Anomalous distributions in heavy ion collisions at high energies

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A multifractal Bernoulli distribution, which appears by a natural way at some morphological phase transition, is introduced and it is shown that this distribution gives a good fit to the data obtained in laboratory experiments and in a numerical simulation of the particle multiproduction in the heavy ions collisions at high energies. [S0556-2813(99)03801-7]

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I. INTRODUCTION

It is expected that particle multiproduction in nuclear collisions at high energies related to phase-transition-like phenomena (see for recent reviews Refs. [1,2]). These phase transitions imply anomalous distributions. Different distributions laws were suggested to interpret experimental data beginning from papers [3]. Simple Bernoulli distribution [4] corresponding to monofractal states was suggested in Ref. [5] using an analogy with second order phase transitions (see also further development of this analogy in Refs. [6] and [7]). The log-normal multiplicity distribution related to self-similar cascade processes [3] as well as the Levy stable distribution and the negative binomial distribution are widely discussed in the literature (see Ref. [1], and references therein).

We, however, believe that an adequate distribution should appear by a natural way when transition from monofractality to multifractality is studied. We also believe that this morphological transition could play a crucial role in the particle multiproduction at high energies (which has a critical nature). In the present paper we perform such an investigation and introduce a new type of statistical distribution—a multifractal Bernoulli distribution—to describe this transition. This new distribution is then used to find some systematics in the data on fractal parameters of the multiparticle spectra observed in collisions of heavy ions.

Three methods (with some modifications) are used at present to obtain multifractal spectra from the experimental data: factorial moments [3], so-called G moments [8], and Takagi methodology [9]. The method of factorial moments has some problems just in the case of heavy ions collisions [10], while the G moments are known to be strongly biased by ''statistical noise,'' particularly important in the small bins where the multiplicity is small. The comparatively new Takagi methodology has been applied to the analysis of the heavy ions collisions data only very recently [11]. Comparing the data obtained by the G-moments method with the data obtained by the Takagi methodology we shall conclude that the G moments give reasonable results for very heavy ions (such as ^{238}U and ^{197}Au) collisions while for *moderate* heavy ions more fine Takagi methodology should be used. In both these cases it is shown that the multifractal Bernoulli distribution can be used to fit the experimental data on the multiparticle production. Moreover, it is shown that the multifractal specific heat for the data obtained in the *very* heavy ions collisions (using G-moments methodology) and in *moderate* heavy ions collisions (using Takagi methodology) have the same gape (equal to 1/4) at the morphological phase transition described by the multifractal Bernoulli distribution. We do not know why the G moments seem to be applicable to obtain the adequate results just for the *very* heavy ions collisions and, therefore, direct calculation of the multifractal spectra for the very heavy ions collisions with Takagi methodology remains an interesting problem for future investigations.

II. MULTIFRACTAL BERNOULLI DISTRIBUTION

Let $\Delta\eta$ be the pseudorapidity interval, and subdivide into M bins each of width $\delta\eta=\Delta\eta/M$. Let N be the number of particles in one event in $\Delta\eta$ interval and k_m be the number of particles in the mth bin. The G_q moments are defined as [8]

$$G_q=\sum_{m=1}^{M} \mu_m^q, \tag{1}$$

where $\mu_m=k_m/N$ is the probability of particles in the mth bin for one event and q is any real number. The summation is carried out over nonempty bins only. If the particle production process exhibit self-similar behavior then the moment follow the power law

$$G_q \propto (\delta\eta/\Delta\eta)^{\tau(q)}. \tag{2}$$

The generalized dimension spectrum is then given by

$$D_q=\tau(q)/(q-1). \tag{3}$$

Then, if one uses standard averaging one obtains

$$\langle\mu^q\rangle=\frac{\Sigma_{i=1}^{M}[\mu_i(l)]^q}{M}\propto(\delta\eta/\Delta\eta)^{(\tau_q+1)}. \tag{4}$$

Let us define

$$\overline{\mu_i}=\mu_i/\max_i\{\mu_i\}.$$

Then

$$\langle\overline{\mu}^p\rangle=\frac{1}{M}\sum_i \overline{\mu}_i^p. \tag{5}$$

The simplest structure, that can be used for fractal description, is a system for which $\overline{\mu_i}$ can take only two values 0 and 1. It follows from Eq. (5) that for such a system (with $p>0$)

$$\langle \overline{\mu}^p \rangle = \langle \overline{\mu} \rangle \tag{6}$$

and fluctuations in this system can be identified as Bernoulli fluctuations [4]. The Bernoulli probability law with parameter $\langle \overline{\mu} \rangle$ is specified by the probability mass function $P(x)$ given by $P(x)=\langle \overline{\mu} \rangle$ for $x=1$, $P(x)=1-\langle \overline{\mu} \rangle$ for $x=0$, and $P(x)=0$ otherwise. It is clear that the Bernoulli distribution can be monofractal only (see also Ref. [14]). The characteristic function of the Bernoulli distribution is

$$\chi(\lambda)=1+\langle \overline{\mu} \rangle (e^{i\lambda}-1). \tag{7}$$

Generalization of Eq. (6) in the form of a generalized scaling

$$\langle \overline{\mu}^p \rangle \sim \langle \overline{\mu} \rangle^{f(p)} \tag{8}$$

can be used to describe more complex (multifractal) systems. We use invariance of the generalized scaling (8) with dimension transform [12]

$$\overline{\mu_i} \to \overline{\mu_i}^{\lambda}$$

to find $f(p)$. This invariance means that

$$\langle (\overline{\mu}^{\lambda})^p \rangle \sim \langle (\overline{\mu}^{\lambda}) \rangle^{f(p)} \tag{9}$$

for all positive λ. Then, it follows from Eqs. (8) and (9) that

$$\langle (\overline{\mu})^{\lambda p} \rangle \sim \langle \overline{\mu} \rangle^{f(\lambda p)} \sim \langle \overline{\mu} \rangle^{f(\lambda) f(p)}. \tag{10}$$

Hence,

$$f(\lambda p)=f(\lambda) f(p). \tag{11}$$

The general solution of functional equation (11) is

$$f(p)=p^{\gamma}, \tag{12}$$

where γ is a positive number. It should be noted that case $\gamma=1$ corresponds to Gauss fluctuations [13]. We, however, shall consider limit $\gamma \to 0$ (i.e., transition to the Bernoulli fluctuations). This transition is nontrivial. Indeed, let us consider generalized scaling

$$F_{qm} \sim F_{km}^{\alpha(q,k,m)}, \tag{13}$$

where

$$F_{qm}=\langle \overline{\mu}^q \rangle / \langle \overline{\mu}^m \rangle. \tag{14}$$

Substituting Eq. (8) into Eqs. (13), (14) and using Eq. (12) we obtain

$$\alpha(q,k,m)=\frac{q^{\gamma}-m^{\gamma}}{k^{\gamma}-m^{\gamma}}.$$

Hence,

$$\lim_{\gamma \to 0} \alpha(q,k,m)=\frac{\ln(q/m)}{\ln(k/m)}. \tag{15}$$

If there is ordinary scaling

$$\langle \overline{\mu}^p \rangle \sim \langle \delta\eta / \Delta\eta \rangle^{\zeta p}, \tag{16}$$

then

$$\alpha(q,k,m)=\frac{\zeta_q-\zeta_m}{\zeta_k-\zeta_m}. \tag{17}$$

From a comparison of Eqs. (15) and (17) we obtain at the limit $\gamma \to 0$

$$\frac{\zeta_q-\zeta_m}{\zeta_k-\zeta_m}=\frac{\ln(q/m)}{\ln(k/m)}. \tag{18}$$

The general solution of functional equation (18) is

$$\zeta_q=a+c \ln q, \tag{19}$$

where a and c are some constants.

If we use the relationship

$$\max_i \{\mu_i\} \sim (\delta\eta/\Delta\eta)^{D_\infty} \tag{20}$$

(see, for instance, Ref. [14]), then it follows from Eqs. (3)–(5) and (16), (19), (20) that

$$D_q=D_\infty+c \frac{\ln q}{(q-1)} \tag{21}$$

for the multifractal Bernoulli fluctuations (i.e., for the fluctuations which appear at the limit $\gamma \to 0$).

III. GAP OF THE MULTIFRACTAL SPECIFIC HEAT

From Eqs. (8), (16), and (19) we can find $f(p)$ corresponding to the multifractal Bernoulli fluctuations

$$f(p)=1+\frac{c}{a} \ln p, \tag{22}$$

where $a=d-D_\infty$. One can see that for finite c the dimension-invariance is broken at the limit $\gamma \to 0$.

Let us find the characteristic function of the multifractal Bernoulli distribution. It is known that the characteristic function $\chi(\lambda)$ can be represented by the following series (see, for instance, Ref. [4]):

$$\chi(\lambda)=\sum_{p=0}^{\infty} \frac{(i\lambda)^p}{p!} \langle \overline{\mu}^p \rangle. \tag{23}$$

Then using Eqs. (8) and (22) we obtain from Eq. (23)

$$\chi(\lambda)=1+\langle \overline{\mu} \rangle \sum_{p=1}^{\infty} \frac{(i\lambda)^p}{p!} p^{\beta}, \tag{24}$$

where

$$\beta=\frac{c}{(d-D_\infty)} \ln \langle \overline{\mu} \rangle. \tag{25}$$

The characteristic function (24) gives complete description of the multifractal Bernoulli distribution. When $c=0$ distri-

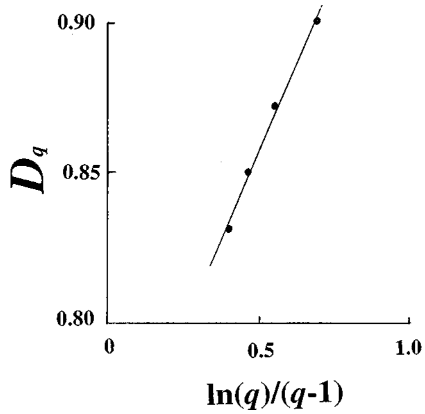

FIG. 1. Generalized dimension spectrum (in pseudorapidity space) for ^{197}Au collisions on 10.6A GeV (dots). Data taken from Ref. [17]. The straight line is drawn for comparison with the multifractal Bernoulli representation (21).

bution (24), (25) coincides with the simple Bernoulli distribution (7). The multifractality-monofractality phase transition (with $\gamma \rightarrow 0$) corresponds to a gap from $c=0$ to a finite nonzero value of c. If we use a thermodynamic interpretation of the multifractality represented in Ref. [15], then the constant c can be interpreted as multifractal specific heat of the system. The gap of the multifractal specific heat at the multifractality-monofractality transition (i.e., with $\gamma \rightarrow 0$) allows us to consider this transition as a thermodynamic phase transition [16].

IV. LABORATORY DATA ON VERY HEAVY IONS COLLISIONS

Let us compare these theoretical results with laboratory data. Figure 1 shows a generalized dimension spectrum D_q against variable $\ln(q)/(q-1)$. This experimental spectrum (dots) was calculated in a recent paper [17] using the pseudorapidity phase space for the shower particles produced in the interactions of ^{197}Au emulsion at 10.6A GeV. The straight line in this figure indicates good agreement between the data and the multifractal Bernoulli representation (21). Analogous data on ^{28}Si ions collisions (also represented in Ref. [17]) do not give such clear indication of the morphological phase transition. This trend is confirmed by the data represented in Ref. [18] and obtained for projective fragments in nuclear collisions at $(1-2)A$ GeV. Figure 2 shows generalized dimension spectra calculated in Ref. [18] for ^{238}U at 0.96A GeV both in the pseudorapidity (lower set of dots) and in the azimuthal (upper set of dots) phase spaces. Again the straight lines drawn in this figure indicate good agreement between the data and the multifractal Bernoulli representation (21). Analogous data calculated in Ref. [18] for ^{84}Kr and for ^{56}Fe ion collisions do not give such a clear indication.

If we calculate the multifractal specific heat from Fig. 1 we obtain for the ^{197}Au reactions value $c \simeq \frac{1}{4}$. Figure 2 also gives for the ^{138}U reactions value $c \simeq \frac{1}{4}$ both in pseudorapidity and in azimuthal phase spaces. One can see that the value of the multifractal specific heat does not practically depend on the type of very heavy ions as well as on their energy or on the type of phase space. This observation can be considered as an indication of an universal nature of this value of the multifractal specific heat (see next section).

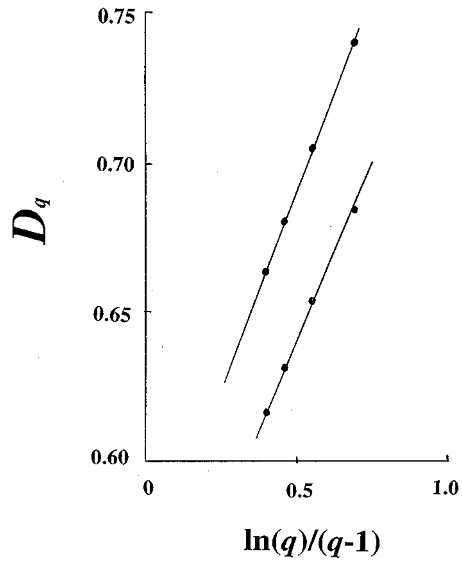

FIG. 2. Generalized dimension spectra for ^{238}U collisions at 0.96A GeV (dots). Data taken from Ref. [18]. The lower set of dots corresponds to pseudorapidity phase space and the upper set of dots corresponds to azimuthal phase space. The straight lines are drawn for comparison with the multifractal Bernoulli representation (21).

V. TAKAGI METHODOLOGY AND LABORATORY DATA FOR MODERATE HEAVY IONS COLLISIONS

Let us recall briefly the formalism introduced by Takagi [9]. A single event contains n particles distributed in the interval $y_{\min}<y<y_{\max}$ in the rapidity (y) space. The multiplicity n changes from event to event according to the distribution $P_n(y)$ where $y=y_{\max}-y_{\min}$. Divide the full rapidity interval of length y into ν bins of equal size $\delta y=y/\nu$. The multiplicity distribution for a single bin is denoted as $P_n(\delta y)$. Particles produced in Ω independent events are distributed in $\Omega\nu$ bins of size δy. Let K be the total number of particles produced in the Ω events and n_{ai} the multiplicity of particles in the ith bin of ath event. If the multifractal approach is applicable for the system then the quantity

$$T_q(\delta y)=\ln \sum_{a=1}^{\Omega} \sum_{i=1}^{\nu} (p_{ai})^q \tag{26}$$

(where $p_{ai}=n_{ai}/K$) behaves similar to a linear function of the logarithm of the ‘‘resolution’’ $R(\delta y)$,

$$T_q(\delta y)=A_q+B_q \ln R(\delta y). \tag{27}$$

If such a behavior is observed for a considerable range of $R(\delta y)$, the generalized dimension may be determined as $D_q=B_q/(q-1)$. For sufficiently large Ω, one has [9]

$$\sum_{a=1}^{\Omega} \sum_{i=1}^{\nu} (p_{ai})^q=\sum_{n=0}^{\infty} \Omega_\nu P_n(\delta y)(n/K)^q=\frac{\langle n^q\rangle}{K^{(q-1)}\langle n\rangle}, \tag{28}$$

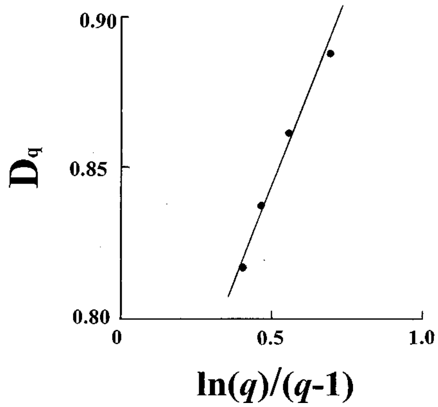

FIG. 3. Generalized dimension spectra for ^{32}S-AgBr interactions at 200*A* GeV [data (dots), taken from Ref. [11]]. The data obtained in azimuthal phase space using Takagi methodology. The straight line indicates agreement with multifractal Bernoulli representation (21).

where $\langle \int(n)\rangle=\Sigma_{n=0}^{\infty}\int(n)P_n(\delta y)$, and $\Omega\nu=K/\langle n\rangle$. Then the D_q compute from the experimental data using the relation

$$\ln\langle n^q\rangle=A_q+[(q-1)D_q+1]\ln\langle n\rangle. \tag{29}$$

Figure 3 [data (dots) taken from Ref. [11]] shows data of multiparticle production obtained in azimuthal phase space using the Takagi methodology for dynamical fluctuation of target evaporated particles in ^{32}S-AgBr interactions at 200*A* GeV. The straight line indicate agreement with multifractal Bernoulli representation (21). It is interesting that the multifractal specific heat calculated from this figure $c\simeq\frac{1}{4}$ as for *very* heavy ions (see previous section).

Figure 4 (adapted from Ref. [19]) shows the data obtained (using the Takagi methodology) in the pseudorapidity phase space at ^{12}C-AgBr interactions at 4.5*A* GeV for ''hot'' and ''cold'' events as characterized by two temperatures (40 and 10 MeV correspondingly). The straight lines in this figure indicate agreement between the data (dots) and the multifractal Bernoulli representation (21). The multifractal specific heat $c\simeq\frac{1}{3}$ both for ''hot'' and for ''cold'' events. It should

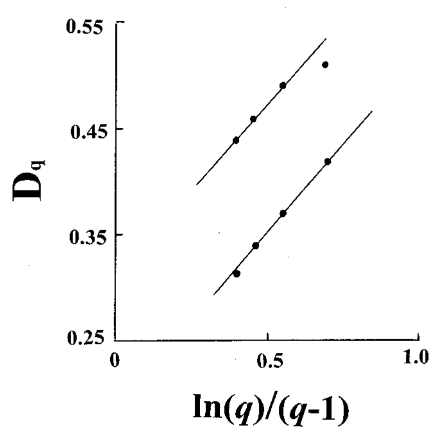

FIG. 4. The generalized dimensions D_q against $q \ln q/(q-1)$ for ''hot'' (upper set of dots) and for ''cold'' (lower set of dots) events in ^{12}C-AgBr collisions at 4.5*A* GeV (adapted from Ref. [19]). The straight lines are drawn for comparison with representation (21).

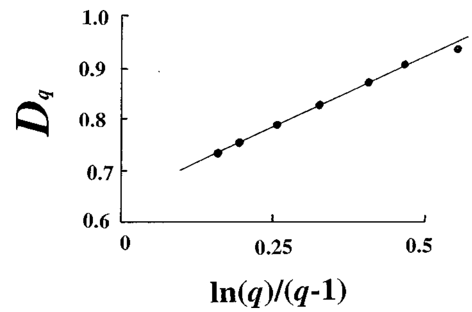

FIG. 5. Generalized dimension spectrum for one single-jet event (adapted from Ref. [20]). The straight line is drawn for comparison with the multifractal Bernoulli representation (21).

be noted that for the relatively ''light'' heavy-ions ^{12}C the experimentally observed value of the multifractal specific heat is different from the value $c\simeq\frac{1}{4}$ observed in interactions of very and moderate heavy ions such as ^{197}Au, ^{238}U, and ^{32}C (also see Sec. VII).

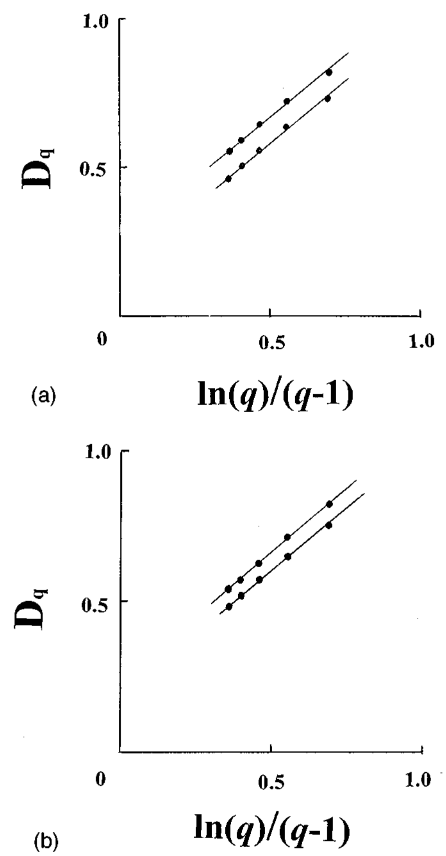

FIG. 6. (a) Generalized dimension spectra for ^{16}O-AgBr interactions at 60 GeV/nucleon (data taken from Ref. [21]). The upper set of the data corresponds to pseudorapidity phase space and the lower set of the data corresponds to azimuthal phase space. Data in this figure were obtained in the interval $2<M<21$. The straight lines are drawn for comparison with the multifractal Bernoulli representation (21). (b) Generalized dimension spectra for ^{16}O-AgBr interactions at 60 GeV/nucleon (data taken from Ref. [21]). The upper set of the data corresponds to pseudorapidity phase space and the lower set of the data corresponds to azimuthal phase space. Data in this figure were obtained in interval $2<M<35$. The straight lines are drawn for comparison with the multifractal Bernoulli representation (21).

VI. DATA OF A NUMERICAL EXPERIMENT

It is also interesting to check whether numerical simulations based on some models of underlying dynamics of the heavy ions reactions exhibit the multifractal Bernoulli distribution law as well. It is pointed out in a recent paper [20] that just the expectation of a phase transition from hadronic matter to quark-gluon plasma at high temperatures or density has created excitement in the field of ultrarelativistic heavy ions collisions. The authors of Ref. [20] proposed to look for a generalized dimension spectrum in azimuthal phase space to check this possibility. Namely, a large number of particles emerging within a narrow azimuthal angular range is described as a jetlike event. Then, the generalized dimensions D_q with $q>1$ should be dominated just the localized (in the phase space) events with large concentration of the particles (see, for instance, Ref. [14]). The authors of Ref. [20] claim that the jettiness can be a signature of phase transition in the multifractal formalism. They study jetlike central events in heavy ion collisions to see the consequences in the azimuthal plane. In particular, they simulate single jetlike events with very high multiplicity. Figure 5 (adapted from Ref. [20]) show an example of the generalized dimension spectra (dots) obtained in Ref. [20] for a single-jet event in the azimuthal phase space. Again the straight line indicates good agreement of the data with the multifractal Bernoulli distribution (21). In this case, however, this distribution is applicable for $q>3$ only. The constant multifractal specific heat, calculated from this figure, $c\simeq 0.56$ is different from the experimentally observed values.

VII. DISCUSSION AND SUMMARY

If one compares Figs. 3 and 4 (obtained using Takagi methodology) with Figs. 1 and 2 (obtained by G-moments methodology for *very* heavy ions) one can conclude that the G moments turn out to be applicable to obtain an adequate information just for *very* heavy ions collisions, while for *moderate* heavy ions one should use the more fine Takagi methodology. Moreover, using the Takagi methodology for the moderate heavy ions we obtain an additional indication of universality of the multifractal specific heat value $c\simeq \frac{1}{4}$ for very and moderate heavy ions, while for the relatively ''light'' heavy ions (such as ^{12}C) the multifractal specific heat could have different values. It is interesting to check this conclusion for relatively ''light'' heavy ions using also data obtained with the corrected factorial moments. Figures 6(a) and 6(b) [data (dots), taken from Ref. [21]] show data obtained using the scaled factorial moments of large local fluctuations of multiparticle production observed in ^{16}O-AgBr interactions at 60 GeV/nucleon. Upper sets of the data correspond to pseudorapidity phase space and lower sets of the data correspond to azimuthal phase space. Data in Fig. 6(a) were obtained in interval $2<M<21$ whereas data in Fig. 6(b) were obtained in interval $2<M<35$. The straight lines are drawn for comparison with the multifractal Bernoulli representation (21). The multifractal specific heat in this interaction is $c\simeq \frac{5}{6}$.

Thus, we can conclude that (1) the data considered in the present paper of different laboratory and numerical investigations exhibit the multifractal Bernoulli fluctuations and (2) it seems from comparison between data obtained with different methods that G moments could be used for *very* heavy ions reactions while for *moderate* heavy ions reactions the more fine Takagi methodology should be used. Moreover, the value of multifractal specific heat obtained for very and moderate heavy ions reactions $c\simeq \frac{1}{4}$ seems to be universal for this type of reactions, while for relatively ''light'' heavy ions reactions we do not observed such universality.

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