of the potentials determined from the inverse-scattering formalism are in accord with the actual ones in all three cases. Since the Klein-Gordon equation is effectively a Schrödinger equation with an energy-dependent potential, the method may, in the appropriate cases, be suitable for the latter case. [S0556-2813(99)05502-8]

PACS number(s): 25.80.Dj, 11.80.−m, 24.10.−i

I. INTRODUCTION

In this paper, we present an inverse-scattering theory at a fixed energy for the scattering problems governed by the Klein-Gordon equation. The method presented here is suitable for the scattering of spinless particles such as pions by spinless targets such as even-even nuclei at relativistic energies. The method developed here is based on the Hooshyar-Razavy method formulated in connection with the inverse-scattering problem in geophysics [1] and applied to problems of neutron scattering by nuclei [2,3] described by the Schrödinger equation. The Hooshyar-Razavy method has been further developed by Alam and Malik [4] to incorporate Coulomb interaction for scattering processes described by the Schrödinger equation. They have successfully applied it to determine the potential between two alpha particles [5] at low energies and between ^{12}C and ^{12}C nuclei at energies just above the Coulomb barrier [6]. It is established as an important method to determine the general nature of the potential between two particles in case information on the phase shifts for a large number of partial waves is available. In view of this success, it is important to extend the method to scattering problems governed by the Klein-Gordon equation which describes the motion of spinless particles at relativistic energies.

The investigation on the inverse scattering problem described by the Klein-Gordon equation is not new in physics. In 1954 Corinaldesi [7] developed a theory to construct the scattering potential from *s*-wave phase shift and binding energy of two-particle-system using the Klein-Gordon equation. In a different approach, Desgasperis [8] has carried out a study to derive relationships among functions characterizing the relativistic scattering process and the parameters governing the interaction. Weis and Scharf [9] have extended this work to develop an inverse-scattering theory pertinent to the Klein-Gordon equation. However, all these approaches are *not at a fixed energy* and require a knowledge of phase shifts as a function of energy, thereby severely restricting their applicability to determine the potential because phase shifts in a large domain of energy is not easily available. On the other hand, the inverse-scattering theory presented in Sec. II of this paper requires the knowledge of phase shifts for all partial waves only at a fixed energy which is usually easier to obtain. In Sec. III, we present applications of the theory to three different complex potentials which may be pertinent for charge less pion scattering by spinless nuclei. The theory developed in Sec. II is valid for Coulomb interaction which is, however, not included in the application.

II. THE THEORY

One may derive the Klein-Gordon equation for the scattering of a spinless particle starting from the following relativistic energy-momentum relation:

$$E^2-\vec{P}^{\,2}c^2=m^2c^4, \tag{1}$$

where E, m, and $\vec{P}$ are the total energy, rest mass, and momentum of the particle, respectively. One may incorporate the electromagnetic scalar, and vector, potentials $\phi(r)$ and $\vec{A}(\mathbf{r})$, respectively, as external potential in the treatment by noting the following relation:

$$(E-e\phi)^2-\left(\vec{P}-\frac{e}{c}\vec{A}\right)^2c^2=m^2c^4. \tag{2}$$

For a purely electrostatic potential, $\vec{A}=0$, one obtains

$$(E-e\phi)^2-\vec{P}^2c^2=m^2c^4. \tag{3}$$

$e\phi(r)$ being a scalar potential, plays the role of an external spherical symmetric time independent potential, $V(r)$. Thus,

$$e\phi=V(r). \tag{4}$$

To obtain the Klein-Gordon equation, $\vec{P}$ may be represented by the familiar differential operator,

$$\vec{P}=-i\hbar\nabla \tag{5}$$

and Eq. (3) is then an operator equation acting on the wave function $\Phi(\vec{r})$. Thus, the Klein-Gordon equation in the presence of an external spherical symmetric potential is the following:

$$(-\hbar^2c^2\nabla^2+m^2c^4)\Phi(\vec{r})=[E-V(r)]^2\Phi(\vec{r}). \tag{6}$$

Since the potential is spherically symmetric, the radial part of $\Phi(\vec{r})$, $R_{nl}(r)$, satisfies the following equation:

$$\left[-\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)+\frac{l(l+1)}{r^2}\right]R_{nl}(r) =\left[\frac{\{E-V(r)\}^2-m^2c^4}{\hbar^2c^2}\right]R_{nl}(r), \tag{7}$$

which includes the centrifugal potential term, $l(l+1)/r^2$. Writing $R_{nl}=\psi_{nl}(r)/r$, one may obtain the equation for the r times the radial wave function, $\psi_{nl}(r)$,

$$\left[\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}\right]\psi_l(r)=-\left[\frac{\{E-V(r)\}^2-m^2c^4}{\hbar^2c^2}\right]\psi_l(r), \tag{8}$$

where we have dropped the subscript n for convenience.

The scalar potential, $V(r)$, may consist of an external potential falling off faster than Coulomb potential, $V_N(r)$ and a Coulomb potential $V_c(r)$

$$V(r)=V_N(r)+V_c(r), \tag{9}$$

where the Coulomb part, $V_c(r)$, is taken to be that of a uniformly charged spheres and is given by

$$V_c(r)=\begin{cases}(Z_1Z_2e^2/2R_c)(3-r^2/R_c^2) & \text{for } r\leqslant R_c\\ Z_1Z_2e^2/r & \text{for } r>R_c.\end{cases} \tag{10}$$

In Eqs. (10) and (11), Z_1 and Z_2 are the atomic numbers of the two colliding nuclei and R_c is the Coulomb radius. Equation (8) may be rewritten as

$$\left[\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}\right]\psi_l(r)=-\left[\frac{[E-V(r)]^2-m^2c^4}{\hbar^2c^4}\right]\psi_l(r) =-\left[\frac{E^2-m^2c^2}{\hbar^2c^2}\right]\psi_l(r)+\frac{2E}{\hbar^2c^2} \times\left[V(r)-\frac{V^2(r)}{2E}\right]\psi_l(r). \tag{11}$$

Defining

$$K^2=\frac{E^2-m^2c^4}{\hbar^2c^2} \tag{12}$$

and

$$U(r)=\frac{2E}{\hbar^2c^2}\left[V(r)-\frac{V^2(r)}{2E}\right]=\frac{2E}{\hbar^2c^2}V_{\text{eff}}(r), \tag{13}$$

where

$$V_{\text{eff}}(R)=V(r)-\frac{V^2(r)}{2E} \tag{14}$$

one obtains from Eq. (12) the following:

$$\left[\frac{d^2}{dr^2}+K^2-U(r)-\frac{l(l+1)}{r^2}\right]\psi_l(r)=0. \tag{15}$$

This equation has the same mathematical form as the one obtained by Alam and Malik [4] for the nonrelativistic case. As a result, we proceed to solve the inverse problem in a similar fashion. For potentials less singular than $1/r^2$ at the origin, $\psi_l(r)$ satisfies the following boundary condition:

$$\lim_{r\to 0}\psi_l(r)\cong(kr)^{l+1}. \tag{16}$$

Introducing

$$\varphi_l(r)=(Kr)^{-(l+1)}\psi_l(r), \tag{17}$$

which satisfies the boundary condition $\varphi(0)=1$, one gets

$$\left[\frac{d^2}{dr^2}+\frac{2(l+1)}{r}\frac{d}{dr}+K^2-U(r)\right]\varphi_l(r)=0. \tag{18}$$

The differential operators may now be replaced by the following central differences:

$$\frac{d}{dr}\varphi_l\to\frac{\varphi_{n+1}-\varphi_{n-1}}{2\Delta}, \tag{19}$$

$$\frac{d^2}{dr^2}\varphi_l\to\frac{\varphi_{n+1}+\varphi_{n-1}-2\varphi_n}{\Delta^2}, \tag{20}$$

where the dependence of φ_n's on l is suppressed. By dividing the non-Coulomb potential range, R to N equal parts, each of length Δ, i.e., $R=N\Delta$, and writing $r=n\Delta$, with n being an integer, one may get the following difference equation:

$$\varphi_{n+1}=A_n(l)B_n(l)\varphi_n+C_n(l)\varphi_{n-1}, \quad n=1,2,\ldots,N, \tag{21}$$

where

$$A_n(l)=2-\Delta^2K^2+\Delta^2U_n, \tag{22}$$

$$B_n(l)=\frac{n}{l+1+n}, \tag{23}$$

and

$$C_n(l)=\frac{l+1-n}{l+1+n}. \tag{24}$$

It is clear that $A_n(l)$ is the most important term since it contains all the information we are looking for, i.e., U_n.

For the calculation of scattering amplitude or phase shifts for a given potential, directly, one may choose a point R such that $V_N(R)=0$ and calculate at that point logarithmic derivative for each partial wave l:

$$Z_N(l)=r\frac{(d\varphi_l)/dr}{\varphi_l}\bigg|_{r=R}. \tag{25}$$

Replacing the derivative in Eq. (25) by central difference at $R=N\Delta$, one obtains

$$Z_N(l)=\left(\frac{N}{2}\right)\left(\frac{\varphi_{N+1}-\varphi_{N-1}}{\varphi_N}\right). \tag{26}$$

To evaluate Eq. (26) for a given potential, one needs the wave functions at φ_N, φ_{N-1} and φ_{N+1}. They may be obtained by noting that for $n=N$, Eq. (21) reduces to

$$\frac{\varphi_{N+1}}{\varphi_N}=A_N(l)B_N(l)+C_N(l)/(\varphi_N/\varphi_{N-1}). \tag{27}$$

One may replace $(\varphi_N/\varphi_{N-1})$ by the corresponding equation and repeat the process to obtain the following expansion of $(\varphi_N/\varphi_{N+1})$:

$$\frac{\varphi_{N-1}}{\varphi_N}=A_N(l)B_N(l)+\left[\frac{C_N(l)}{A_{N-1}(l)B_{N-1}(l)+}\,\frac{C_{N-1}(l)}{A_{N-2}(l)B_{N-2}(l)+}\cdots\frac{C_3(l)}{A_2(l)B_2(l)}\,\frac{C_2(l)}{A_1(l)B_1(l)+C_1(l)/(\varphi_1/\varphi_0)}\right]. \tag{28}$$

Since $C_m(l)=0$, for $m=l+1$, the last term $C_1(l)/(\varphi_1/\varphi_0)$, in the continued fraction expression for $(\varphi_{N+1}/\varphi_N)$, does not enter in actual calculations. One may calculate $(\varphi_{N-1}/\varphi_N)$ similarly, and finally the logarithmic derivative $Z_N(l)$ at the point where $V_N(r)=0$.

One may connect $Z_N(l)$ to phase shifts by matching it to the logarithmic derivative of the external solution ψ_l. For example, in the absence of a Coulomb potential, $V_c=0$ and the solution for $r>R$ is given by

$$\psi_l=Kr\{j_l(Kr)+T_l[\,\eta_l(Kr)+ij_l(Kr)]\}, \tag{29}$$

where $j_l(Kr)$ and $\eta_l(Kr)$ are the spherical Bessel and Neumann functions, respectively. [In case $V_c\neq 0$ for $r>R$, $j_l(Kr)$ and $\eta_l(Rr)$ are replaced by Coulomb functions that are regular and singular at $r=0$, respectively.] The matrix T_l is related to the phase shift δ_l and the S-matrix S_l by

$$T_l=\frac{e^{2i\delta_l}-1}{2i}=\frac{S_l-1}{2i}. \tag{30}$$

In case $V_N(R)$ is complex, δ_l is complex.

The inverse problem in the absence of a Coulomb potential may be stated as follows: From given logarithmic derivatives for all partial waves at a fixed energy, i.e., K^2 at a point where $V_N(r)=0$, one is to determine the potential. To examine this point, one notes that at $r=N$, where $V_N(R)=0$, the value of $A_N=2-\Delta^2K^2$ and hence, is known. Combining Eqs. (26) and (27), one obtains

$$\frac{\varphi_N}{\varphi_{N-1}}=\frac{[C_N(l)-1]}{\left[\frac{2}{N}Z_N(l)-A_N(l)B_N(l)\right]}, \tag{31}$$

where $l=0,1,2,\ldots,L$ and $N=L+1$, L being the largest of the partial waves.

One may define a particular l_n which makes the corresponding C_n zero. Thus, for

$$l_n=N-1+n,\quad n=1,2,\ldots,N-1. \tag{32}$$

One has

$$C_n(l_{N-n})=0, \tag{33}$$

which follows from Eq. (24). With this and for $n=N-1$, the difference equation (22) reads

$$A_{N-1}(l_1)=\frac{1}{B_{N-1}(l_1)}\,\frac{\varphi_N(l_1)}{\varphi_{N-1}(l_1)}. \tag{34}$$

This inward iteration process can be continued to find all A_{N-j} at the points for $j=2,3,\ldots,N-1$,

$$A_{N-j}(l_j)=\frac{1}{B_{N-j}(l_j)}\left[\frac{C_{N+1-j}(l_j)}{-A_{N+1-j}(l_j)B_{N+1-j}(l_j)+}\cdots\times\frac{C_{N-2}(l_j)}{-A_{N-2}(l_j)B_{N-2}(l_j)+}\times\frac{C_{N-1}(l_j)}{-A_{N-1}(l_j)B_{N-1}(l_j)+\varphi_N(l_j)/\varphi_{N-1}(l_j)}\right],$$
$$j=2,3,\ldots,N-1. \tag{35}$$

The function U_n at the point n, where A_{N-j} is known, can be evaluated by using the relation

$$U_n=\frac{1}{\Delta^2}[A_n-2+\Delta^2K^2]. \tag{36}$$

This leads to the determination of the effective potential at point n $(V_{\rm eff})_n$

$$(V_{\rm eff})_n=\frac{\hbar^2c^2}{2E}U_n=V_N(r)-\frac{V_N^2(r)}{2E}. \tag{37}$$

For a complex V_N it can be expressed as the sum of imaginary (V_I) and real (V_R) parts,

$$V_N=iV_I+V_R. \tag{38}$$

Then the effective imaginary part of potential, $(V_{\rm eff})_{\rm Imag}$, is given by

$$(V_{\rm eff})_{\rm Imag}=V_1-\frac{V_1V_R}{E} \tag{39}$$

and the effective real part of potential, $(V_{\rm eff})_R$, is given by

$$(V_{\rm eff})_R=V_R-\frac{(V_R^2-V_1^2)}{2E}. \tag{40}$$

Thus, given $Z_N(l)$, one can determine the values of the potential at points $(1,2,\ldots N-1)$. The number of points is limited by the relation (32) to $(l+1)$, i.e., by the number of partial waves that contributes to the cross section at a given energy.

Despite the similarity between Eq. (15) and the Schrödinger equation, there are three important distinctions that merit investigation of the inverse problem for potentials relevant to pion-nucleus scattering. First, the real and imaginary part of the actual potential is very different from the effective potential given by Eqs. (39) and (40). Both the real and the imaginary part of the effective potential are determined by the magnitude of the real and imaginary part of the actual potential. For example, a strong imaginary part of the actual potential could make the real part of the effective potential repulsive, even though the real part of the actual potential is attractive. Secondly, the relation (37) implies, that near the surface, the effective and actual potentials are the same, but not in the interior. Thirdly, the effective potential (37) is actually energy-dependent and hence, the inverse problem is that of an energy-dependent potential, which has not been examined before.

With these considerations in mind, we apply the method to three cases having some relevance to pion-nucleus and relativistic spin-zero nucleus-nucleus scattering.

III. APPLICATION AND DISCUSSION

We test the theory for three types of complex potentials: for a shallow and a deep monotic potential and a nonmonotomic one. All calculations have been performed for 163.3 MeV incident uncharged pion on ^{40}Ca target. For this purpose, $Z_l(N)$ is calculated from a given potential by solving Eq. (15). These are, then, used as inputs for the inverse problem to determine the potential at $(l+1)$ points. The first potential chosen is relevant to pion scattering by ^{40}Ca and used by Satchler [10] to fit the data at angles other than backward ones. It is given by

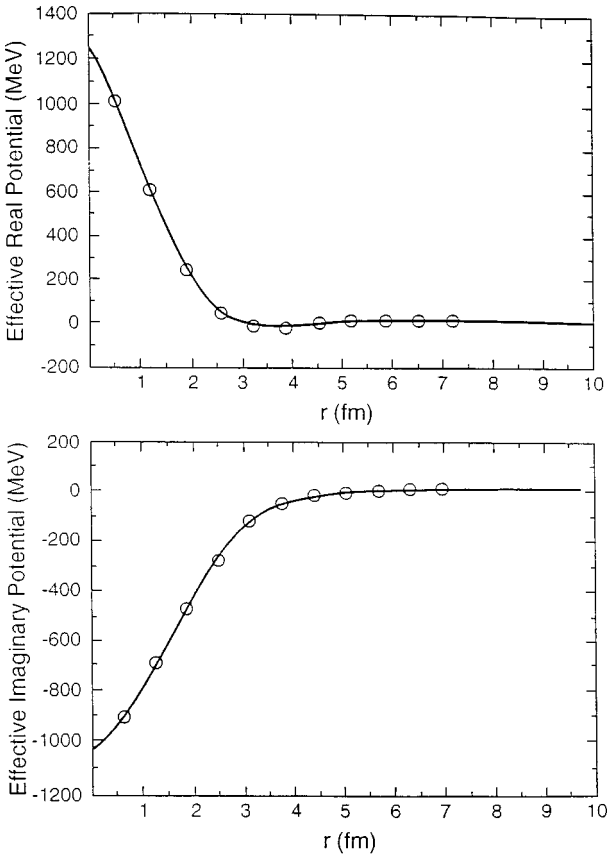

FIG. 1. The real and imaginary parts of the effective potential $V_1(r)$ given by Eq. (41) are shown as solid lines. The open circles are the calculated points of the potential obtained from the solution of the inverse problem.

$$V_1(r)=\frac{V}{1+\exp[(r-R_0)/a_0]}+i\,\frac{W}{1+\exp[(r-R_1)/a_1]}, \tag{41}$$

with

$$V=-45\ \text{MeV},\ R_0=4.271\ \text{fm},\ a_0=0.305\ \text{fm},$$

$$W=-1000\ \text{MeV},\ R_1=1.688\ \text{fm},\ a_1=0.763\ \text{fm}.$$

The real and imaginary part of the corresponding effective potential are calculated from Eqs. (40) and (39), respectively, and shown by solid lines in Fig. 1. It is interesting to note that the effective real part of the potential is repulsive at short distances, although the potential itself is attractive. Only ten partial waves contribute to the scattering process and one can calculate ten values of $Z_N(l)$. For the inverse problem, these ten values of $Z_N(l)$ are given as inputs. From these, the effective potential at 11 points has been determined and are shown in Fig. 1 as open circles. These follow closely the input potential. Thus, one can determine the points of this type of potentials rather accurately by this inverse scattering method.

The second potential considered has a deep real part and moderate imaginary part and is taken to be

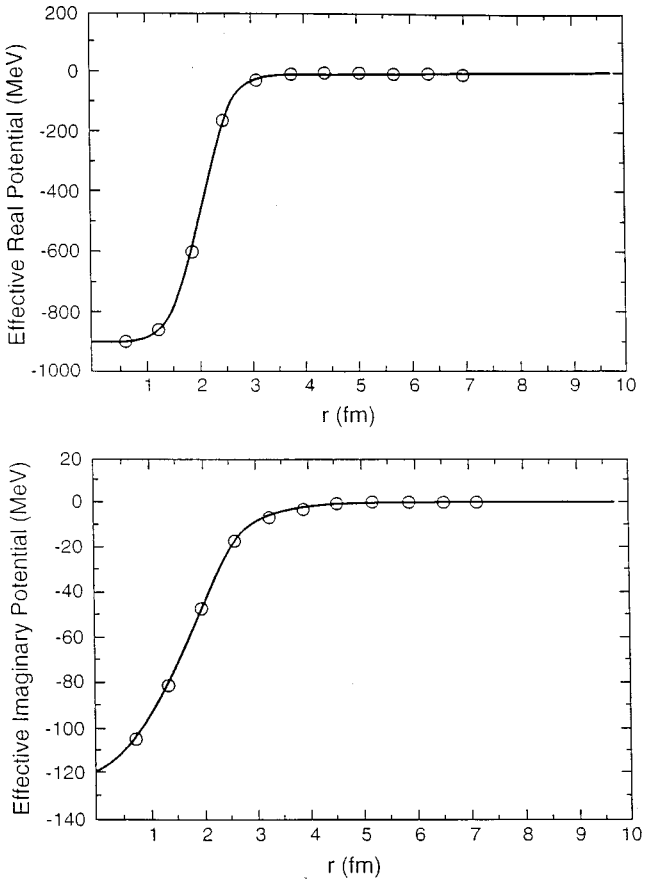

FIG. 2. The real and imaginary parts of the effective potential $V_2(r)$ given by Eq. (42) are shown as solid lines. The open circles are the calculated points of the potential obtained from the solution of the inverse problem.

$$V_2(r)=\frac{V}{1+\exp[(r-R_0)/a_0]}+i\frac{W}{1+\exp[(r-R_1)/A_1]}, \tag{42}$$

with

$$V=-500 \text{ MeV}, \quad R_0=2.271 \text{ fm}, \quad a_0=0.305 \text{ fm},$$

$$W=-50 \text{ MeV}, \quad R_1=1.688 \text{ fm}, \quad a_1=0.763 \text{ fm}.$$

The real and imaginary parts of the effective potential for $V_2(r)$ are plotted in Fig. 2 as solid lines. The effective real part is strongly attractive. $Z_l(N)$ calculated from this potential is used as inputs for the inverse problem. The calculated values of the potential determined at eleven points are shown as open circles in Fig. 2 and once again the agreement is satisfactory.

The third potential, $V_3(r)$ is chosen to be a nonmonotonic one with moderate absorption and given by

$$V_3(r)=\frac{V_0}{1+\exp[(r-R_0)/a_0]}+V_1\exp(-r/R_2)+i\frac{W}{1+\exp[(r-R_1)/a_1]}, \tag{43}$$

with

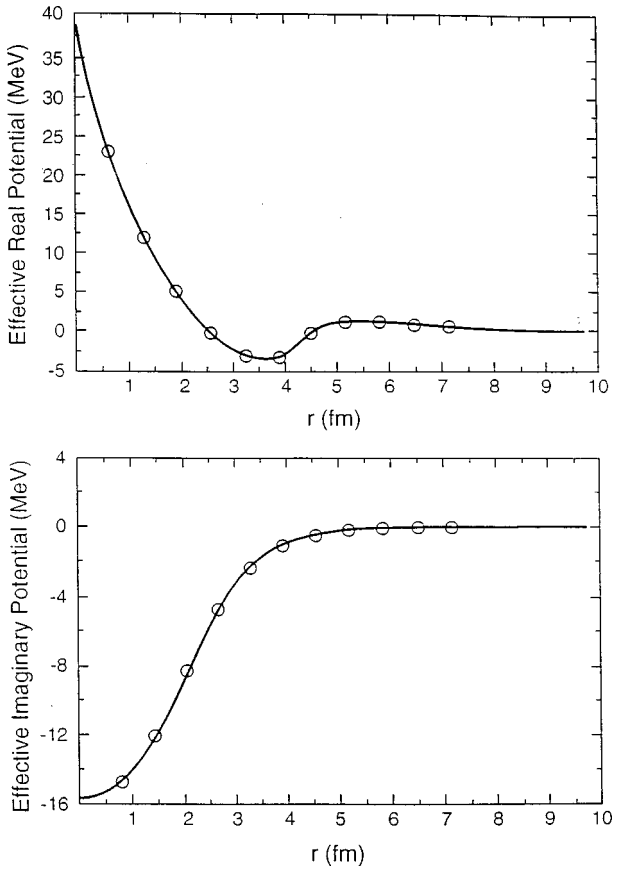

FIG. 3. The real and imaginary parts of the effective potential $V_3(r)$ given by Eq. (43) are shown as solid lines. The open circles are the calculated points of the potential obtained from the solution of the inverse problem.

$$V_0=-9 \text{ MeV}, \quad R_0=4.271 \text{ fm}, \quad a_0=0.305 \text{ fm},$$

$$V_1=50 \text{ MeV},$$

$$R_2=1.50 \text{ fm}, \quad W=-20 \text{ MeV}, \quad R_1=1.688 \text{ fm},$$

$$a_1=0.763 \text{ fm}.$$

The real and imaginary part of the effective potential for $V_3(r)$ are shown in Fig. 3 as solid lines. It is interesting to note that the real and imaginary parts are of moderate strengths. $Z_l(N)$ is then calculated for each partial wave and used as an input for the inverse problem. The potential determined from the inverse problem are shown as open circles in Fig. 3. Once again the points of the potential determined from the inverse problem are very close to the actual ones.

At this point, it is appropriate to discuss the strength and the limitation of the method. The number of points of a potential function determined by the method is equal to one plus the number of partial waves contributing to the cross section. If the number of partial wave is small compared to the range of the potential, the step size, i.e., the interval between two adjacent points of the potential is large and the replacement of differential equation by difference equation is not a good approximation [3–6]. On the other hand, if the number of partial waves are large resulting into very small but a large number of step sizes, the error at each point, even

though small, accumulates. As noted in the analysis of Leeb [11] and in [3–6], this propagation of error may, in some cases, make the determination of the points in the interior region uncertain. For a large number of partial waves each having very small phase shift, the error propagation may also lead to the loss of information in the interior region as noted by Leeb.

The method is best suited for the incident energy range that provides information on 10 to 20 partial waves for potentials having 6 to 10 F range. In general, the method is useful in determining the surface and overall shapes of potentials. For example, the application of the method to the α-^{12}C system has ruled out the potential to be a deep one [5] and to the ^{12}C-^{12}C system points towards a complex molecular potential for that system [6]. The information about the interior part of the potential is to be treated with caution.

IV. CONCLUSION

The inverse scattering method at a fixed energy applied to the Klein-Gordon equation can determine reasonable information about spherical symmetric potentials at least in the energy range considered here. The success of the method depends on the number of partial waves involved and accurate determination of the logarithmic derivative, $Z_l(N)$, at the boundary. Since the Klein-Gordon equation is effectively a Schrödinger equation with an energy-dependent potential, the method may, in some cases, be suitable for energy (or momentum)-dependent potentials.

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